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A random cellular automaton related to the noisy Burgers equation

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Abstract

We derive a random cellular automaton (CA) model from the noisy Burgers equation. The method for that derivation is a nonanalytic limiting procedure called ‘ultra-discretization method’ (UDM) proposed by Tokihiro et al. (Phys. Rev. Lett. 76 (1996) 3247). Special attention is paid to the noise term. It is the intention of this paper to propose a random CA model which could be used as a starting point to understand on a CA basis the precise mechanism leading from deterministic evolutionary equations which trigger chaos such as the Kuramoto–Shivashinsky (KS) equation to equivalent stochastic large-scale descriptions. We investigate our CA model to ensure that it belongs to the same universality class as the noisy Burgers (and thereby also KS) equation itself. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Referring back to Newton the classical approach to formulate physical laws relies on differential equations. On the other hand, physicists, too, have always longed for simplicity. With the introduction of computers the possibility to simulate simplified particle models obeying abstract rules rather than full differential equations became possible. In this approach, the cellular automation (CA) approach, the abstract rules are formulated to only capture the essentials of a physical phenomena thus being as simple as possible. Recently, for many problems in physics CA models have been developed as an independent alternative to the classical descriptions [1]. Through their simplicity they have sometimes given insight into underlying microscopic mechanisms.

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Now, it seems to be of great interest to link this approach to the classical by giving a systematic method to derive CA models from classical ones. For a special class of evolutionary equations this was achieved by Tokihiro et al. [2]. To date CA equivalents are known for many of the famous integrable partial differential equations [3] and also for some ordinary differential equations such as the Painlevé equations [4]. However, no results for evolutionary equations including noise are recorded as yet. Nevertheless, this seems to be a challenging problem from some other point of view: For the KS equation its equivalence to the large scale properties of the Kardar–Parisi–Zhang (KPZ) equation in one dimension have been established in several numerical studies [5–8]. Thus deterministic and the noise-driven descriptions of this chaotic system with many degrees of freedom can be considered to be equivalent. Nevertheless, the precise mechanism leading from one to the other is not well understood. To understand the large-scale stochastic dynamics for KS, Hohenberg and Shraiman [9,10] have suggested to address this problem on the level of the motion and interaction of *cells* which constitute the actual patterns of the spatiotemporal chaos.

Here we pursue a different approach. We will show that we can formulate a random CA related to the noisy Burgers equation. The noisy Burgers equation is equivalent to the KPZ equation for a velocity field $\boldsymbol{v} = -(1/\lambda)\boldsymbol{\nabla} h$, where h is the height field of the corresponding KPZ. Thus they belong to the same universality class and the noisy Burgers equation can be considered to equal the large-scale properties of KS as well. Formulating the noisy Burgers CA then puts the question of large-scale dynamics for KS in the context of understanding the relationship between deterministic and probabilistic CA and gives a link to existing theory on probabilistic CA [1].

In this paper we proceed as follows. In the next section we will derive the noisy Burgers CA from the noisy Burgers equation. Section 3 is devoted to some investigation of the noisy Burgers CA to ensure that it belongs to the same universality class as the noisy Burgers equation itself. We will conclude with some discussion in which we also address open questions.

2. Derivation of a random CA from the noisy Burgers equation

The noisy Burgers equation is known as a paradigm for turbulence studies. In one dimension it is given by

$$v_t = 2vv_x + v_{xx} + \nabla\eta\,, \qquad (1)$$

where η is uncorrelated in space and time:

$$\langle \eta(x,t) \rangle = 0\,, \qquad (2)$$

$$\langle \eta(x,t)\eta(x',t') \rangle = 2D\delta(x-x')\delta(t-t')\,. \qquad (3)$$

