Matching of Matrix Elements and Parton Showers
Lecture 1: QCD

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Plan of the lectures

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Lectures on matrix element/parton shower matching

1. Introduction to soft/collinear QCD and parton showers
   Tutorial 1: Study parton showers from Pythia and effects of jet definitions
2. Matrix element/Parton shower matching in e^+e^- collisions
   Tutorial 2: First studies of matching using MG-Pythia or Sherpa
3. Matrix element/Parton shower matching in hadronic collisions
   Tutorial 3: More studies of matching in different processes

Introduction: The big picture

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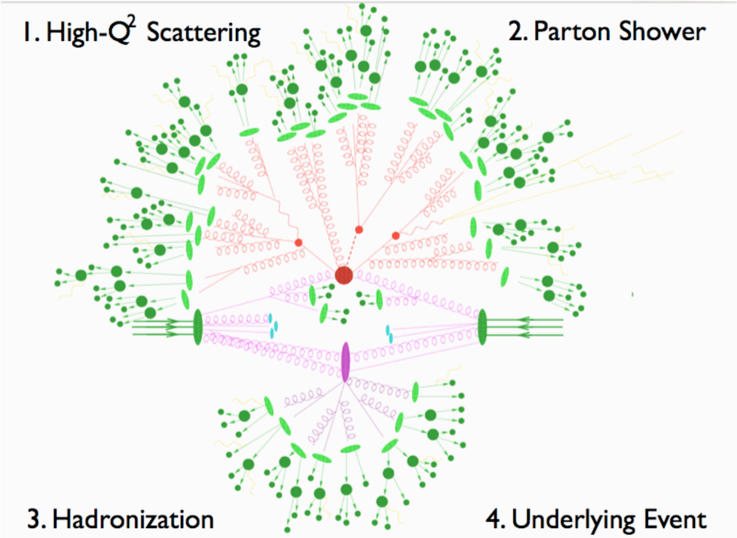
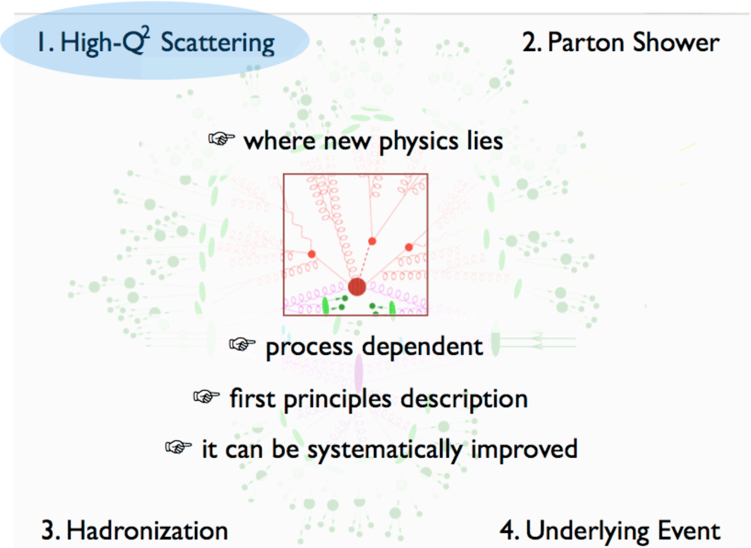


Introduction: The big picture



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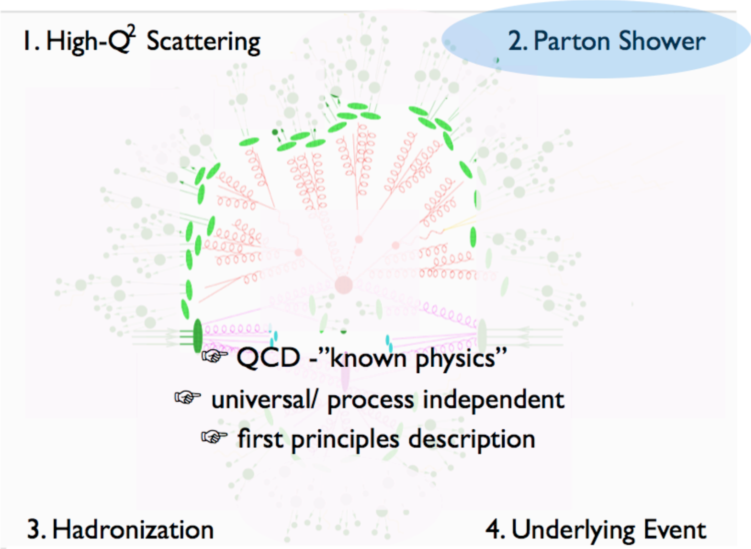

