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Solid State Physics II

Chapter 7 Microscopic Theory of Superconductivity

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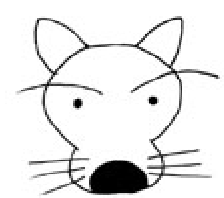

Microscopic Theory of Superconductivity

BCS Trial Wavefunction

As we discussed in the previous chapter, the superconducting ground state needs to be described in terms of the Cooper pair states, which can have two accessible states, i.e., $\{|o\rangle, |\mathbf{k}\uparrow, \mathbf{k}\downarrow\rangle$. A possible choice of the ground state is

$$|\psi_{\rm BCS}\rangle = \prod_k (u_k + v_k c^+_{k\uparrow} c^+_{-k\downarrow})|0\rangle$$

where the parameters u_k and v_k are to be determined by the variational principles.

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where the parameters u_k and v_k are to be determined by the variational principles.

> Note the difference between the BCS trial state $|\psi_{\rm BCS}\rangle$ and the Hartree-Fock trial state $|\psi_{\rm HF}\rangle = \prod_{k<k_F,\sigma} c^+_{k\sigma}|0\rangle$.

Mean Field Theory

Basic Idea: Heisenberg Model as an example

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + g\mu_B \mathbf{H} \cdot \sum_i \mathbf{S}_i$$

For *free* spins in an external field $\mathbf{H}$,

$$\mathcal{H}_0 = g\mu_B \mathbf{H} \cdot \sum_i \mathbf{S}_i$$

we can obtain an exact solution for the thermodynamic quantities from the partition function $\mathcal{Z}_0$:

$$\mathcal{Z}_0 = \mathrm{tr} e^{-\beta \mathcal{H}_0}.$$

(For more details, please refer the standard **textbook** on *Statistical Mechanics*.)

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In general, there is no exact solution for the interacting spins. To understand the ground (or finite-temperature) state of such *interacting* spin systems, one can introduce an idea of **"mean field"**, or an effective field acting on a spin at each site.

Introducing a **Mean field** $\langle \mathrm{S}_i \rangle$, which is a c-number, we can describe the quantum operator as follows:

$$\mathbf{S}_i = \langle \mathbf{S}_i \rangle + (\mathbf{S}_i - \langle \mathbf{S}_i \rangle) = \langle \mathbf{S}_i \rangle + \delta \mathbf{S}_i$$

By the definition, we know

$$\langle \delta \mathbf{S}_i \rangle = 0,$$

but in general

$$\langle \delta \mathbf{S}_i \cdot \delta \mathbf{S}_j \rangle \neq 0.$$

However, if we can assume that the fluctuation in the system is much smaller than the average, i.e.,

$$\sqrt{\langle (\delta \mathbf{S}_i)^2 \rangle} \ll \langle \mathbf{S}_i \rangle,$$

then we can approximate the interacting spins by the *free* spins interacting with an *effective* external field.

$$\mathbf{S}_i \cdot \mathbf{S}_j \approx \mathbf{S}_i \cdot \langle \mathbf{S}_j \rangle + \langle \mathbf{S}_i \rangle \cdot \mathbf{S}_j - \langle \mathbf{S}_i \rangle \cdot \langle \mathbf{S}_j \rangle$$

Note that the effective "mean" field can be defined

(i) either by an average expectation for the zero-temperature ground state

$$\langle \mathcal{S} \rangle = \langle \psi_S | \mathcal{S} | \psi_S \rangle$$

(ii) or by an ensemble average for the finite-temperature state.

$$\langle \mathcal{S} \rangle = \mathrm{tr}\,(\rho \mathcal{S})$$

with the density matrix $\rho = e^{-\beta \mathcal{H}} = \sum_m e^{-\beta E_m} |m\rangle\langle m|$.

Mean field solution:

Introducing an effective field $\mathbf{H}_{\text{eff}}$,

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + g\mu_B \mathbf{H} \cdot \sum_i \mathbf{S}_i = g\mu_B \sum_i \mathbf{S}_i \cdot \mathbf{H}_{\text{eff}}$$

where the effective mean field

$$\mathbf{H}_{\text{eff}} = \mathbf{H} - \frac{1}{g\mu_B} \sum_j J_{ij} \mathbf{S}_j$$

and the average magnetization

$$\langle \mathbf{S}_i \rangle = \frac{V}{N} \frac{\mathbf{M}}{g\mu_B}$$

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + \lambda \mathbf{M}$$

$$\lambda = \frac{V}{N} \frac{J_o}{(g\mu_B)^2}$$

$$M = -\frac{N}{V} \frac{\partial F}{\partial H} = M_o\left(\frac{H_{\text{eff}}}{T}\right)$$

- For the case of $H = 0$,

  one can fine the magnetization M by solving the equation:

$$M(T) = M_o(\frac{\lambda M}{T})$$

$$\chi_o(T) = \left(\frac{\partial M_o}{\partial H}\right)_{H=0} = \frac{M_o'(0)}{T}$$

  where the Curie's constant is determined to be $C_o = M_o'(0)$.

- For the case of $H \neq 0$,

$$\chi = \frac{\partial M}{\partial H} = \frac{\partial M_o}{\partial H_{\text{eff}}} \frac{\partial H_{\text{eff}}}{\partial H} = \chi_o(1 + \lambda\chi)$$

$$\chi = \frac{\chi_o}{1 - \lambda\chi_o} = \frac{C_o}{T - T_c}$$

  where the critical temperature T_c becomes

$$T_c = \frac{N}{V} \frac{(g\mu_B)^2}{3k_B} S(S+1)\lambda = \frac{S(S+1)}{3k_B} J_o$$

Mean Field Theory for the interacting electron systems

- Mean field for the Hartree-Fock (normal state) trial solution:

$$\langle\psi_{\mathrm{HF}}|c^{+}_{k\uparrow}c_{k\uparrow}|\psi_{\mathrm{HF}}\rangle = n_{k\uparrow} \neq 0$$

$$\langle\psi_{\mathrm{HF}}|c_{k\sigma}c_{k'\sigma}|\psi_{\mathrm{HF}}\rangle = 0$$

- Mean field for the BCS (superconducting state) trial solution:

$$\langle\psi_{\mathrm{BCS}}|c^{+}_{k\uparrow}c_{k\uparrow}|\psi_{\mathrm{BCS}}\rangle = n_{k\uparrow} \neq 0$$

$$\langle\psi_{\mathrm{BCS}}|c_{-k\downarrow}c_{k\uparrow}|\psi_{\mathrm{BCS}}\rangle = A_k \neq 0$$

The BCS Hamiltonian:

$$\mathcal{H} = \sum_{k\sigma} \xi_k c_{k\sigma}^{+} c_{k\sigma} - V_0 \sum_{kk'} c_{k\uparrow}^{+} c_{-k\downarrow}^{+} c_{-k'\downarrow} c_{k'\uparrow}$$

Following the mean-field procedure, we can introduce

$$c_{-k\downarrow} c_{k\uparrow} = A_k + (c_{-k\downarrow} c_{k\uparrow} - A_k)$$