Such η can for example be realized by Gaussian white noise. We can rewrite the equation to yield:

$$v_t + \zeta_1 = 2vv_x + v_{xx} + \zeta_2 \,, \tag{4}$$

with both ζ's also being uncorrelated. This is the equation we want to apply the UDM to. In a first step we discretize Eq. (4). To do so we introduce discrete u_j^t and relate them to the continuous function $v(x,t)$ as follows:

$$u_j^t = c \cdot e^{\Delta x \cdot v(j\Delta x, t\Delta t)} \,, \tag{5}$$

where c is just a constant. This yields the following discrete noisy Burgers equation:

$$u_j^{t+1} = u_{j-1}^t \frac{1 + \frac{1-2\delta}{c\delta} u_j^t + \frac{1}{c^2} u_j^t u_{j+1}^t + \frac{\zeta_{2j}^t}{u_{j-1}^t}}{1 + \frac{1-2\delta}{c\delta} u_{j-1}^t + \frac{1}{c^2} u_{j-1}^t u_j^t + \frac{\zeta_{1j}^t}{u_j^{t+1}}} \tag{6}$$

with $\delta = \Delta t/(\Delta x)^2$. Eq. (1) can be checked to be its continuous limit by taking $\Delta x \to 0$ and $\Delta t \to 0$. At this point ζ_{1j}^t and ζ_{2j}^t should be imagined as discrete 'reductions' of continous Gaussian distributions. Introducing new variables

$$u_j^t = e^{x_j^t/\varepsilon} \,, \tag{7}$$

$$\frac{1-2\delta}{c\delta} = e^{-M/\varepsilon} \,, \tag{8}$$

$$c^2 = e^{L/\varepsilon} \,, \tag{9}$$

$$\zeta_{1j}^t = e^{\tilde{\zeta}_{1j}^t} \,, \tag{10}$$

$$\zeta_{2j}^t = e^{\tilde{\zeta}_{2j}^t} \,. \tag{11}$$

We rewrite Eq. (6) to give

$$\begin{aligned} x_j^{t+1} = x_{j-1}^t &+ \varepsilon \log(1 + e^{(x_j^t - M)/\varepsilon} + e^{(x_j^t + x_{j+1}^t - L)/\varepsilon} + e^{(\tilde{\zeta}_{2j}^t - x_{j-1}^t)/\varepsilon}) \\ &- \varepsilon \log(1 + e^{(x_{j-1}^t - M)/\varepsilon} + e^{(x_{j-1}^t + x_j^t - L)/\varepsilon} + e^{(\tilde{\zeta}_{1j}^t - x_j^{t+1})/\varepsilon}) \,. \end{aligned} \tag{12}$$

Now, we impose the important limiting procedure which is the key in obtaining CA descriptions from discrete evolutionary equations. It relies on the following identity:

$$\lim_{x \to +0} \varepsilon \ln(e^{\frac{A}{\varepsilon}} + e^{\frac{B}{\varepsilon}} + ...) = \max[A; B; .....] \,. \tag{13}$$

Utilizing this identity and taking the limit $\varepsilon \to 0$ we obtain

$$\begin{aligned} x_j^{t+1} = x_{j-1}^t &+ \max[0; x_j^t - M; x_j^t + x_{j+1}^t - L; \tilde{\zeta}_{2j}^t - x_{j-1}^t] \\ &- \max[0; x_{j-1}^t - M; x_{j-1}^t + x_j^t - L; \tilde{\zeta}_{1j}^t - x_j^{t+1}] \,. \end{aligned} \tag{14}$$

This can be rewritten to yield

$$x_j^{t+1} = x_j^t + \min[M; x_{j-1}^t; L - x_j^t; x_j^{t+1} + x_{j-1}^t - \tilde{\zeta}_{1j}^t] \\ - \min[x_j^t; M; L - x_{j+1}^t; x_{j-1}^t + x_j^t - \tilde{\zeta}_{2j}^t] \qquad (15)$$

with the following two identities:

$$\max[A; B; \ldots] = -\min[-A; -B; \ldots]\,, \qquad (16)$$

$$\min[A; B; \ldots] + X = \min[A + X; B + X; \ldots]\,. \qquad (17)$$

In Eq. (15) $\tilde{\zeta}_{1j}^t$ and $\tilde{\zeta}_{2j}^t$ should also be uncorrelated. However, they also have to fit in this 'ultra-discrete' picture. This can be realized by choosing the $\tilde{\zeta}_j^t$'s randomly to be -1 or 1 with equal probability.