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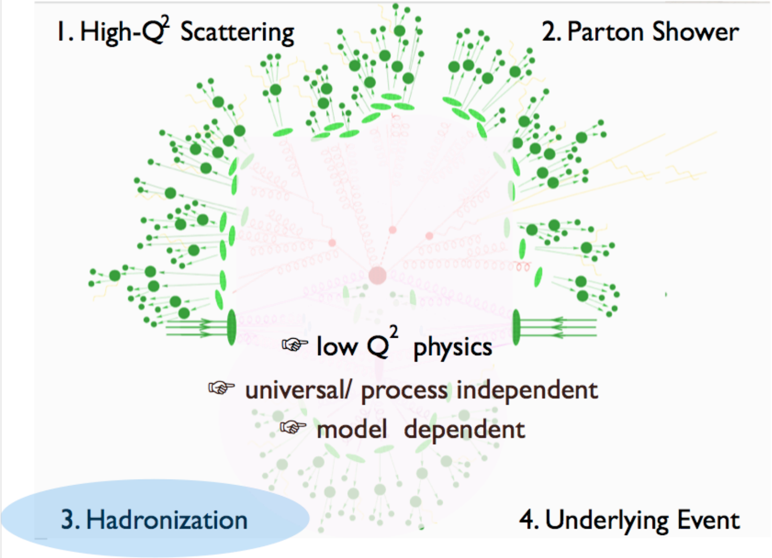
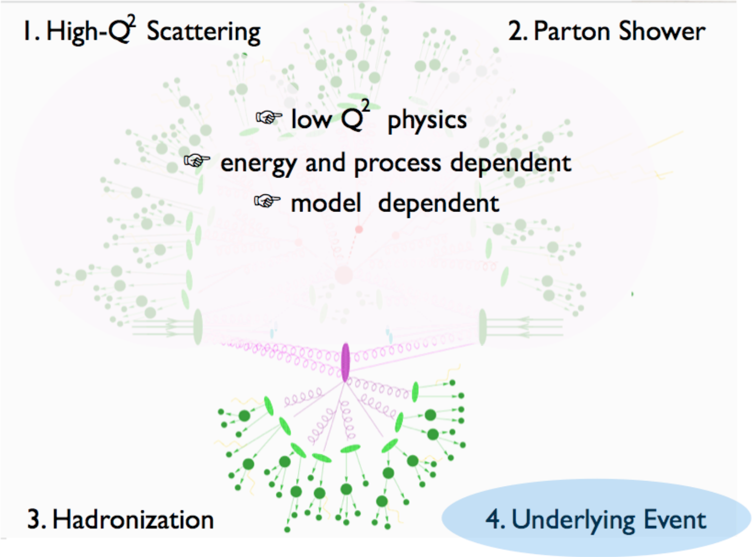
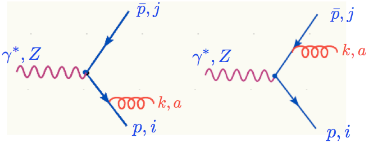


Introduction: The big picture



Infrared Behaviour of QCD

e^+e^- at NLO: Soft gluon emission



Consider the real gluon emission corrections to the process $e^+e^- \to q\bar{q}$. The full calculation is a little bit tedious [EXERCISE], but here we are only interested in the issues arising in the infra-red, so let's go immediately to that approximation.

$$A = \bar{u}(p)\, \not\epsilon(-ig_s)\frac{-i}{\not p + \not k}\Gamma^\mu v(\bar{p})t^a + \bar{u}(p)\Gamma^\mu \frac{i}{\not{\bar{p}} + \not k}(-ig_s)\, \not\epsilon v(\bar{p})t^a$$
$$= -g_s\left[\frac{\bar{u}(p)\, \not\epsilon(\not p + \not k)\Gamma^\mu v(\bar{p})}{2p\cdot k} - \frac{\bar{u}(p)\Gamma^\mu(\not{\bar{p}} + \not k)\, \not\epsilon v(\bar{p})}{2\bar{p}\cdot k}\right] t^a$$

The denominators $2p\cdot k = p_0 k_0(1-\cos\theta)$ give singularities for collinear ($\cos\theta \to 1$) and soft ($k_0 \to 0$) emissions.

Let's neglect k in the numerators and use the Dirac equation to get:

$$A_{\rm soft} = -g_s t^a\left(\frac{p\cdot\epsilon}{p\cdot k} - \frac{\bar{p}\cdot\epsilon}{\bar{p}\cdot k}\right)A_{\rm Born}, \quad A_{\rm Born} = \bar{u}(p)\Gamma^\mu v(\bar{p})$$

Factorization: Independence of long-wavelength (soft) emission from the hard (short-distance) process; Soft emission is universal!

By squaring the amplitude we obtain

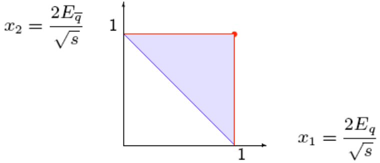
$$\sigma_{q\bar{q}g} = C_F g_s^2 \sigma_{q\bar{q}}^{\mathrm{Born}} \int \frac{d^3k}{2k^0(2\pi)^2} 2 \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$
$$= C_F \frac{\alpha_s}{2\pi} \sigma_{q\bar{q}}^{\mathrm{Born}} \int d\cos\theta \frac{dk^0}{k^0} \frac{4}{(1-\cos\theta)(1+\cos\theta)}$$

Two collinear divergencies and a soft divergence.

