HOME WORK #8

(due date: Sat., 15 December 2007)

Consider a degenerate Fermion system with residual attractive interaction, which is described by the BCS Hamiltonian:

$$\mathscr{H} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{+}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

where the pairing interaction $V_{\mathbf{k}\mathbf{k}'}$ is given by

$$V_{\mathbf{k}\mathbf{k}'} = V_o \theta(\omega_o - |\xi_{\mathbf{k}}|)\theta(\omega_o - |\xi_{\mathbf{k}'}|)$$

and the one-particle energy $\xi_{\mathbf{k}}$ is defined relative to the Fermi energy ε_F,

$$\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \varepsilon_F.$$

1. Two-particle eigenstate: If there is no residual interaction, i.e., $V_o = 0$, the ground state can be represented by a simple Fermi sphere $|\Phi_0\rangle$ such that

$$\mathscr{H}|\Phi_0\rangle = E_o|\Phi_0\rangle.$$

As we discussed during the class, the two-particle scattering process with zero total momentum, i.e., $\mathbf{P}_{\text{tot}} = 0$, may have a singularity corresponding to the pairing instability. Thereby, one has to consider the two-particle eigenstate as a basis for the description of many-particle ground state of $\mathscr{H}$ instead of usual single-particle eigenstates. Let us consider a two-particle state by adding two extra electrons to the "unperturbed" ground state $|\Phi_0\rangle$ satisfying the two-particle eigenvalue equation:

$$\mathscr{H}|\psi_2\rangle = E|\psi_2\rangle \tag{1}$$

As for the two-particle (lowest-energy) eigenstate, one may represent $|\psi_2\rangle$ as a superposition of two-particle states with total momentum $\mathbf{P}_{\text{tot}} = 0$ so that

$$|\psi_2\rangle = \sum_{|\mathbf{k}|>k_F} A_{\mathbf{k}} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow}|\Phi_0\rangle \tag{2}$$

> **Question:** Why do we choose the sum over $|\mathbf{k}| > k_F$? What should we do if we want to consider the case of $|\mathbf{k}| < k_F$?

From Eq. (1), show that the two-particle state can have a bound state, so-called the Cooper pair, when the residual interaction is attractive, i.e, $V_o < 0$, with the binding energy Δ in a form of

$$\Delta = 2\omega_o \exp(-1/\lambda).$$

Also, show that $\lambda = N(0)|V_o|/2$ where $N(0)$ is the density-of-state at ε_F.

> Note that the summation over $\mathbf{k}$ can be rewritten by the energy integral:
>
> $$\sum_{\mathbf{k}} = \int d\xi \; N(\xi)$$
>
> where $N(\xi)$ is the density-of-state of one-particle eigenstates.

2. Coherence length: The two-particle state can be represented in a real-space form:

$$\psi_2(\mathbf{x}_1, \mathbf{x}_2) = \langle \mathbf{x}_1, \mathbf{x}_2 | \psi_2\rangle$$

From Eq. (2), one can have the expression

$$\psi_2(\mathbf{x}_1, \mathbf{x}_2) = \sum_{k>k_F} A_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}$$

Alternatively, we have

$$\psi_2(\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2) = \sum_{k>k_F} A_k e^{i\mathbf{k}\cdot\mathbf{x}}.$$

For the real-space pair wavefunction, estimate the size of Cooper pair or the coherence length ξ_o as given by the definition:

$$\xi_o^2 = \langle \psi_2(\mathbf{x}) | \mathbf{x}^2 | \psi_2(\mathbf{x}) \rangle = \int d\mathbf{x}\, \mathbf{x}^2 |\psi_2(\mathbf{x})|^2$$

and show that the coherence length ξ_o becomes

$$\xi_o = \frac{\hbar v_F}{\Delta},$$

where the one-particle energy ξ near ε_F can be approximated by

$$\xi = v_F (k - k_F).$$

3. Variational Solution of the BCS Hamiltonian: As we discussed during the class, the BCS trial (variational) wave function can be represented by a direct-product and superposition of the empty states $|0_{\mathbf{k}}\rangle$ and the pair states $|1_{\mathbf{k}}\rangle = c^+_{k\uparrow} c^+_{-k\downarrow})|0\rangle$:

$$|\psi^0_{\mathrm{BCS}}\rangle = \prod_k (u_k |0_{\mathbf{k}}\rangle + v_k |1_{\mathbf{k}}\rangle)$$

(a) Calculate the expectation value of the ground state energy E^0_{BCS} as a function of θ_k:

$$E^0_{\mathrm{BCS}}(\{\theta_k\}) = \langle \psi^0_{\mathrm{BCS}} | \mathscr{H} | \psi^0_{\mathrm{BCS}} \rangle$$

where $u_k = \cos(\theta_k/2)$ and $v_k = \sin(\theta_k/2)$ are determined through the normalization of $|\psi^0_{\mathrm{BCS}}\rangle$, and the BCS Hamiltonian is given by

$$\mathscr{H} = \sum_{k\sigma} \xi_k c^+_{k\sigma} c_{k\sigma} - \sum_{kk'} V c^+_{k\uparrow} c^+_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}.$$

Here the cutoff energy of V is ω_0, i.e.,

$$V = V_0 \theta(\omega_0 - |\xi_k|) \theta(\omega_0 - |\xi_{k'}|).$$

(b) From the minimum energy principle, by minimizing E^0_{BCS} with respect to θ_k, obtain the parameters u_k and v_k in terms of Δ and $E_k = \sqrt{\xi_k^2 + \Delta^2}$ where Δ is defined by

$$\Delta = V_0 \sum_{k'} u_{k'} v_{k'} = \frac{V_0}{2} \sum_{k'} \sin\theta_k.$$

(c) Guess a trial wave function $|\psi^1_{\mathrm{BCS}}\rangle$ for the excited state. One may assume that only one of the pairs with $\mathbf{k}$ get excited, the state of which should be orthogonal to the ground state $|\psi^0_{\mathrm{BCS}}\rangle$. Based on the results obtained above, show that the minimal excitation energy in the superconducting state becomes

$$\Delta E_{\mathrm{excitation}} = 2\Delta.$$