Eq. (15) itself cannot be evaluated in this form since x_j^{t+1} appears on its right-hand side. Therefore, we approximate it by the following statistical description:

$$x_j^{t+1} = x_j^t + \min[M; x_{j-1}^t; L - x_j^t; \langle x_j^{t+1} \rangle + x_{j-1}^t - \tilde{\zeta}_{1j}^t] \\ - \min[x_j^t; M; L - x_{j+1}^t; x_{j-1}^t + x_j^t - \tilde{\zeta}_{2j}^t]\,, \qquad (18)$$

where $\langle x_j^{t+1} \rangle$ stands for the statistical average of the x_j^{t+1}'s in the automaton's evolution. Thereby the imposed problem can be solved by simulation: we insert various values for x_j^{t+1} on the RHS of the equation, run the simulation, evaluate $\langle x_j^{t+1} \rangle$ and compare it with the value inserted for x_j^{t+1} in the beginning. As one solution we obtain:

$$\langle x_j^{t+1} \rangle = -1 \quad \text{and} \quad x_j^{t+1} \in \{-3; -2; -1; 0; 1\}\,.$$

Thus,

$$x_j^{t+1} = x_j^t + \min[M; x_{j-1}^t; L - x_j^t; -1 + x_{j-1}^t - \tilde{\zeta}_{1j}^t] \\ - \min[x_j^t; M; L - x_{j+1}^t; x_{j-1}^t + x_j^t - \tilde{\zeta}_{2j}^t] \qquad (19)$$

constitutes a generalized CA which we call noisy Burgers CA. We give physical meaning to it as follows: The Burgers equations without noise is known to evolve in 'shock waves' (Fig. 1). We consider the states x_j^t can take as discrete reminders of those 'shock waves' disorganized by the noise term.

3. Investigation of the noisy Burgers CA

3.1. Fractal dimension

In this section we investigate the noisy Burgers CA to insure that the universality class is maintained through application of the UDM. A first approach, which can of course only give a very rough idea, utilizes the fractal dimension. If we visualize

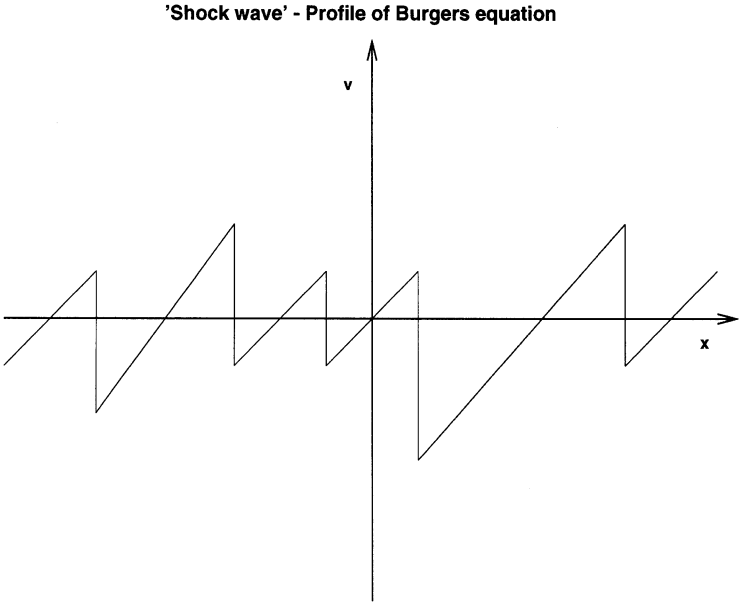

Fig. 1. The picture shows the ‘shock wave’-profile in which the deterministic Burgers equation is known to evolve. We refer to it in Section 2 to give physical meaning to the set of values which form the states of the noisy Burgers CA.