Usually expressed in $x_1 = 2E_q/\sqrt{s}$ and $x_2 = 2E_{\bar{q}}/\sqrt{s}$, the fraction of energies of the quark and anti-quark:

$$x_1 = 1 - x_2 x_3 (1 - \cos\theta_{23})/2$$
$$x_2 = 1 - x_1 x_3 (1 - \cos\theta_{13})/2$$
$$x_1 + x_2 + x_3 = 2$$
$$0 \le x1, x2 \le 1 \,\mathrm{and}\, x1 + x2 \le 1$$

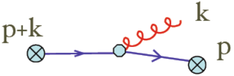
where $x_3 = 2E_g/\sqrt{s}$



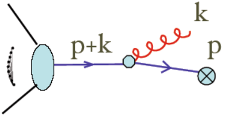
Note that the divergent part of the cross-section is canceled by the divergent part of the virtual contribution to the NLO calculation.

Soft-gluon emission Feynman rules

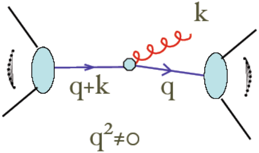
- Soft emission is universal, i.e. does not depend on the hard process
- Soft emission is spin independent
- Only external legs give rise to soft divergencies



$$\bar{\psi}(p)\,\gamma_\mu\psi(p+k)\,\varepsilon^\mu(k) \xrightarrow{k\to 0} \overbrace{\bar{\psi}(p)\,\gamma_\mu\psi(p)}^{\text{charge current of a free fermion}}\varepsilon^\mu(k) = 2p\cdot\varepsilon$$



$$\frac{1}{\not{p}+\not{k}}\gamma_\mu\psi(p+k)\,\varepsilon^\mu(k) \xrightarrow{k\to 0} \frac{1}{2p\cdot k}\not{p}\gamma_\mu\psi(p)\varepsilon^\mu(k) = \frac{p\cdot\varepsilon}{p\cdot k}$$



$$\frac{1}{\not{q}+\not{k}}\gamma_\mu\frac{1}{\not{q}}\varepsilon^\mu(k) \xrightarrow{q^2\neq 0,\,k\to 0} \frac{1}{q^2}\not{q}\gamma_\mu\not{q}\frac{1}{q^2}\varepsilon^\mu(k)$$

=> finite

What do we learn from soft-gluon emissions?

- Infrared divergencies $\Longrightarrow$ soft gluons “all over the place”
- Exclusive observables (like single quarks) impossible to calculate
- Danger: Definition of observables that change by insertion of soft particles
- Must ensure that physical observables are infra-red safe

Infra-red safe observables

- Quantities insensitive to soft or collinear branchings
- Determined by hard, short-distance physics
- Definition: $\mathcal{S}_{n+1}(p_1^\mu, \ldots, (1-\lambda)p_n^\mu, \lambda p_n^\mu) = \mathcal{S}_{n+1}(p_1^\mu, \ldots, p_n^\mu)$ for $\lambda \to 0$ or $0 < \lambda < 1$
- Total cross-section
- Shape parameters (thrust, planarity,...)
- Careful jet definitions

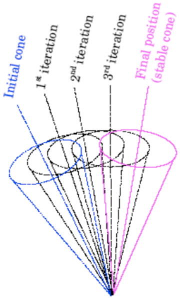
Jet Definitions

Jet Definitions

- Precise definition of procedure to cut multi-jet events
- Crucial when comparing experiment with theory
- Must be infrared-safe (not always the case!)
- In hadronic colliders: Must be boost-insensitive in z direction

Cone algorithms

- Jet: A sufficient amount of hadronic energy found within a cone of a specified radius in $R = \sqrt{\eta^2 + \phi^2}$
- Uses calorimeter information
- Need initial cone direction ("seed"), then iterate to final direction
- Difficulties when jets are found in overlapping cones



Cluster algorithms

- Cluster particles, starting with softest and most collinear particles to get mother-particles up to some distance
- Perform clustering using some particle distance definition (see next slide)
- Uses tracker information
- Jet boundaries more complicated than cones
- More compatible with QCD splitting structure (as will be seen...)

Clustering jet algorithms

Recipe:

1. Choose a jet distance measure d_{ij} (usually $\sim k_{T,ij}$)
2. Find the two particles closest in the distance measure
3. Combine them into one pseudo-particle
4. Repeat steps 2-3 until distance $\geq$ specified distance d_{cut}

e^+e^- colliders

- JADE algorithm: $d_{ij}^2 = 2E_iE_j(1-\cos\theta_{ij})$ (IR unsafe!)
- Durham algorithm: $d_{ij}^2 = 2\min\left(E_i^2, E_j^2\right)(1-\cos\theta_{ij}) \simeq k_T^2$
- Various others (LUCLUS, GENEVA, CAMBRIDGE, DICLUS,...)