Neglecting the fluctuation term,

$$\langle (c_{k\uparrow}^{+} c_{-k\downarrow}^{+} - A_k^{*})(c_{-k'\downarrow} c_{k'\uparrow} - A_{k'}) \rangle \approx 0$$

the original interacting Hamiltonian can be reduced to a mean-field Hamiltonian

$$\mathcal{H}_{\mathrm{MF}} = \sum_{k\sigma} \xi_k c_{k\sigma}^{+} c_{k\sigma} - V_0 \sum_{kk'} (A_k^{*} c_{-k'\downarrow} c_{k'\uparrow} + c_{k\uparrow}^{+} c_{-k\downarrow}^{+} A_{k'}) + V_0 \sum_{kk'} A_k^{*} A_{k'}$$

Variational Method

Since the trial BCS wave function is given as a function of variational parameters $\{u_k, v_k\}$, we can apply the minimum energy principle to find the solution.

$$E(\{u_k, v_k\}) = \langle \psi_{\rm BCS} | \mathcal{H} | \psi_{\rm BCS} \rangle / \langle \psi_{\rm BCS} | \psi_{\rm BCS} \rangle$$

$$\frac{\partial E}{\partial u_k} = 0, \qquad \frac{\partial E}{\partial v_k} = 0$$

- Normalization:

$$\langle\psi_{\rm BCS}|\psi_{\rm BCS}\rangle = \prod_k (|u_k|^2 + |v_k|^2) = 1$$

$$|u_k| = \cos(\theta_k/2), \quad |v_k| = \sin(\theta_k/2)$$

- Pairing order parameter A_k:

$$A_k = \langle\psi_{\rm BCS}|c_{-k\downarrow}c_{k\uparrow}|\psi_{\rm BCS}\rangle = u_k^* v_k = \frac{1}{2}\sin\theta_k$$

$$\langle 0| \left\{ \prod_{k_1} (u_{k_1}^* + v_{k_1}^* c_{-k_1\downarrow}c_{k_1\uparrow}) \right\} (c_{-k\downarrow}c_{k\uparrow}) \left\{ \prod_{k_2} (u_{k_2} + v_{k_2} c_{k_2\uparrow}^+ c_{-k_2\downarrow}^+) \right\} |0\rangle = u_k^* v_k$$

- In a normal (HF) state, we have $u_k = \theta(\xi_k)$ and $v_k = \theta(-\xi_k)$, or, alternatively, $\theta_k = 0$ for $\xi_k > 0$ and π for $\xi_k < 0$. Consequently, the anomalous pairing order parameter becomes zero for all k, i.e., $\sin(2\theta_k) = 0$,

$$A_k = 0.$$

Canonical Transformation

There is another way of treating the minimum energy principle. Reminding that the operator approach to the solution of quantum harmonic oscillator requires a lower bound in energy such that

$$a|G\rangle = 0.$$

Since the mean-field BCS Hamiltonian is quadratic in c_k and c_k^+, one should be able to find a canonical transformation to a new set of operator γ_k and γ_k^+, satisfying the **same** commutation rule as c_k and c_k^+,

$$[\gamma_k, \gamma_{k'}^+]_+ = \delta_{kk'}$$

$$\gamma_k = \cos(\theta_k/2)c_k - \sin(\theta_k/2)c_{-k}^+$$

and

$$c_k = \cos(\theta_k/2)\gamma_k + \sin(\theta_k/2)\gamma_{-k}^+$$

Through the canonical transformation, the mean-field Hamiltonian can be represented by

$$\mathcal{H}_{\mathrm{MF}} = \sum_{ka} E_k \gamma^{+}_{ka} \gamma_{ka} + E_G$$

where the quasi-particle energy E_k is determined to be

$$E_k = \sqrt{\xi_k^2 + |\Delta|^2}$$

$$\mathcal{H}_{\mathrm{MF}} = \frac{1}{2} \sum_k (c^{+}_{k\uparrow} c_{-k\downarrow} c_{k\downarrow} c^{+}_{-k\uparrow}) \begin{pmatrix} \xi_k & -\Delta & 0 & 0 \\ -\Delta^* & -\xi_{-k} & 0 & 0 \\ 0 & 0 & \xi_k & -\Delta \\ 0 & 0 & -\Delta^* & -\xi_{-k} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{+}_{-k\downarrow} \\ c^{+}_{k\downarrow} \\ c_{-k\uparrow} \end{pmatrix}$$

with the pairing potential Δ:

$$\Delta = V_0 \sum_k A_k$$

It is noted that one can view the BCS ground state in terms of the quasi-particle operators:

$$|\psi_{\mathrm{BCS}}\rangle = \prod_k \gamma_{-k}\gamma_k|0\rangle,$$

which is equivalent to:

$$|\psi_{\mathrm{BCS}}\rangle = \prod_k (u_k + v_k c^{+}_{k\uparrow} c^{+}_{-k\downarrow})|0\rangle$$

Thermodynamic Derivation of London Equation

Consider a metal with a free-electron-like band,

$$\mathcal{F} = \int f_s d\mathbf{r} + E_K + E_{\rm mag}$$

where

- f_s = energy of SC electrons with $\mathbf{j}_s = 0$
- E_K = kinetic energy due to the supercurrent $\mathbf{j}_s(\mathbf{r})$
- $E_{\rm mag}$ = electromagnetic energy.

  Note that the SC (superconducting) state can be characterized by (i) the macroscopic quantum state of the pairs and (ii) the quasi-particle (excitation) states of electrons.

Defining $\mathbf{v}(\mathbf{r})$= drift velocity of superconducting electrons (i.e., pairs) at $\mathbf{r}$,

$$\mathbf{j}_s(\mathbf{r}) = n_s e \mathbf{v}(\mathbf{r})$$

where n_s refers to the no. of SC electrons.

$$E_K = \int d\mathbf{r} \;\; \frac{1}{2} m \mathbf{v}^2 n_s$$

assuming $\mathbf{v}(\mathbf{r})$ be a slowly-varying function of $\mathbf{r}$ compared to $n_s(\mathbf{r})$.

$$E_{\rm mag} = \int d\mathbf{r} \, \frac{\mathbf{h}^2}{8\pi}$$

From the Maxwell's equation,

$$\nabla \times \mathbf{h} = \frac{4\pi}{c}\mathbf{j}_s = \left(\frac{4\pi e n_s}{c}\right)\mathbf{v}$$

we can rewrite the free energy by

$$F = F_o + \frac{1}{8\pi}\int d\mathbf{r}\ \left[\mathbf{h}^2 + \lambda_L^2(\nabla \times \mathbf{h})^2\right]$$

where

$$F_o = \int d\mathbf{r}\ f_s$$

$$\lambda_L = \left(\frac{mc^2}{4\pi n_s e^2}\right)$$; London penetration depth

Minimizing F w.r.t. **h**,

$$\delta F = \frac{1}{4\pi}\int d\mathbf{r}\ \left[\mathbf{h}\cdot\delta\mathbf{h} + \lambda_L^2(\nabla\times\mathbf{h})\cdot(\nabla\times\delta\mathbf{h})\right]$$

