HOME WORK #7

(due date: Wed., 5 December 2007)

Consider a degenerate Fermion system with a ground state $|\Phi_o\rangle$ given by

$$|\Phi_o\rangle = \prod_{\alpha\sigma}^{occ} c^+_{\alpha\sigma}|0\rangle = \prod_{\alpha\sigma} \theta(\epsilon_F - \epsilon_\alpha) c^+_{\alpha\sigma}|0\rangle$$

where the basis function $\{\phi_\alpha\}$ is defined by the single particle Schrodinger equation

$$h_o|\phi_\alpha\rangle = \epsilon_\alpha|\phi_\alpha\rangle.$$

Now consider another basis set of $\{\phi_i\}$ such that ϕ_i is not an eigenstate of h_o, and $\{\phi_\alpha\}$ and $\{\phi_i\}$ are related by the unitary matrix U:

$$|\phi_\alpha\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\phi_\alpha\rangle = \sum_i |\phi_i\rangle U_{i\alpha}\,.$$

1. Express the expectation value P^σ_{ij} defined by
$$P^\sigma_{ij} = \langle\Phi_o|c^+_{i\sigma}c_{j\sigma}|\Phi_o\rangle$$
in terms of the unitary matrix U.
2. Show that
$$\langle\Phi_o|c^+_{i\sigma}c^+_{k\sigma'}c_{l\sigma'}c_{j\sigma}|\Phi_o\rangle = P^\sigma_{ij}P^{\sigma'}_{kl} - \delta_{\sigma\sigma'}P^\sigma_{il}P^\sigma_{kj}\,.$$
3. We can represent the one-particle operator $F_1 = \sum_a f_1(\mathbf{r}_a)$ in the second quatization form by
$$\hat{F}_1 = \sum_{ij}\langle i|f_1|j\rangle c^+_i c_j$$
where
$$\langle i|f_1|j\rangle = \int \phi^*_i(\mathbf{r}_a)f_1(\mathbf{r}_a)\phi_j(\mathbf{r}_a)d\mathbf{r}_a\,.$$
Similarly, the two particle operator $F_2 = \sum_{a\neq b} f_2(\mathbf{r}_a,\mathbf{r}_b)$ can be represented by
$$\hat{F}_2 = \sum_{ijlm}\langle ij|f_1|lm\rangle c^+_i c^+_j c_m c_l$$
where
$$\langle ij|f_1|lm\rangle = \int \phi^*_i(\mathbf{r}_a)\phi^*_j(\mathbf{r}_b)f_2(\mathbf{r}_a,\mathbf{r}_b)\phi_l(\mathbf{r}_a)\phi_m(\mathbf{r}_b)d\mathbf{r}_a d\mathbf{r}_b\,.$$
Considering a free electron system with hamiltonian $h_o = -(1/2)\nabla^2$ with $\phi_{\alpha=k}(\mathbf{r}) = (1/\sqrt{V})\exp(i\mathbf{k}\cdot\mathbf{r})$, show that the number density operator $\rho(\mathbf{r}) = \sum_a \delta(\mathbf{r}-\mathbf{r}_a)$ can be written by
$$\hat{\rho}(\mathbf{r}) = \sum_{k,q,\sigma} e^{-i\mathbf{q}\cdot\mathbf{r}} c^+_{\mathbf{k}\sigma}c_{\mathbf{k}-\mathbf{q}\,\sigma}\,.$$
Also, calculate the expectation value of the number density
$$\rho(\mathbf{r}) = \langle\Phi_o|\hat{\rho}(\mathbf{r})|\Phi_o\rangle$$
where
$$|\Phi_o\rangle = \prod_{\mathbf{k}\sigma}\theta(k_F - |\mathbf{k}|)c^+_{\mathbf{k}\sigma}|0\rangle\,.$$
4. Repeat the exercise #3 above for the two particle density operator,
$$\rho(\mathbf{r},\mathbf{r}') = \sum_{a\neq b}\delta(\mathbf{r}-\mathbf{r}_a)\delta(\mathbf{r}'-\mathbf{r}_b)\,.$$