Hadronic colliders

The k_T clustering algorithm:

$$\begin{cases} d_{iB} = p_{T,i}^2 & \text{between particle and beam} \\ d_{ij} = \min(p_{T,i}^2, p_{T,j}^2) R_{ij}^2 & \text{between two particles} \end{cases}$$

where

$$R_{ij}^2 = f(\eta_i - \eta_j, \phi_i - \phi_j) \to (\eta_i - \eta_j)^2 + (\phi_i - \phi_j)^2 \text{ as } |\eta_i - \eta_j|, |\phi_i - \phi_j| \to 0$$

Common choices for R_{ij}^2:

$$R_{ij}^2 = (\eta_i - \eta_j)^2 + (\phi_i - \phi_j)^2$$
$$R_{ij}^2 = 2\left[\cosh(\eta_i - \eta_j) - \cos(\phi_i - \phi_j)\right]$$

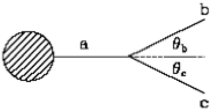
The latter theoretically attractive due to form of eikonal multiparton QCD matrix elements.

Parton Showers

Parton branchings

The leading soft and collinear enhanced terms in QCD matrix elements can be identified and resummed to all orders, enabling a description of parton emissions which is exact at small branching angles.

Consider a splitting of an outgoing parton a into b and c:



- Assume $p_b^2, p_c^2 \ll p_a^2 \equiv t$. Opening angle is $\theta = \theta_a + \theta_b$, energy fraction is

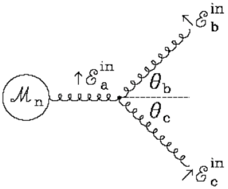
$$z = E_b/E_a = 1 - E_c/E_a$$

- For small angles,

$$t = 2E_bE_c(1-\cos\theta) = z(1-z)E_a^2\theta^2,$$

$$\theta = \frac{1}{E_a}\sqrt{\frac{t}{z(1-z)}} = \frac{\theta_b}{1-z} = \frac{\theta_c}{z}$$

Let's first consider $g \to gg$ branching:



- Amplitude has triple-gluon vertex factor

$$gf^{ABC}\epsilon_a^\alpha\epsilon_b^\beta\epsilon_c^\gamma\left[g_{\alpha\beta}(p_a-p_b)_\gamma+g_{\beta\gamma}(p_b-p_c)_\alpha+g_{\gamma\alpha}(p_c-p_a)_\beta\right]$$

where ϵ_i^μ is the polarization vector for gluon i and all momenta are defined as outgoing, so $p_a=-p_b-p_c$. Use this and $\epsilon_i \cdot p_i = 0$ to get

$$-2gf^{ABC}\left[(\epsilon_a\cdot\epsilon_b)(\epsilon_c\cdot p_b)-(\epsilon_b\cdot\epsilon_c)(\epsilon_a\cdot p_b)-(\epsilon_c\cdot\epsilon_a)(\epsilon_b\cdot p_c)\right]$$

- Resolve the polarization vectors into ϵ_i^{in} in the plane of branching and ϵ_i^{out} normal to the plane, so that

$$\epsilon_i^{\text{in}}\cdot\epsilon_j^{\text{in}}=\epsilon_i^{\text{out}}\cdot\epsilon_j^{\text{out}}=-1$$
$$\epsilon_i^{\text{in}}\cdot\epsilon_j^{\text{out}}=\epsilon_i^{\text{out}}\cdot p_j=0$$

- For small θ (neglecting terms of order θ^2) we get

$$\epsilon_a^{\text{in}}\cdot p_b=-E_b\theta_b=-z(1-z)E_a\theta$$
$$\epsilon_b^{\text{in}}\cdot p_c=+E_c\theta=(1-z)E_a\theta$$
$$\epsilon_c^{\text{in}}\cdot p_b=-E_b\theta=-zE_a\theta$$

- Collinear singularity $1/\theta$ from vertex factor $\propto \theta\times$ propagator factor $\propto 1/t \propto 1/\theta^2$
- $(n+1)$-parton matrix element-squared given (in small-angle region) from n-parton matrix element:

$$|\mathcal{M}_{n+1}|^2 \sim \frac{4g^2}{t} C_A F(z;\epsilon_a,\epsilon_b,\epsilon_c)|\mathcal{M}_n|^2$$

where $C_A = 3$ (color factor from $F^{ABC}F^{ABC}$ and the functions F are given by:

ϵ_a	ϵ_b	ϵ_c	$F(z;\epsilon_a,\epsilon_b,\epsilon_c)$
in	in	in	$(1-z)/z + z/(1-z) + z(1-z)$
in	out	out	$z(1-z)$
out	in	out	$(1-z)/z$
out	out	in	$z/(1-z)$

- Sum/avarage over polarizations gives

$$C_A\langle F\rangle \equiv \widehat{P}_{gg}(z) = C_A\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right]$$

This is the (unregularized) gluon splitting function.

- Soft divergencies for $z\to 0$ and $z\to 1$ due to soft gluon polarized in plane of branching.

Now look at $g \to q\bar{q}$ branching:

- The vertex factor is

$$-ig\bar{u}^b \gamma_\mu \epsilon_a^\mu v^c$$

  where u^b and v^c are the quark and antiquark spinors.
- The spin-averaged splitting function is

$$T_R \langle F \rangle \equiv \widehat{P}_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right]$$

- No soft singularities (only for gluon emission)

Finally, $q \to qg$ branching:

- The spin-averaged splitting function is

$$C_F \langle F \rangle \equiv \widehat{P}_{qq}(z) = C_F \frac{1+z^2}{1-z}$$

- Helicity conservation ensures that the quark doesn't change helicity in branching.