Therefore, we have the London equation:

$$\mathbf{h} + \lambda_L^2\nabla\times(\nabla\times\mathbf{h}) = 0$$

Physical Implication of the London Equation

Considering SC electrons as "an incompressible non-viscous charge fluid", we can apply a standard classical fluid dynamic model with the relation:

$$\mathbf{j}_s(\mathbf{r}, t) = n_s e \mathbf{v}(\mathbf{r}, t)$$

Simple charge fluid model

- Continuity Eq.:

$$\nabla \cdot \mathbf{j} = \nabla \cdot \mathbf{v} = 0$$

- Newton's Eq.:

$$\frac{d\mathbf{v}}{dt} = +\frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{h}\right)$$

Define: $\mathbf{Q} \equiv \nabla \times \mathbf{v} + e\mathbf{h}/mc$,

$$\frac{\partial \mathbf{Q}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{Q})$$

$$\rightarrow \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla(v^2/2) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

$$\rightarrow \frac{\partial \mathbf{v}}{\partial t} - \frac{e}{m}\mathbf{E} + \nabla(v^2/2) = \mathbf{v} \times (\nabla \times \mathbf{v} + \frac{e}{mc}\mathbf{h})$$

Consider a bulk superconductor in zero field so that $\mathbf{Q} = 0$. Then, the above equation implies that $\mathbf{Q} = 0$ for the rest of the time sequence, even if $\mathbf{h} \neq 0$, independent of how the final state is reached.

The fundamental assumption for superconductors under all circumstances:

$$\mathbf{Q} \equiv \nabla \times \mathbf{v} + \frac{e\mathbf{h}}{mc} = 0$$

Now the question is **why**!

For superconductor as an imcompressible charge fluid, we have two equations (London equations):

$$\mathbf{Q} \equiv \nabla \times \mathbf{v} + \frac{e\mathbf{h}}{mc} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{v^2}{2} \right) = \frac{e\mathbf{E}}{m}$$

Using $\mathbf{j}_s = n_s e \mathbf{v}$ and $\nabla \times \mathbf{h} = \frac{4\pi}{c}\mathbf{j}_s$,

$$\mathbf{h} = -\frac{mc}{n_s e^2} \nabla \times \mathbf{j}_s$$

$$\rightarrow \mathbf{h} = -\frac{mc^2}{4\pi n_s e^2} \nabla \times (\nabla \times \mathbf{h}) = +\frac{1}{\lambda_L^2} \nabla^2 \mathbf{h}$$

London Penetration Depth λ_L

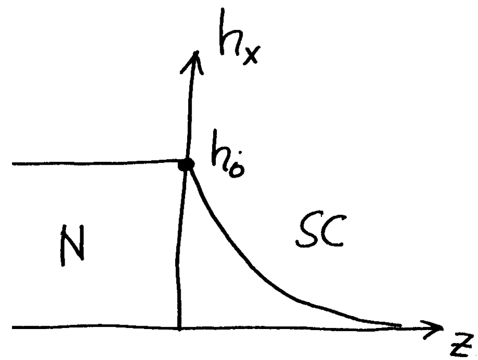
- $\mathbf{h} = h_z \hat{z}$,

$$\frac{\partial h_z}{\partial z} = 0 \quad \rightarrow \quad h_z = \text{const.}, \quad j_z = 0$$

$$\therefore h_z = 0$$

- $\mathbf{h} = h_x \hat{x}$,

$$\therefore h_x(z) = h_o e^{-z/\lambda_L}$$



Quantization of Fluxoid

Suppose that $\nabla(v^2/2) = 0$, then

$$\frac{e\mathbf{E}}{m} = \frac{\partial \mathbf{v}}{\partial t} = \frac{1}{n_s e}\frac{\partial \mathbf{j}_s}{\partial t}$$

From the Maxwell's Eq.,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{h}}{\partial t}$$

$$\int d\mathbf{s} \cdot \frac{\partial \mathbf{h}}{\partial t} = -c\int d\mathbf{s} \cdot \nabla \times \mathbf{E} = -c\oint_C d\mathbf{l} \cdot \mathbf{E}$$

$$\therefore \frac{\partial}{\partial t}\left(\int d\mathbf{s} \cdot \mathbf{h} + \frac{mc}{n_s e^2}\oint_C d\mathbf{l} \cdot \mathbf{j}_s\right) = 0$$

Since $\nabla \times \mathbf{A} = \mathbf{h}$, we can rewrite $\mathbf{Q} = 0$ by

$$m\mathbf{Q} = \nabla \times \left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right) = \nabla \times \mathbf{p} = 0$$

Magnetic Flux

$$\Phi = \frac{c}{e}\int d\mathbf{s}\cdot(\nabla\times\mathbf{p}) = \frac{c}{e}\oint_C d\mathbf{l}\cdot\mathbf{p} = \frac{hc}{e}n$$

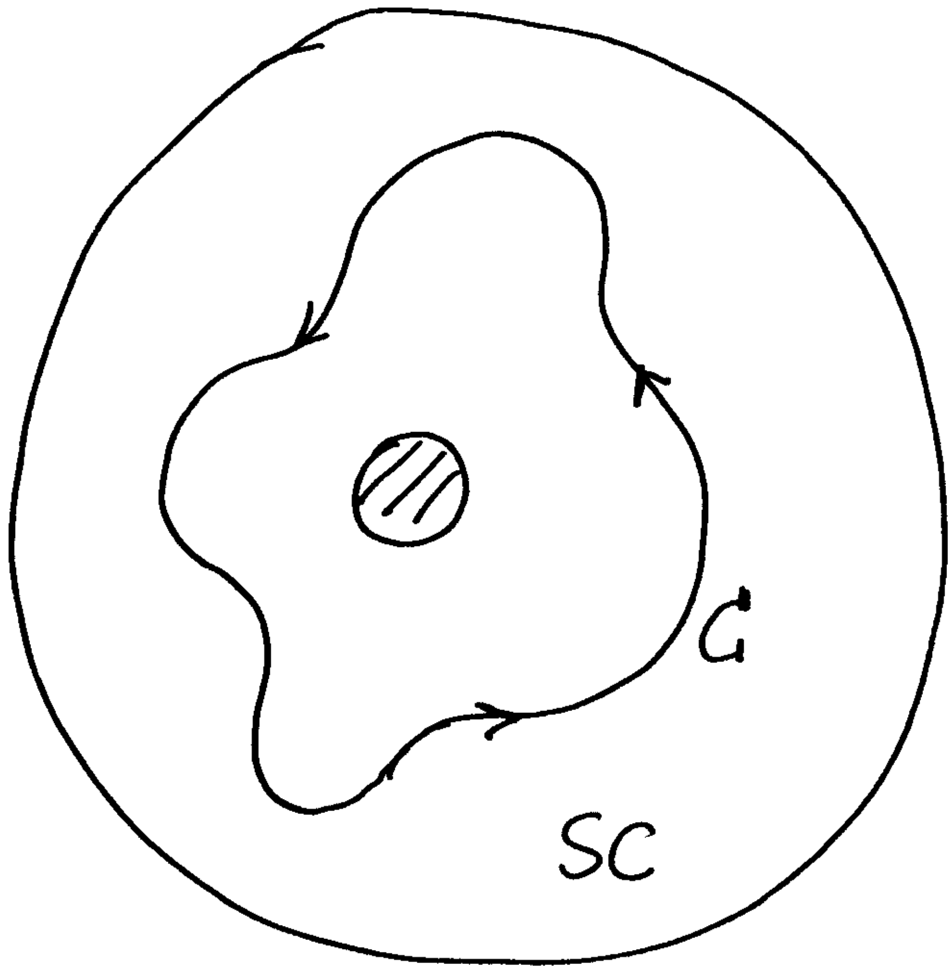
Note that the Sommerfeld quantization rule:

$$\oint_C d\mathbf{l}\cdot\mathbf{p} = nh$$

Quantization of the magnetic flux inside superconductor:

$$\phi_o = \frac{hc}{e^*}$$

For Cooper pairs, we should put $e \to e^* \to 2e$!!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SC