evolution according to Eq. (19) by denoting all values $\geqslant 0$ with a black dot and all values <0 with a white dot we obtain a random fractal (Fig. 2). The generated patterns are obviously self-affine, thus we can relate the fractal dimension to the roughness exponent α by

$$d_f = 2 - \alpha \,, \tag{20}$$

where d_f is the fractal dimension. We evaluated d_f for Fig. 2 as box-counting-dimension and obtained

$$d_f = 1.5 \Rightarrow \alpha = 0.5 \,.$$

We checked our box-counting-algorithm by also evaluating the fractal dimension of Fig. 3. Fig. 3 shows evolution according to the ballistic deposition model which is also considered as a model belonging to the KPZ/noisy Burgers universality class. Here we got

$$d_f = 1.6 \Rightarrow \alpha = 0.4 \,.$$

Numerically, the roughness exponent α for the ballistic deposition model in one dimension has been evaluated as $\alpha = 0.42$ [11] and $\alpha = 0.47$ [12]. We refrain from making more precise statements for α since the method of evaluating the fractal dimension by the box-counting-algorithm is known to be not very accurate. Thus, here we only want

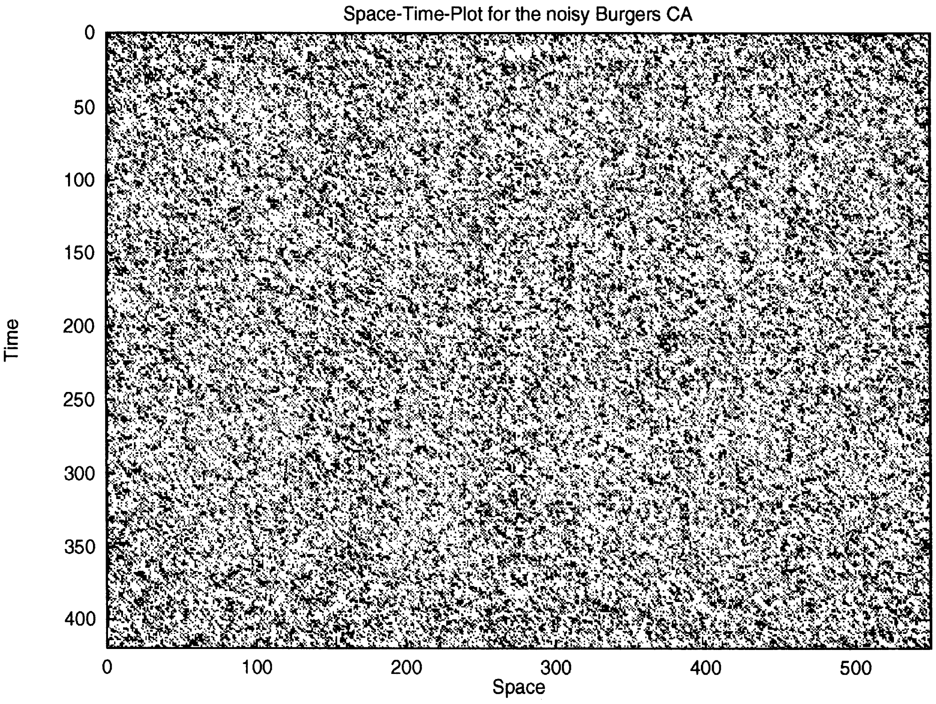

Fig. 2. The space-time-plot for the noisy Burgers CA. Evolution in a system of 550 cells is shown. Time runs from top to bottom. Each black dot corresponds to a cell with positive (including 0) value. White space corresponds to negative values. We use this evolutionary pattern to define a fractal dimension.

to state that our box-counting-algorithm seems to be reliable in the accuracy range of this method and can give a first idea that α for the noisy Burgers CA is indeed 0.5 just as for the noisy Burgers equation.