Summary of splitting functions

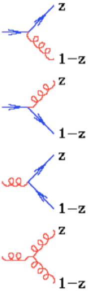
$$\widehat{P}_{qq}(z) = C_F \left[\frac{1+z^2}{1-z}\right],$$

$$\widehat{P}_{gq}(z) = C_F \left[\frac{1+(1-z)^2}{z}\right],$$

$$\widehat{P}_{qg}(z) = T_R \left[z^2+(1-z)^2\right],$$

$$\widehat{P}_{gg}(z) = C_A \left[\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right]$$

where $C_F = \frac{4}{3}$, $C_A = 3$, $T_R = \frac{1}{2}$.



Note that these are unregulated splitting probabilities, since they contain singularities at $z = 1$ and $z = 0$.

The cross-sections before and after splitting are related by

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_s}{2\pi} \widehat{P}_{ba}(z)$$

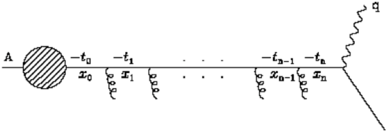
after integration over the azimuthal angle ϕ.

Evolution equations and parton densities

In the relation between the cross-section before and after a splitting is a factor $dt/t \Longrightarrow$logarithmic divergence after integration

These divergences can be resummed through evolution equations.

Consider successive small-angle gluon emission in deep inelastic scattering (hadron-virtual photon collisions):

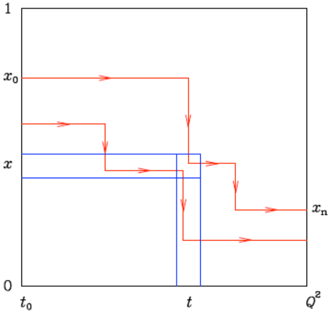


Assume that the quark is found in the hadron with a initial probability f_0 at a virtuality scale $t_0 = -p_0^2 > 0$. After one gluon emission, the probability to find the quark at a virtuality $t > t_0$ will be:

$$f(x,t) = f_0(x) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} \hat{P}(z)\, f_0\left(\frac{x}{z}\right)$$

At every gluon emission, the incoming quark moves to higher virtual mass t and lower momentum fraction x.

To see what happens at multiple gluon emissions, let's introduce a pictorial representation of evolution in t and x:



- Represent a sequence of branchings by path in (t, x)-space. Each branching corresponds to a step downwards in x at a given value of $t = -p^2$ for initial-state showers
- At $t = t_0$, paths have a distribution of starting points $f_0(x)$ characteristic of the target hadron and quark type (at that scale)
- The change in the parton distribution $f(x, t)$ when t is increased to $t + \delta t$ is the number of paths arriving in element $(\delta t, \delta x)-$ number of paths leaving the element, divided by δx

The number of paths arriving is the branching probability times the parton density integrated over all higher momenta $x' = x/z$:

$$\begin{aligned}\delta f_{\rm in}(x,t) &= \frac{\delta t}{t}\int_x^1 dx'\,dz\,\frac{\alpha_s}{2\pi}\widehat{P}(z)f(x',t)\delta(x-zx') \\ &= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_s}{2\pi}\widehat{P}(z)f(x/z,t)\end{aligned}$$

For the number of paths leaving the element we must integrate over lower momenta $x' = zx$:

$$\begin{aligned}\delta f_{\rm out}(x,t) &= \frac{\delta t}{t}f(x,t)\int_x^1 dx'\,dz\,\frac{\alpha_s}{2\pi}\widehat{P}(z)\delta(x'-zx) \\ &= \frac{\delta t}{t}f(x,t)\int_0^1 \frac{dz}{z}\frac{\alpha_s}{2\pi}\widehat{P}(z)\end{aligned}$$

So the change of population of the element is

$$\begin{aligned}\delta f(x,t) &= \delta f_{\rm in}(x,t) - \delta f_{\rm out}(x,t) \\ &= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_s}{2\pi}\widehat{P}(z)\left[\frac{1}{z}f(x/z,t) - f(x,t)\right]\end{aligned}$$

Let's introduce the plus-prescription with definition

$$\int_0^1 dz f(z) g(z)_+ = \int_0^1 dz [f(z) - f(1)] g(z)$$

We can then define the regularized splitting function

$$P(z) = \widehat{P}(z)_+$$

to obtain the DGLAP (Dokshitzer-Gribov-Lipaton-Altarelli-Parisi) evolution equation

$$t \frac{\partial}{\partial t} f(x,t) = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f(x/z, t)$$

(Note that $\int_x^1 dz f(z) g(z)_+ = \int_0^1 dz \Theta(z-x) f(z) g(z)_+ = \int_x^1 dz [f(z) - f(1)] g(z) - f(1) \int_0^x dz g(z)$)

We have now been looking at space-like emissions (initial-state radiation), where $f(x,t)$ represents the momentum fraction distribution of a quark in a hadron, probed at a certain scale $t > t_0$. For time-like emissions (final-state radiation), $f(x,t)$ instead represents the momentum fraction distibution of an outgoing parton found at a $t < t_0$, where t_0 is the virtuality of the parton leaving the hard interaction.