London equation

Using the London gauge,

$$\nabla \cdot \mathbf{A} = 0$$

$$\mathbf{A} \cdot \hat{n} = 0 \text{on boundaries}$$

the London's equation $\mathbf{h} = -(mc/n_s e^2)\nabla \times \mathbf{j}_s$ becomes

$$\mathbf{j}_s(\mathbf{r}) = -\frac{n_s e^2}{mc}\mathbf{A}(\mathbf{r})$$

In this picture, the penetration depth should depend only on fundamental constants and n_s. This solution is valid when $\mathbf{h}$ and $\mathbf{v}$ are *"slowly varying function of* $\mathbf{r}$*"* in the scale of coherence length ξ_o, that is,

$$\xi_o \ll \lambda_L$$

Coherence Length ξ_o

The Cooper pairs are formed by the electrons within the energy range:

$$E_F - \Delta < \frac{p^2}{2m} < E_F + \Delta$$

$$\delta p \sim \frac{2\Delta}{v_F}$$

$$\delta x \to \xi_o = \frac{\hbar}{\delta p} \approx \frac{\hbar v_F}{\pi \Delta}$$

Two kinds of superconductors

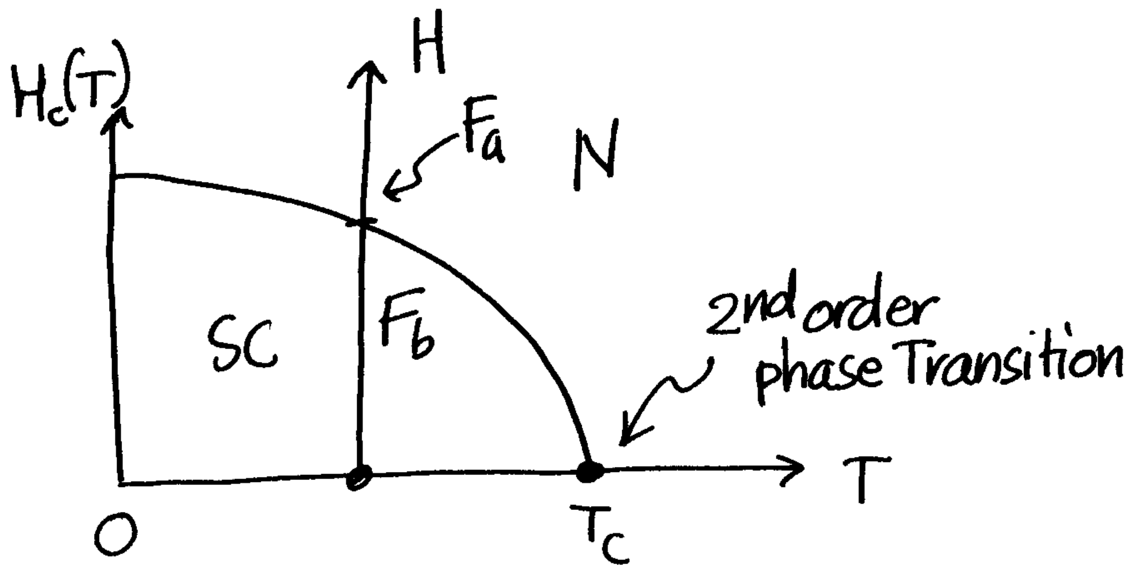
- Type I: $\lambda_L < \xi_o$
- Type II: $\lambda_L > \xi_o$

Free Energy of Superconductors

Remind that the free energy of a system under the external field $\mathbf{h}$ can be expressed by

$$F = F_o + \frac{1}{8\pi}\int \left[\mathbf{h}^2 + \lambda_L^2(\nabla\times\mathbf{h})^2\right]$$

where $F_o = F_n$ for the normal state and $F_o = F_s$ for the superconduting state.



- $\mathcal{F}_a$: a free energy with $h > H_c$ (i.e., normal state)

$$\mathcal{F}_a = F_n + V \cdot \frac{h^2}{8\pi}$$

- $\mathcal{F}_b$: a free energy with $h < H_c$ (i.e., superconducting state)

$$\mathcal{F}_b = F_s + 0$$

(Here we neglect the surface region of λ.)

Here note that external WORK is needed to apply the field from 0 to H_c or above,

$$\mathcal{W}_{ba} = \mathcal{F}_a - \mathcal{F}_b = \int_b^a I \frac{d\Phi}{dt} dt = V \cdot \frac{h^2}{4\pi}$$

Superconducting Condensation Energy:

$$\therefore F_n - F_s = \frac{H_c^2}{8\pi}$$

- Entropy:

  From the relation $S = -(\partial F/\partial T)$,

$$S_n - S_s = -\frac{1}{4\pi} H_c \frac{H_c}{dT}$$

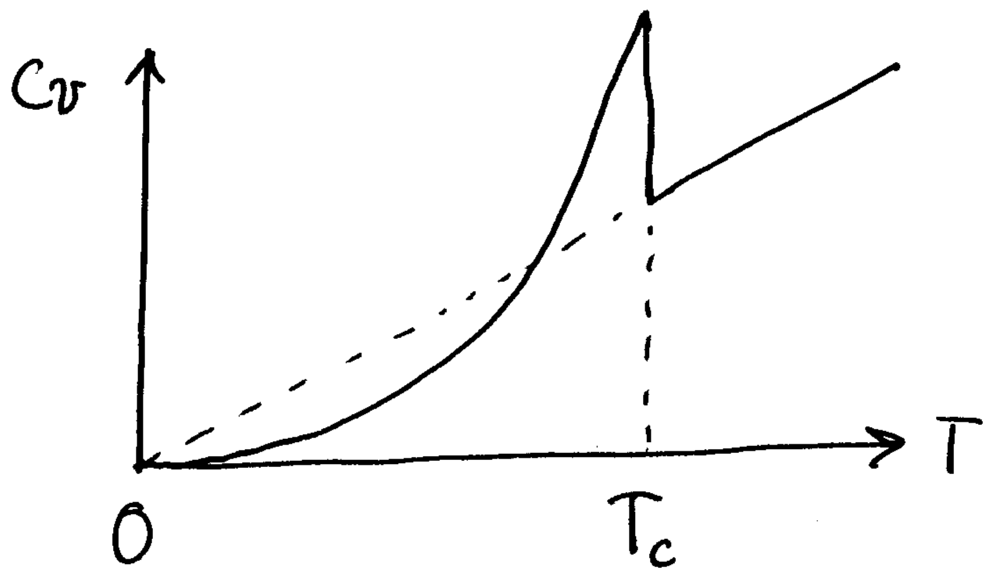
Latent heat:

$$L = T(S_n - S_s) = -\frac{T}{4\pi} H_c \frac{H_c}{dT} > 0$$

Since $H_c(T_c) = 0$ at $T = T_c$, $L = 0$, i.e., 2nd order phase transition.

