Relativistic Fermions in Condensed Matter Physics

"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."

— Sydney Coleman.

Quantum field theory is a (perhaps, the most powerful) theoretical framework to describe microscopic physical phenomena. As Sydney remarked, quantum field theory actually consists of an infinite number of harmonic oscillators at every spacetime points: for the description of microscopic physics, two kinds of quantum variables are used, i.e.,

(i) bosonic variables a_i, $a_i^\dagger$ satisfying

$$[a_j, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \ , \ \ [a_i, a_j^\dagger] = \delta_{ij} \ ,$$

(ii) fermionic variables c_i, $c_i^\dagger$ satisfying

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 \ , \ \ \{c_i, c_j^\dagger\} = \delta_{ij} \ .$$

The basis for the Hilbert space can be identified with

$$\left\{ |n_1, n_2, \cdots\rangle \equiv \prod_i \frac{1}{\sqrt{n_i!}} (a_i^\dagger)^{n_i} |0\rangle \ ; \ n_i = 0, 1, 2, \cdots \right\}$$

or

$$\left\{ |n_1, n_2, \cdots\rangle \equiv \prod_i (c_i^\dagger)^{n_i} |0\rangle \ ; \ n_i = 0, 1. \right\} \ .$$

Here $|0\rangle$ satisfying $a_i|0\rangle = 0$ (or $c_i|0\rangle = 0$) for all i is the (naive) vacuum state.

In terms of these building blocks, various filed theory systems can be constructed. For a bosonic example, 1-D lattice vibration can be described by a field theory whose Lagrangian has the form (massless Klein-Gordon type Lagrangian)

$$L = \int dx \left\{ \frac{\mu}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 \right\} \ .$$

A fermionic example is the Dirac QFT:

$$\mathcal{L} = -\bar{\psi}\left(\gamma^m u \frac{1}{i}\partial_\mu + m\right)\psi \, .$$

(Here γ^μ ($mu = 0, 1, \cdots, d$) ($d = D - 1$ is the number of spatial dimensions) are appropriate matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$ ($g_{\mu\nu}$ is the spacetime metric).) The corresponding Hamiltonian is given by

$$H = \int d^d x \, \psi^\dagger \left(\vec{\alpha} \cdot \frac{1}{i}\vec{\nabla} + m\gamma^0\right)\psi \, .$$

Massless (="gapless excitations") Dirac fermion appears in condensed matter physics

- from the fermion chain/spin system $\Leftarrow$ discussed below ,
- from the Schrödinger type QFT (Luttinger model) $\Leftarrow$ not discussed here .

The Hamiltonian defined on a 1-D (tight-binding) chain (periodic boundary condition is assumed)

$$H = \mathcal{E}\sum_i c_i^\dagger c_i + T\sum_i (c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}) \tag{1}$$

have the energy spectrum

$$E_l = \mathcal{E} + 2T\cos\left(\frac{2\pi}{N}l\right) \, , \; (l = 0, 1, \cdots, N-1) \, ,$$

with corresponding energy eigenstates (Bloch states)

$$|\Phi^{(l)}\rangle = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{i\frac{2\pi l}{N}(n-1)} c_n^\dagger |0\rangle \, .$$

Actually, it is possible to find the map (Jordan-Wigner transformation) connecting our system to the spin-$\frac{1}{2}$ (isotropic) XY model defined by the Hamiltonian

$$H = J = \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) - 2h\sum_i S_i^z \, .$$

Furthermore, the model (1) with $\mathcal{E} = 0$ and $T = J/2$ in continuum limit

$$H_0 = \frac{J}{2}\sum_i (c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} (J\cos k) c^\dagger(k) c(k)$$

in its ground state, i.e., at the Fermi points, is gapless since, for modes near the Fermi points $\cos k \approx \pm k$ where k is measured from the respective Fermi points. These modes are associated with the $(1+1)$-D massless Dirac field excitations.

Defining

$$b(n) \equiv i^{-n} c_n \ ,$$

H_0 becomes

$$H_0 = \frac{J}{2} \sum_n i b^\dagger(n) [b(n+1) - b(n-1)] \ ,$$

and if one introduces the 'spinor' operators $\phi_\alpha(s)$

$$\phi_1(s) = b(n = 2s) \ , \quad \phi_2(n = 2s+1) \quad (s = 0, 1, \cdots, N/2)$$

whose anti-commutation relations are given by

$$\{\phi_\alpha(s), \phi_\beta(s')\} = \{\phi_\alpha^\dagger(s), \phi_\beta^\dagger(s')\} = 0 \ , \ \{\phi_\alpha^\dagger(s), \phi_\beta(s')\} = \delta_{ss'} \delta_{\alpha\beta} \ ,$$

H_0 can be rearranged as follows:

$$H_0 = \frac{J}{2} i \left\{ \sum \phi_1^\dagger(s) [\phi_2(s) - \phi_2(s-1)] + \sum \phi_2^\dagger(s) [\phi_1(s+1) - \phi_1(s)] \right\} \ .$$

The continuum limit of this with $\psi_\alpha(x = 2as) = \frac{1}{\sqrt{2a}} \phi_\alpha(s)$ ($a \sim L/N$ is the lattice spacing) is the desired massless Dirac Hamiltonian:

$$\tilde{H}_0 = \frac{1}{Ja} H_0 = \int dx \ \psi^\dagger(x) \alpha \frac{1}{i} \partial_x \psi(x) \ , \quad \alpha = -\sigma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ ,$$

and

$$\{\psi_\alpha(x), \psi_\beta(x')\} = \{\psi_\alpha^\dagger(x), \psi_\beta^\dagger(x')\} = 0 \ , \ \{\psi_\alpha^\dagger(x), \psi_\beta(x')\} = \delta_{\alpha\beta} \delta(x - x') \ .$$

Therefore, the physics of some fermion chain and spin systems can be studied by examining the massless Dirac (relativistic) field theory.