For several different types of partons, we must take into account the different processes by which partons of type i can enter or leave the element $(\delta t, \delta x)$. We therefore get coupled DGLAP equations of the form

$$t\frac{\partial}{\partial t}f_i(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_s}{2\pi}P_{ij}(z)f_j(x/z,t)$$

where $i,j = q,\bar{q},g$.

After some algebra, considering all possible ways the partons can enter or exit the space element and the definition of the plus prescription, we get:

Regulated splitting functions

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right],$$

$$P_{gq}(z) = P_{g\bar{q}}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right],$$

$$P_{qg}(z) = P_{\bar{q}g}(z) = T_R\left[z^2+(1-z)^2\right],$$

$$P_{gg}(z) = 2C_A\left[\frac{1-z}{z}+\frac{z}{(1-z)_+}+z(1-z)\right]+\frac{1}{6}(11C_A-4N_f T_R)\delta(1-z)$$

The role of the plus prescription is here to ensure conservation of probability.

Logarithmic resummation

The DGLAP evolution equation is said to resum large collinear logarithms. So where are these logarithsm, and where is the resummation?

Let's perform the integration of the DGLAP equation and expand the result:

$$
\begin{aligned}
f(x,t) &= f_0(x) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\alpha_s(t')}{2\pi} \int_x^1 \frac{dz}{z} P(z)\, q\left(\frac{x}{z}, t'\right) \\
&= f_0(x) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P(z) \Bigg\{ f_0\left(\frac{x}{z}\right) + \\
&\qquad + \int_{t_0}^{t'} \frac{dt''}{t''} \frac{\alpha_s}{2\pi} \int_{x/z}^1 \frac{dz'}{z'} P(z') \left[f_0\left(\frac{x}{zz'}\right) + \ldots \right] \Bigg\} \\
&= f_0(x) + \frac{\alpha_s}{2\pi} \ln\left(\frac{t}{t_0}\right) \int_x^1 \frac{dz}{z} P(z)\, f_0\left(\frac{x}{z}\right) + \\
&\qquad + \frac{1}{2!} \left[\frac{\alpha_s}{2\pi} \ln\left(\frac{t}{t_0}\right)\right]^2 \int_x^1 \frac{dz}{z} P(z) \int_{x/z}^1 \frac{dz'}{z'} P(z')\, f_0\left(\frac{x}{zz'}\right) + \ldots
\end{aligned}
$$

As suggested by the last step, it is indeed a resummation of all terms proportional to $\left[\frac{\alpha_s}{2\pi} \ln\left(\frac{t}{t_0}\right)\right]^n$.

Sudakov form factors

While the DGLAP equation is convenient to describe the evolution of parton distributions, a slightly different form is useful to study emissions from final-state particles.

Consider the probability for a quark to emit a gluon at a virtuality t:

$$\mathcal{P}_{\text{branching}}(t) = \frac{\delta t}{t} \int_{\epsilon(t)}^{1-\epsilon(t)} dz \frac{\alpha_s(t)}{2\pi} \widehat{P}(z)$$

The probability for not emitting a gluon within a virtuality element δt is then

$$\mathcal{P}_{\text{no-branching}}(t) = 1 - \mathcal{P}_{\text{branching}}(t)$$

This reminds us of the probability for nuclear decay at a time t, and just like in that case, the probability exponentiates. Assuming that the quark was generated at a virtuality t_1, the probability that there has been no emission between t_1 and $t_2 < t_1$ is given by the (infinitesimal) product of $\mathcal{P}_{\text{no-branching}}(t)$ for all virtuality elements between t_1 and t_2, giving

$$\mathcal{P}_{\text{no-branching}}(t_1, t_2) \equiv \Delta(t_1, t_2) = \exp\left[- \int_{t_2}^{t_1} \frac{dt'}{t'} \int_{\epsilon(t)}^{1-\epsilon(t)} dz \frac{\alpha_s(t)}{2\pi} \widehat{P}(z) \right]$$

This quantity is called the Sudakov form factor.

- The Sudakov is sometimes written as $\Delta(t) \equiv \Delta(t, t_{\text{cut}})$ for a given cut-off scale t_{cut}.
- To leading log (LL) accuracy, $\Delta(t_1, t_2) = \Delta(t_1, t_3)/\Delta(t_2, t_3)$, i.e. the probability factorizes. However, this is no longer true at next-to-leading log (NLL) accuracy, due to angular ordering of subsequent gluon emissions
- Studies have suggested that the best choice for the scale of α_s is the relative $k_T^2 \simeq z(1-z)t$
- Note that here it is the unregularized splitting function which is used. That is why we need to introduce the explicit infrared cutoff $\epsilon(t) > 0$. Branchings below this cutoff is considered as unresolvable: The emitted parton is too soft to detect
- The infrared cutoff $\epsilon(t)$ depends on what we classify as a resolvable emission. For timelike branchings, a natural resolution limit is given by demanding that the virtual mass-squared t of the products are above a given t_{cut}. When the parton energies are much larger than their virtualities, the transverse momentum in $a \to bc$ is:

$$p_T^2 = z(1-z)p_a^2 - (1-z)p_b^2 - zp_c^2 > 0$$

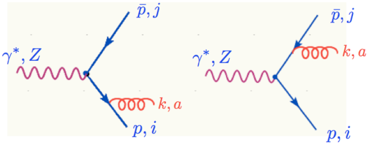
Hence for $p_a^2 = t$ and $p_b^2, p_c^2 > t_{\text{cut}}$ we require $z(1-z) > t_{\text{cut}}/t$, giving

$$z, 1-z > \epsilon(t) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4t_{\text{cut}}/t} \simeq t_{\text{cut}}/t$$