- At zero field,

$$C_n - C_s = T\frac{d}{dT}(S_n - S_s)|_{T=T_c} = -\frac{T_c}{4\pi}\left(\frac{dH_c}{dT}\right)^2_{T=T_c} < 0$$



Thermodynamics

So far we discussed the free energy of electrons by

$$F_e = U_e - TS$$

where the internal energy U_e is given by

$$U_e = \sum_i \left[\frac{1}{2m} (\mathbf{p}_i - e\mathbf{A}_i/c)^2 + v_i \right] + \sum_{i>j} v_{ij}$$

Thus, including the free energy of $\mathbf{h}$-field, the total free energy becomes

$$\mathcal{F} = F_e + \int d\mathbf{r}\, \frac{\mathbf{h}^2}{8\pi}$$

$$F_e = \int f_s(\mathbf{r}) d\mathbf{r}$$

Induction B

From the local field $\mathbf{h}(\mathbf{r})$,

$$\mathbf{B} = \begin{cases} \langle \mathbf{h}(\mathbf{r}) \rangle_{\Delta v} & \text{inside SC} \\ \mathbf{h}(\mathbf{r}) & \text{outside} \end{cases}$$

Thermodynamic Field H

$$\delta\mathcal{F} = \int d\mathbf{r}\frac{\mathbf{H}(\mathbf{r})}{4\pi}\cdot\delta\mathbf{B}(\mathbf{r})$$

(i) Outside SC,

$$\mathbf{h} = \mathbf{B}$$

$$\delta\mathcal{F} = \frac{\mathbf{h}\cdot\delta\mathbf{h}}{4\pi} = \frac{\mathbf{H}\cdot\delta\mathbf{B}}{4\pi}$$

$$\therefore \mathbf{h} = \mathbf{H} = \mathbf{B}$$

(ii) Inside SC, we expect $\mathbf{j}_s \neq 0$ so that

$$\langle\mathbf{j}\rangle = \langle\mathbf{j}_s\rangle + \mathbf{j}_{\text{ext}}$$

$$\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j}_{\text{ext}}$$

Thermodynamic Potential $\mathcal{G}$

at fixed T and $\mathbf{j}_{\rm ext}$:

$$\delta\mathcal{F} = \frac{1}{4\pi}\int d\mathbf{r}\mathbf{H}\cdot\delta\mathbf{B} - S\delta T$$

Define:

$$\mathcal{G} = \mathbf{F} - \int d\mathbf{r}\frac{\mathbf{B}\cdot\mathbf{H}}{4\pi}$$

$$\delta\mathcal{G} = -\int\frac{\mathbf{B}\cdot\delta\mathbf{H}}{4\pi}d\mathbf{r} - S\delta T$$

Using $\nabla\cdot\mathbf{B}$ and $\mathbf{B} = \nabla\times\mathbf{A}$,

$$S\delta T + \delta\mathcal{G} = -\frac{1}{4\pi}\int(\nabla\times\mathbf{A}\cdot\delta\mathbf{H})d\mathbf{r} = \frac{1}{c}\int d\mathbf{r}\mathbf{A}\cdot\delta\mathbf{j}_{\rm ext}$$

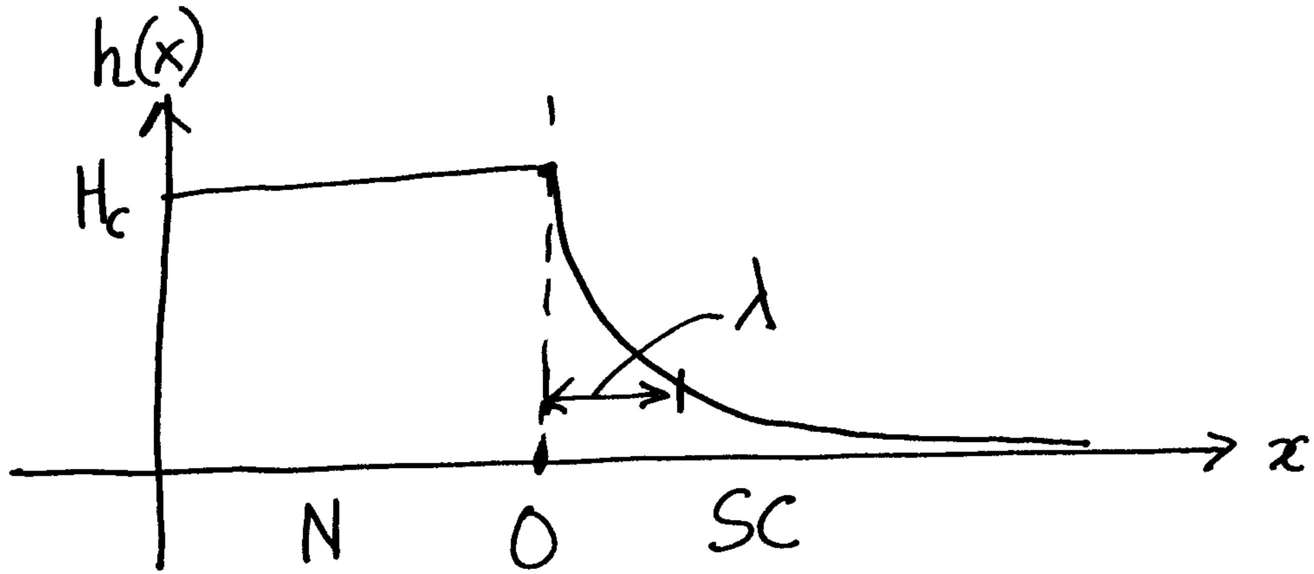
$$\therefore \delta\mathcal{G} = 0$$

at constant T and $\mathbf{j}_{\rm ext}$, minimum of $\mathcal{G}$.