3.2. Introduction of an interface width

Most scaling relations to determine α are not formulated for the velocity appearing in the Burgers equation but for its integral. Integration of the Burgers equation yields the KPZ equation for some height h_j^t. For each point of time t the heights determine the interface width w as its rms fluctuation:

$$w(L,t)=\sqrt{\frac{1}{L}\sum_{i=1}^{L}[h_i^t-\bar{h}^t]^2}\,, \tag{21}$$

where L denotes the length of the system under investigation and $\bar{h}^t$ the mean value of the h_i^t at t. w grows in time until a saturation width w_{sat} is reached. This saturation width depends on L and can be related to α through the following scaling law:

$$w_{sat}(L)\sim L^{\alpha}\,. \tag{22}$$

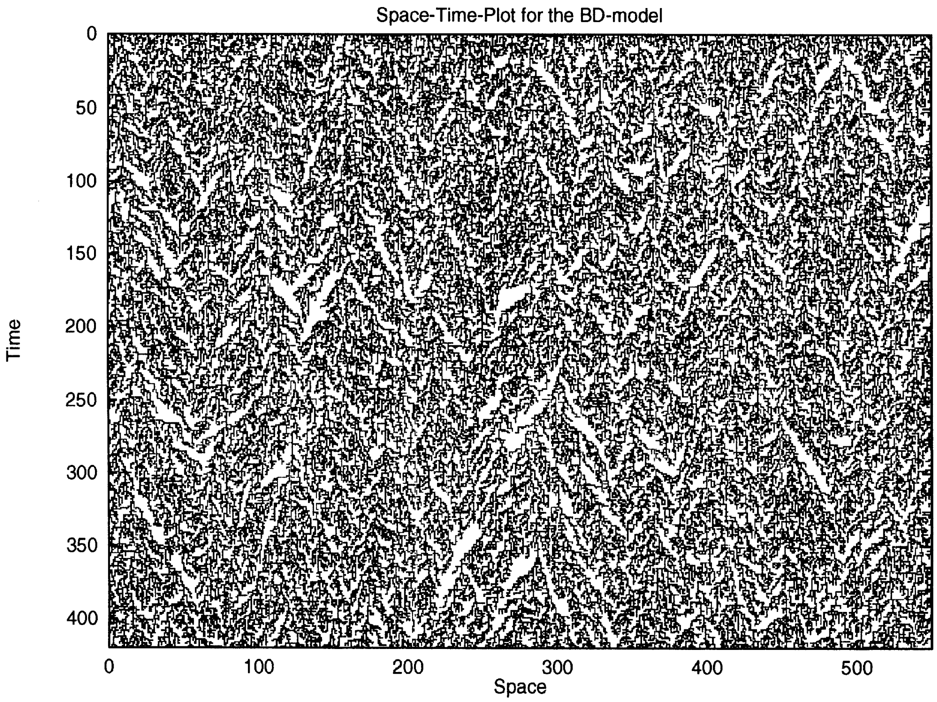

Fig. 3. The space-time-plot for the ballistic deposition model. Again, evolution on 550 cells is shown for 420 time steps running from top to bottom. The evolving pattern merely served as check for our box-counting-algorithm. To evaluate the fractal dimension we did not consider the first 10^4 time steps to let the transient die out.