Angular ordering

Let's return momentarily to soft gluon radiation, to examine an important phenomenon: Angular ordering

Soft gluon emission enhances the cross-section due to a logarithmic divergence. Since there is interference between emission from two parton legs, the total enhancement factor includes a sum over all pairs of external lines (i, j):



$$d\sigma_{n+1} = d\sigma_n \frac{d\omega}{\omega} \frac{d\Omega}{2\pi} \alpha_s 2\pi \sum_{i,j} C_{ij} W_{ij}$$

where ω is the emitted gluon energy, $d\Omega$ its solid angle element, C_{ij} is a color factor and W_{ij} the radiation function given by

$$W_{ij} = \frac{\omega^2 p_i \cdot p_j}{p_i \cdot q p_j \cdot q} = \frac{1 - v_i v_j \cos\theta_{ij}}{(1 - v_i \cos\theta_{iq})(1 - v_j \cos\theta_{jq})}$$

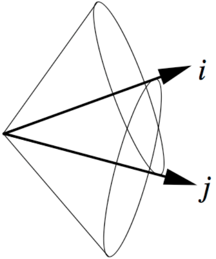
The radiation function can be separated into two parts, containing the collinear singularities along the lines i and j. For simplicity let's consider massless particles with $v_{i,j} = 1$. Then $W_{ij} = W_{ij}^i + W_{ij}^j$ where

$$W_{ij}^i = \frac{1}{2}\left(W_{ij} + \frac{1}{1-\cos\theta_{iq}} - \frac{1}{1-\cos\theta_{jq}}\right)$$

If we write the angular integration in polar coordinates w.r.t. the direction of i, $d\Omega = d\cos\theta iqd\phi_{iq}$ and perform the azimuthal integration, we get

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} W_{ij}^i = \frac{1}{1-\cos\theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}, \text{ otherwise } 0$$

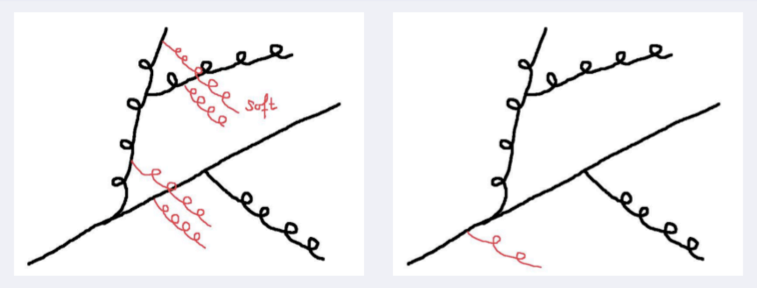
This is the remarkable property of angular ordering.



So, after azimuthal averaging, radiation from the leg i is confined to cone with opening angle to direction j, and vice versa.

This coherence effect is identical to the Chudakov effect in QED, suppressing soft bremsstrahlung from boosted e^+e^- pairs. A photon at larger angles cannot resolve the electron and positron charges separately but sees only the total (neutral) charge of the pair.

More generally, if i and j come from the branching of a parton k with (color) charge $Q_k = Q_i + Q_j$ then radiation outside angular-ordered cones is emitted coherently by i and j and can be treated as coming directly from the (color) charge of k.



NLL Sudakovs

At NLL, next-to-leading logarithmic order (where we take into account also the coherence effects giving a strict ordering of emission angles) the Sudakov form factors for quarks and gluons in the final state can be written as

$$\Delta_q(Q, Q_{\text{cut}}) = \exp\left(-\int_{Q_{\text{cut}}}^{Q} dq\, \Gamma_q(q, Q)\right)$$

$$\Delta_g(Q, Q_{\text{cut}}) = \exp\left(-\int_{Q_{\text{cut}}}^{Q} dq\, \left[\Gamma_g(q, Q) + \Gamma_f(q)\right]\right)$$

with $Q_{\text{cut}} = \sqrt{t_{\text{cut}}}$, $Q = \sqrt{t}$ and the NLL branching probabilities $\Gamma_{q,g,f}$ for $q \to qg$, $g \to gg$ and $g \to q\bar{q}$ respectively, are given by

$$\Gamma_q(q, Q) = \frac{2C_F}{\pi}\frac{\alpha_s(q)}{q}\left(\ln\frac{Q}{q} - \frac{3}{4}\right)$$

$$\Gamma_g(q, Q) = \frac{2C_A}{\pi}\frac{\alpha_s(q)}{q}\left(\ln\frac{Q}{q} - \frac{11}{12}\right)$$

$$\Gamma_f(q) = \frac{N_f}{3\pi}\frac{\alpha_s(q)}{q}$$

The probability for a parton i to evolve from a scale Q to $Q_1 > Q_{\text{cut}}$ without any emission resolvable at the scale Q_{cut} is

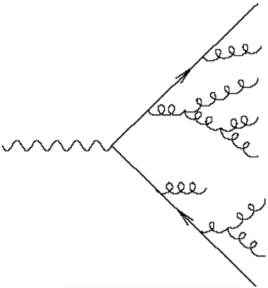
$$\Delta_i(Q, Q_{\text{cut}})/\Delta_i(Q_1, Q_{\text{cut}})$$

Parton showers in Monte Carlos

The Sudakov formalism is very convenient for the implementation of parton showers in Monte Carlo event generators.