Origin of the Surface Energy

- Type I SC, i.e., $\lambda \ll \xi_o$,

$$\Delta \mathcal{G}_s = \gamma \approx -\frac{H_c^2}{8\pi}\lambda A_S + \frac{H_c^2}{8\pi}\xi_o A_S \approx \frac{H_c^2}{8\pi}\xi_o A_S$$



- Type II SC, i.e., $\lambda \gg \xi_o$, Contrary to the case of Type I, here the condensation energy is negligible compared to the magnetic energy:

$$h = \begin{cases} H_c & (x < 0, \text{Normal region}) \\ H_c e^{-x/\lambda} & (x > 0, \text{S}Cregion) \end{cases}$$

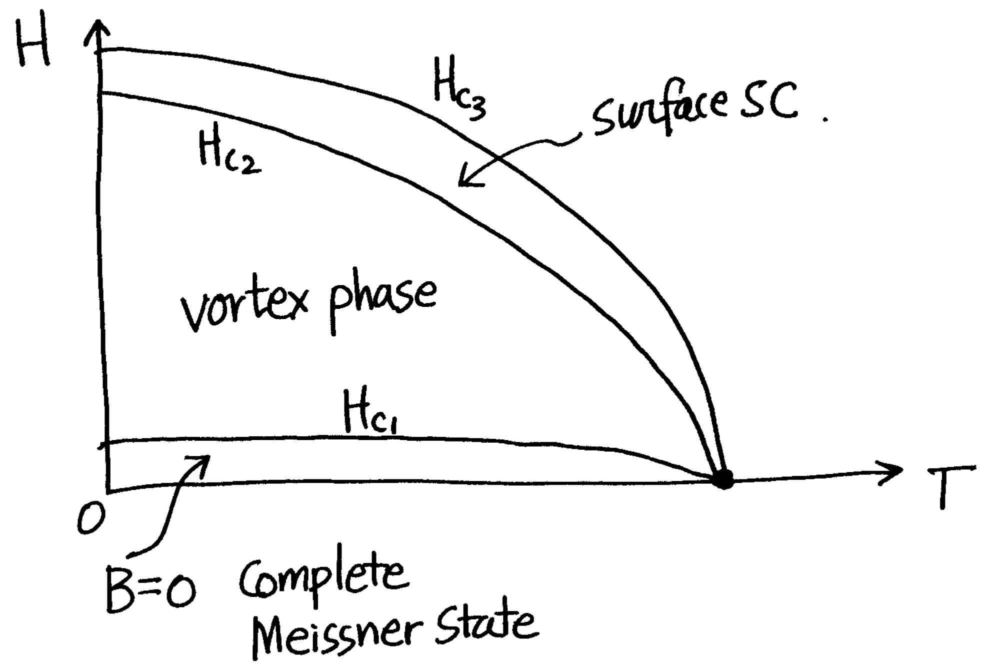
$$\begin{aligned} \mathcal{G}_s &= \int_{x>0} dx \left(F_n - \frac{H_c^2}{8\pi} + \frac{h^2}{8\pi} - \frac{Hh}{4\pi} + \lambda^2 \frac{(dh/dx)^2}{8\pi} \right) \\ &= \int dx (F_n - \frac{H_c^2}{8\pi}) + \gamma A_S \end{aligned}$$

Negative surface energy for the Type II SC:

$$\gamma = \int_0^\infty dx \left[\frac{h^2 + \lambda^2 (dh/dx)^2}{8\pi} - \frac{hH_c}{4\pi} \right] = -\frac{H_c^2}{8\pi} \lambda < 0$$