To utilize this scaling law for our purpose x_j^t in Eq. (19) also has to become subject to some sort of integration. In the 'ultra-discrete' world, integration should correspond to summation. Therefore adopting the idea from [2] and extending the approach further by some weighing we define continuous heights as follows:

$$h_i^t = \sum_{k=0}^{i} (x_k^t)^{1/2} \,. \tag{23}$$

This does not yield a realistic model for the KPZ equation since our definition results in increasing heights with rising position index. However we do get homogen widths when applying Eq. (21) and that is what we are actually interested in. The homogen widths can be identified with the saturation width in relation (22) and we can check for its validity in case of our model. The results are as follows: For small system sizes we encounter a finite size effect (Fig. 4), however, for system sizes greater than 500 units (cells) the scaling law does hold yielding

$$w_{sat}(L) = (0.3615 \pm 0.0015) \cdot L^{0.5} \,. \tag{24}$$

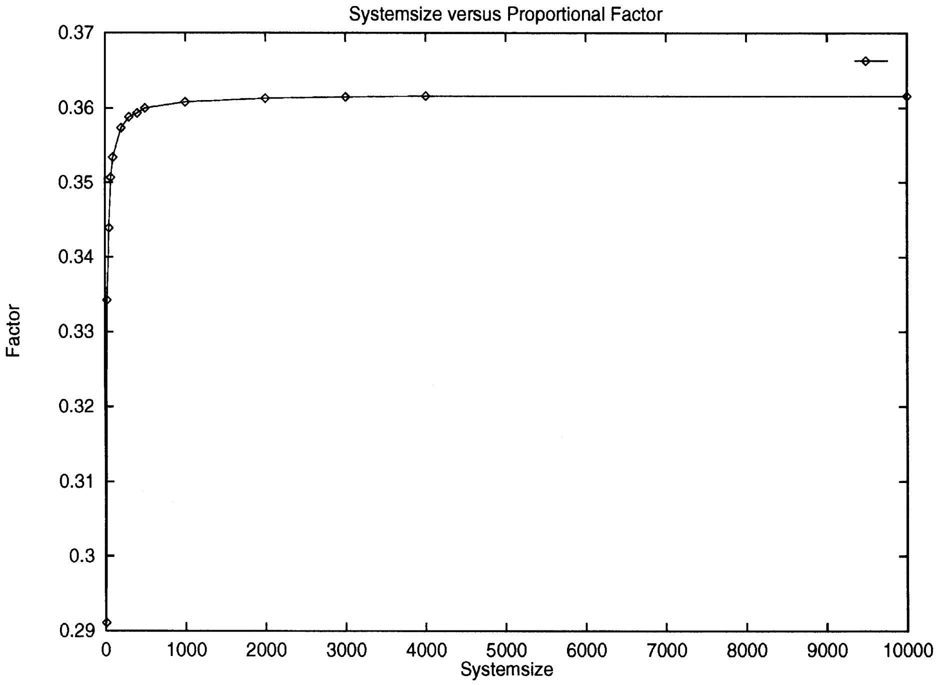

Fig. 4. We show this graph to give an idea of the finite size effect we encountered when determining the proportional factor in scaling relation (22). For system sizes greater than 500 cells the proportional factor can be considered to be constant. Each point in the graph corresponds to averaging over 10^6 time steps where also the first 10^4 steps were not considered.

Again α for the noisy Burgers CA can be identified with 0.5. Thus, we feel confirmed that the UDM maintains universality class.

4. Concluding discussion

In this paper we have derived a noisy Burgers CA from the noisy Burgers equation through UDM. We have verified that UDM maintains the universality class through some investigations related to the roughness exponent α. Here we would like to stress once more that in Section 3.2 we did not intend to propose a realistic model for height growth. The introduction of the heights was merely done to be able to deduce an interface width and check for the scaling relation (22). To our knowledge this paper is the first work to derive a probabilistic CA model from an evolutionary equation with noise and thus links the question of KS stochastic large scale description to the theory of probabilistic CA. We have to admit that the relationship between probabilistic and deterministic CA is not well enough established yet to come to some ad hoc understanding. However, we believe that it is much easier to address the problem from that point of view than on a *cell* level basis because of greater algorithmic simplicity. Moreover, simulation results even for large size systems can be obtained easily and fast on small machines and CA models have proven to give insight in underlying microscopic mechanisms in the past. Presently, we are proceeding with studies involving the equivalence of deterministic and probabilistic CA descriptions which we will record elsewhere.

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