Final-state (time-like) showers are constructed as follows:

1. Start the evolution at the virtual mass scale t_0 and momentum fraction $x_0 = 1$
2. Given a virtual mass scale and momentum fraction (t_1, x_1) at some stage in the evolution, generate the scale of the next emission t_2 according to the Sudakov probability $\Delta(t_1, t_2)$ by solving
$$\Delta(t_1, t_2) = \mathcal{R}$$
where $\mathcal{R}$ is a random number (uniform on $[0, 1]$).
3. If $t_2 < t_{\text{cut}}$ it means that the shower has finished.
4. Otherwise, generate $z = x_2/x_1$ with a distribution proportional to $(\alpha_s/2\pi)P(z)$, where $P(z)$ is the appropriate splitting function.
5. For each emitted particle, iterate steps 2-4 until branching stops.



Due to these successive branchings, the parton cascade or parton shower develops. Each outgoing line is a source of a new cascade, until all outgoing lines have stopped branching. At this stage, which depends on the cutoff scale, outgoing partons have to be converted into hadrons via a hadronization model.

Backward evolution

For initial-state parton showers, Monte Carlo generators use backward evolution: Starting from the x_0 and t_0 of the parton going in to the hard interaction, the parton shower is evolved backwards, towards smaller t and larger x.

The initial x_0 and t_0 determine the parton density weight $f(x_0, t_0)$ at the point of the hard interaction. In the backwards evolution, the Sudakov factor must be weighted at each step by the parton density at that point (t, x), in order to correct for the availability of that type of parton at that point. Therefore the probability for evolving backwards from (t_1, x) to (t_2, x) with no branching reads

$$\Pi(t_1, t_2; x) = \frac{f_b(x, t_2)}{f_b(x, t_1)} \frac{\Delta_b(t_1)}{\Delta_b(t_2)}$$

In the backwards evolution of initial-state showers this is the probability distribution from which one must choose t_2.

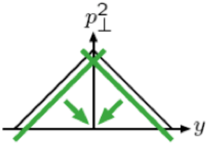
The $z = x/x_2$ value of the branching $a \to bc$ is then generated from a probability distribution proportional to

$$\frac{\alpha_s}{2\pi} \frac{P(z)}{z} \frac{f_a(x/z, t_2)}{f_b(x, t_2)}$$

where $P(z)$ is the appropriate splitting function.

Different approaches to ordering variables in Parton Showers

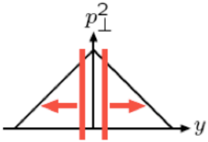
PYTHIA: $Q^2 = m^2$



large mass first
⇒ "hardness" ordered
coherence brute force
covers phase space
ME merging simple
g → q$\bar{\text{q}}$ simple
not Lorentz invariant

ISR: $m^2 \to -m^2$

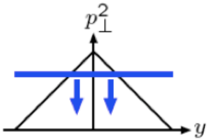
HERWIG: $Q^2 \sim E^2\theta^2$



large angle first
⇒ **hardness not ordered**
coherence inherent
gaps in coverage
ME merging messy
g → q$\bar{\text{q}}$ simple
not Lorentz invariant

ISR: $\theta \to \theta$

ARIADNE: $Q^2 = p_\perp^2$



large $p_\perp$ first
⇒ "hardness" ordered
coherence inherent

covers phase space
ME merging simple
g → q$\bar{\text{q}}$ **messy**
Lorentz invariant

ISR: more messy

That's all for today, folks!

Tutorial 1

Before the first tutorial, you will need on your laptops:

- Root 5
- "MadGraph V4.0" downloaded from `http://madgraph.roma2.infn.it/Downloads/downloads.html`, (Note! The Roma server! Not the same as you already have!) Untar MG_ME_V4.0.tar.gz,
- The Pythia package, downloadable from `http://madgraph.roma2.infn.it/Downloads/downloads.html`, "Pythia and PGS package". Untar the package in the directory MG_ME_V4.0/. Run "make" in the MG_ME_V4.0/ directory.
- The Pythia manual, downloadable from `http://www.thep.lu.se/~torbjorn/Pythia.html`

Please read the README files in MG_ME_V4.0/, Template/ and pythia-pgs/ During the tutorial, I will ask you familiarize yourselves with Pythia and the jet clustering algorithms. Look at properties of the Pythia parton showers by studying e.g. jet distributions. Make comparisons with the results of matrix element calculations by MadEvent, and look at the impact of different jet definitions and different jet parameters.

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