Magnetic Propreties of Type II Superconductors

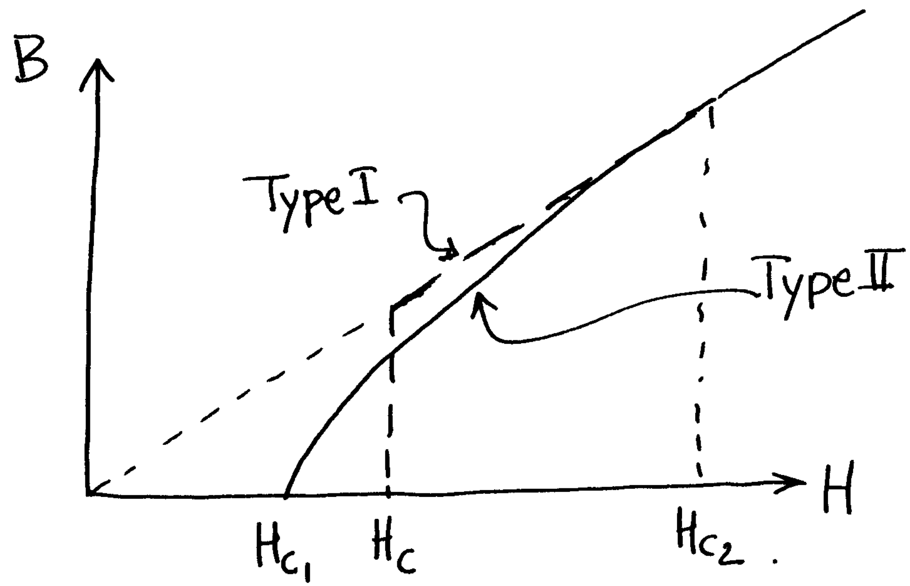
Phase diagram of a type II SC



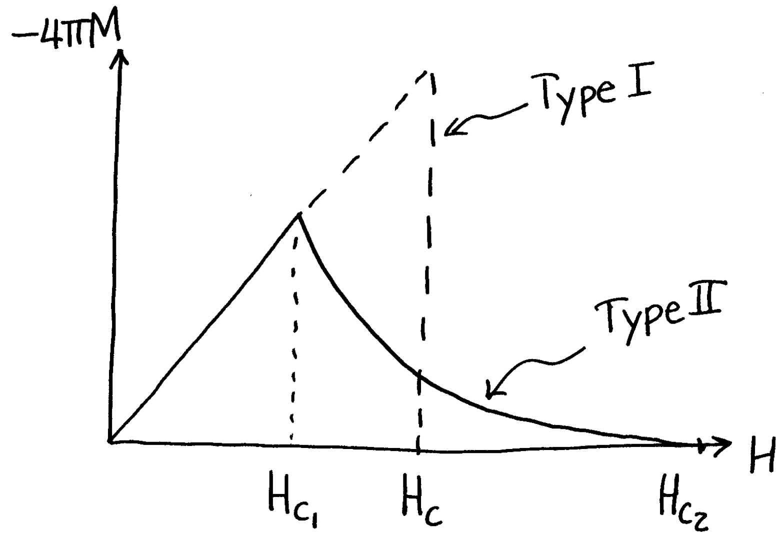
Magnetization of the SC

$$M = \frac{B - H}{4\pi}$$

At a give $T < T_c$,



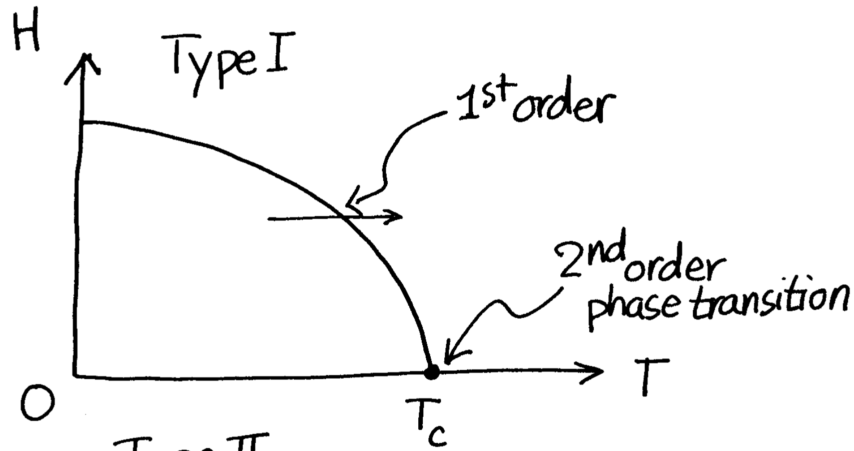
Reversal Magnetization

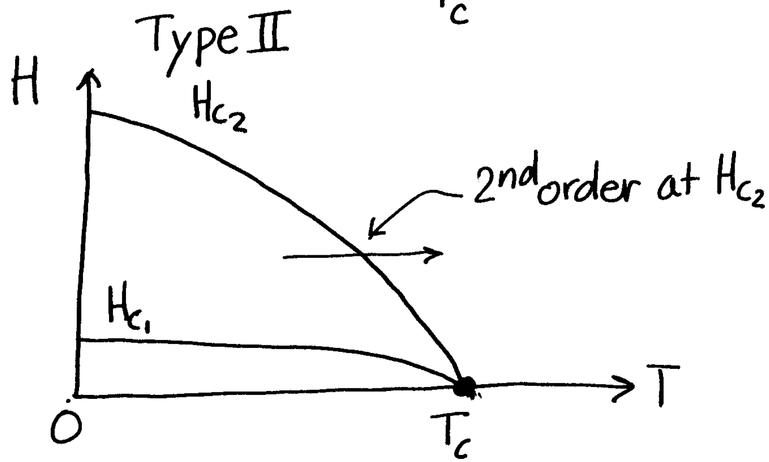


Remark:

$$\int_0^{H_c} M_I dH = \int_0^{H_{c2}} M_{II} dH = -\frac{H_c^2}{8\pi}$$

if both SC's have the same thermodynamic field H_c.


H
Type I
1st order
2nd order
phase transition
O
Tc
T


Type II
H
Hc2
2nd order at Hc2.
Hc1
O
Tc
T

Vortex State

For $\lambda \gg \xi$, the vortex line arises from "negative surface energy".

- Flux quantization with $n = 1$:

$$\phi = \int \mathbf{h} \cdot d\mathbf{s} = n\phi_o$$

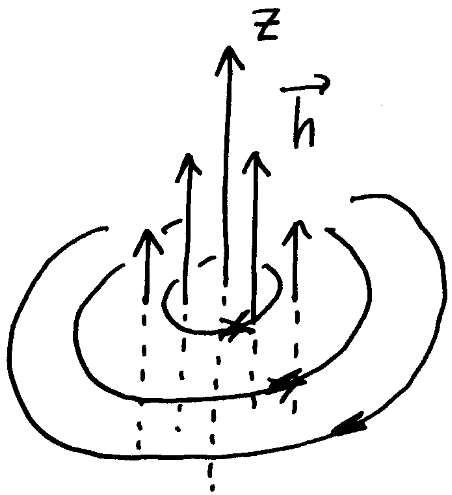
  where $\phi_o = ch/2e = 2 \times 10^{-7}$ G·cm^2.

- String of singularity analogously to the mono-pole field.
  $\Rightarrow$ "Solition" of a complex field with $U(1)$ gauge symmetry.

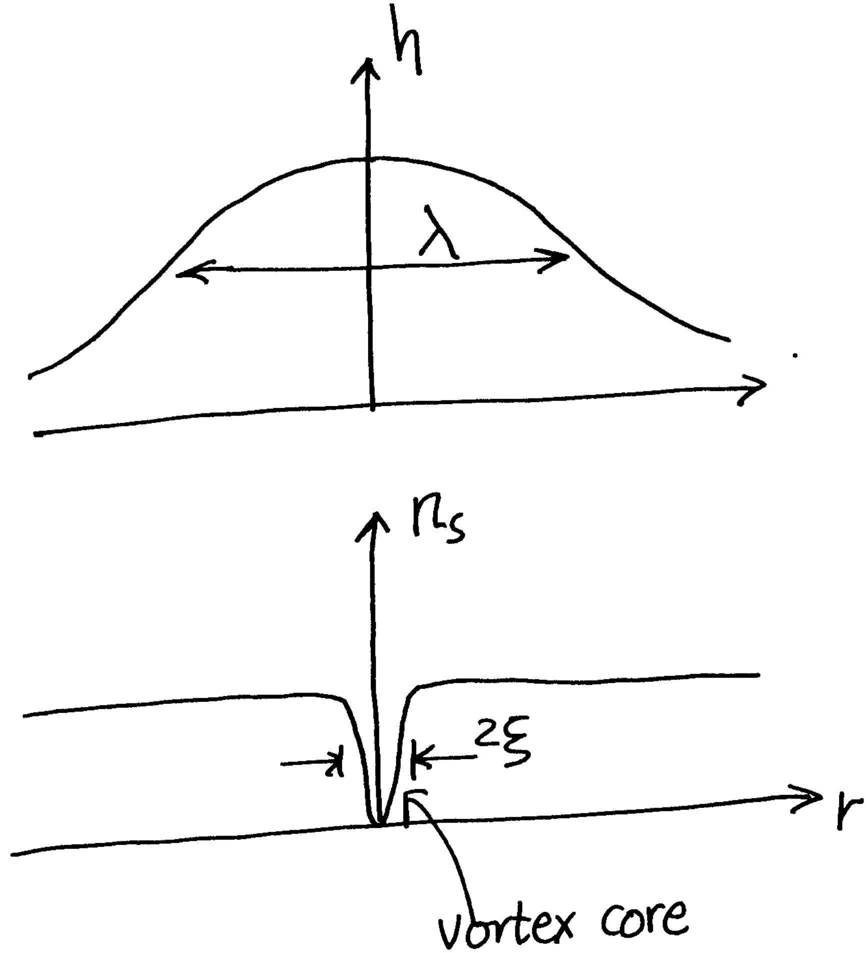
- Analog to the superfluid:

$$\nabla \times \mathbf{v} = 0$$

  But, the range of v in He^4 is $\sim 1/r$ while that of **j** in SC $\sim e^{-r/\lambda}$.



$$\nabla \times \vec{j} = -\frac{ne^2}{mc}\vec{h}$$



Distribution of $\mathrm{h}(\mathrm{r})$ near the vortex

For $r > \xi$, minimizing $\mathcal{F}$ leads to the London equation,

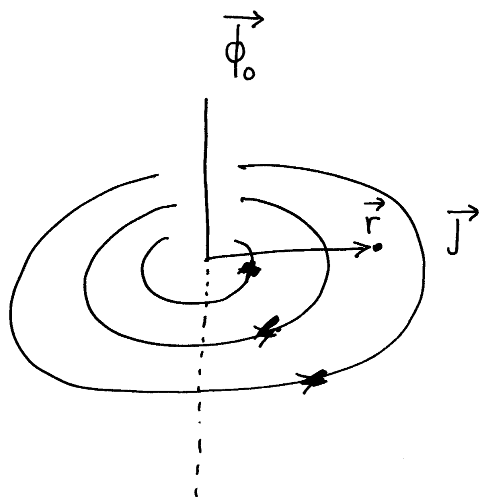
$$\mathbf{h} + \lambda^2 \nabla \times (\nabla \times \mathbf{h}) = 0 \qquad (r > \xi)$$

