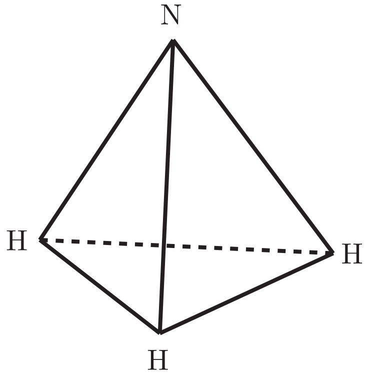
Spring Final Exam (June 24, 2004)

Total=110

1. (a)[10] In the Wigner-Eckardt theorem, there is the reduced matrix element of a vector operator. If we were to calculate the matrix element of the angular momentum operator, we need to calculate the reduced matrix element for $\langle\alpha'; j'||\mathbf{J}||\alpha; j\rangle$. Explain how you would calculate it. What is it if $j = 2$?

(b)[10] Consider $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ where $j_1 = 1$ and $j_2 = \frac{3}{2}$. Obtain $|j = \frac{5}{2}, j_z = -\frac{3}{2}\rangle$ and $|j = \frac{3}{2}, j_z = -\frac{3}{2}\rangle$ in terms of J_{1z} and J_{2z} eigenstates. Choose the phases as you wish but they must be consistent.

2.[15] Consider the ammonia molecule NH_3 whose configuration is as given in the following figure. One can find a symmetry of this system. The mass of the ammonia molecule is large.



To find the information on its structure, note that there is a top-down symmetry around the plane of three hydrogen atoms. You can think of many things. But one consideration from the above symmetry is that the nitrogen atom N can be at the symmetrical point(the mirror image of the above figure) at the bottom of the hydrogen plane which will have the same energy as the one shown in the figure. Now these two states T(shown in the figure) and B(the mirror reflected state) are degenerate.[Reflection of an H atom against an NHH plane gives the same conclusion.] If we stop here, nothing interesting happens. Now consider the possibility of transitions from T to B, and vice versa. Parametrize this transition, which amounts to declaring a quantum mechanical Hamiltonian, and obtain the ground and the excited state energies. Write down the ground state vector. Suppose that we started from

$|T\rangle$ at $t=0$. When do you expect to find the 100 % $|B\rangle$ state for the first time?

3. In the class, the $E1$ transition from $|nlm\rangle = |2;1,m\rangle$ to the ground state $|nlm\rangle = |1;0,0\rangle$ was discussed.

(a)[10] Write down the relevant term in the interaction Hamiltonian and its matrix element for the $E1$ transition.

(b)[10] Calculate the Clebsch-Gordan coefficient $\langle 11mq|11;0,0\rangle$.

(c)[10] Write down Fermi's golden rule, including all constants such as $\hbar$ and c. In 3 dimensional space, calculate the density of final states, using the box normalization with the box size $V = L^3$.

(d)[12] Calculate the transition rate. Here, take the reduced matrix element as A

$$\langle n=1, l'=0||\mathbf{p}^{(1)}||n'=2, l=1\rangle = A\ .$$

and take the photon wave function as $\langle \text{photon}; \mathbf{k}, \lambda|a^\dagger(\mathbf{k},\lambda)|0\rangle = (B/L^{3/2})\hat{\mathbf{k}}_\perp$ where $\hat{\mathbf{k}}_\perp \cdot \hat{\mathbf{k}} = 0$. I will see your method, not the details. But if you get the correct result, it will also help.

4. (a)[5] Consider one dimensional time independent Schrödinger wave equation

$$\left(-\tfrac{\hbar^2}{2m}\tfrac{d^2}{dx^2} + \tfrac{1}{2}m\omega^2 x^2\right)\psi(x) = E\psi(x)$$

where m, ω and the energy parameter E are constants. Using the engineering dimension analysis, define the annihilation and creation operators a and $a^\dagger$ such that $[a, a^\dagger] = 1,\ [a, a] = [a^\dagger, a^\dagger] = 0$, and express the Hamiltonian operator in terms of a and $a^\dagger$.

(b)[10] Considering a Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x})\right)^2, \tag{166}$$

calculate the velocity operator $\mathbf{v} = \dot{\mathbf{x}}$. Here, $\mathbf{A}$ is the vector potential satisfying $\nabla \times \mathbf{A} = \mathbf{B}$ appearing in electromagnetism. Calculate commutators of the velocity operator $\mathbf{v}$, i.e. calculate $[v_i, v_j]$.

(c)[8] An electron is moving in a constant magnetic field $\mathbf{B} = B\hat{z}$ with $B > 0$. If its motion is in the $x - y$ plane, what are the possible energies of this electron?

5.[10] The laser beam is said to be coherent. In the harmonic oscillator problem of **4**(a), the energy eigenstates are labeled by $|n\rangle$ with integer $n \geq 0$. In the harmonic oscillator problem, the coherent state $|z\rangle$ is defined as an eigenstate of the complex operator a.

(a) Express the n^{th} excited state $|n\rangle$ by applying $a^\dagger$'s on the ground state $|0\rangle$.

(b) Show that

$$|z\rangle \propto \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

is a coherent state. In this problem, do not consider the time dependence of the Hamiltonian.

Are the coherent states orthogonal and/or complete? Give a reason(s).