Substituting a small core by a delta function,

$$\mathbf{h} + \lambda^2 \nabla \times (\nabla \times \mathbf{h}) = \vec{\phi_o} \delta_2(\mathbf{r})$$

where $\hat{\phi}_o = \phi_o \hat{z}$ represent the total flux carried by the line.

$$\int \mathbf{h} \cdot d\mathbf{s} + \lambda^2 \oint_C \nabla \times \mathbf{h} \cdot d\mathbf{l} = \phi_o$$



(i) When $r \gg \lambda$, $\mathbf{j}(\mathbf{r}) = (c/4\pi)\nabla \times \mathbf{h} \to 0$,

$$\int \mathbf{h} \cdot d\mathbf{s} = \phi_o$$

(ii) When $\xi \ll r \ll \lambda$, we can neglect

$$\int \mathbf{h} \cdot d\mathbf{s} \approx \left(\frac{r^2}{\lambda^2}\right)^2 \phi_o \to 0$$

$$\lambda^2 |\nabla \times \mathbf{h}| 2\pi r \approx \phi_o$$

$$\therefore |\nabla \times \mathbf{h}| = \frac{\phi_o}{2\pi\lambda^2}\frac{1}{r} \qquad (\xi < r \ll \lambda)$$

For $\mathbf{h} = h\hat{z}$, $|\nabla \times \mathbf{h}| = -dh/dr$.

$$h(r) = \frac{\phi_o}{2\pi\lambda^2}\left[\ln\left(\frac{\lambda}{r}\right) + \text{const.}\right] \qquad (\xi < r \ll \lambda)$$

Complete solution:

$$\mathbf{h}(r) = \frac{\phi_o}{2\pi\lambda^2} K_o(r/\lambda)\hat{z}$$

$$\mathbf{j}(r) = \frac{\phi_o c}{8\pi^2\lambda^3} K_1(r/\lambda)\hat{\theta}$$

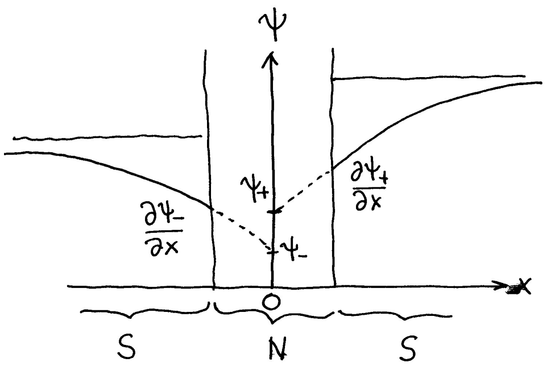
Vortex Line Energy:

$$\mathcal{F}_v = \left(\frac{\phi_o}{4\pi\lambda}\right)^2 \ln\left(\frac{\lambda}{\xi}\right)$$

- $F = F(\xi)$
- $F \propto \phi^2 \rightarrow$ ensures the minimum flux value ϕ_o.

Josephson Effect

S-N-S Junction



When $\mathbf{A} = 0$, the boundary values are related by

$$\psi_+ = M_{11}\psi_- + M_{12}\frac{\partial \psi_-}{\partial x}$$

$$\frac{\partial \psi_+}{\partial x} = M_{21}\psi_- + M_{22}\frac{\partial \psi_-}{\partial x}$$

where $M_{11}M_{22} - M_{12}M_{21} = 1$ (M_{ij}: real).
The supercurrent I_x / unit-area,

$$I_x = \frac{-ie\hbar}{m}\left[\psi^* \frac{\partial \psi}{\partial x} - c.c.\right]$$

Josephson Current:

$$I_x = \frac{-ie\hbar}{m}\left[\psi_-^*\left(\frac{1}{M_12}\psi_+ - \frac{M_{11}}{M_{12}}\psi_-\right) - c.c.\right] = I_x = \frac{-ie\hbar}{M_{12}m}(\psi_-^*\psi_+ - \psi_-\psi_+^*)$$

Since the phase factors can be modified by the presence of $\mathbf{A}$,

$$\psi = |\psi|e^{i\phi}$$

$$I_x = \frac{2e\hbar}{M_{12}m}|\psi_+||\psi_-|\sin(\phi_+ - \phi_-) = I_m \sin(\phi_+ - \phi_-)$$

Similar discussion can be made by using a tunneling model Hamiltonian:

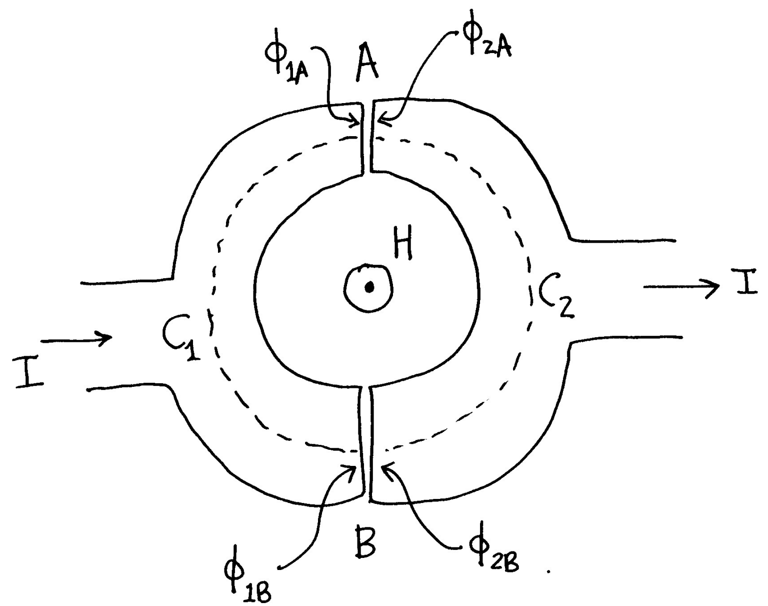
$$i\hbar\frac{\partial\psi_1}{\partial t} = H_{11}\psi_1 + H_{12}\psi_2$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = H_{12}\psi_2 + H_{22}\psi_2$$

SQUID

Superconducting Quantum Interferometer Device

$$I = I_m \left[\sin(\phi_{2A} - \phi_{1A}) + \sin(\phi_{2B} - \phi_{1B})\right]$$



Inside the wire, $\mathbf{j}_s = 0$,

$$\phi_{1B} - \phi_{1A} = \int_{C_1} \frac{2e}{\hbar c}\mathbf{A} \cdot d\mathbf{l}$$

$$\phi_{2A} - \phi_{2B} = \int_{C_2} \frac{2e}{\hbar c} \mathbf{A} \cdot d\mathbf{l}$$

$$(\phi_{1B} - \phi_{2B} + \phi_{2A} - \phi_{1A}) = \frac{2e}{\hbar c} \oint \mathbf{A} \cdot d\mathbf{l} = 2\pi \frac{\Phi}{\phi_o}$$

$$(\phi_{2A} - \phi_{1A}) = -(\phi_{2B} - \phi_{1B}) = \pi \frac{\Phi}{\phi_o}$$

$$\therefore \quad I[\Phi] = 2I_m \left| \sin\left(\pi \frac{\Phi}{\phi_o}\right) \right|$$