Chapter 6: Group Theory

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Abstract

In this chapter, we give a brief review on elementary concepts in group theory. We will try to proceed until introducing continuous groups such as $SU(2)$. Easy readings for undergraduate students can be found in Refs. [1–3].

I. INTRODUCTION

Physical problems can be solved easily if we know the symmetry behind the problems.

For example, the trajectory of a point particle attracted by Earth's gravity can be solved in (2+1) dimension since a translation toward the vertical direction of the trajectory plane must give the same physical result. We have reduced the 3D spatial problem to a 2D spatial problem.

If the physical problem assumes that the solution has a spherical symmetry, then effectively the 3D problem is reduced to a 1D problem, the variable being the radius r from the center of the spherical symmetry.

Symmetries implicitly contain parameters representing the amount of symmetry operation. The symmetry operation is the action on the system after which the physical system remains the same as before. Depending on the parameter set, we distinguish symmetries broadly into two kinds of symmetries: discrete and continuous symmetries. A discrete symmetry corresponds to symmetry operations with finite parameters, and hence the number of physical systems in consideration is finite. On the other hand, a continuous symmetry corresponds to the one with an infinitesimally connected parameter set, implying an infinite number of physical systems connected by the symmetry operation.

Some examples possessing the symmetries are:

- 1. A beautiful temple building possessing left-right symmetry.
- 2. An equilateral triangle which has the same shape and location after 120^o or 240^o rotation, clockwise or anticlockwise.
- 3. A disk, symmetrical by any amount of rotation around its center.
- 4. A sphere.

However, a more important quantity for physical symmetry is the physical law itself. The above examples are realizations of physical systems. What I mean by physical laws are equations of motions, such as Newton's second law, Schrödinger equation, Boltzmann equation, Dirac equation, etc. Talking about equation of motion is equivalent to talking about an action leading to that equation. Thus, the symmetries possessed by the action are the most important clue in simplifying the problem. For example, the second law of Newton is

$$m\frac{d^2}{dt^2}\mathbf{x}(t) = \frac{d}{dt}\mathbf{p} = \mathbf{F}(\mathbf{x}).$$

If the external force is vanishing, the above equation has a symmetry that it remains the same after we translate $\mathbf{x}$ by a constant vector $\mathbf{a}$. Thus, we have the linear momentum conservation. It comes from the symmetry that the action or the equation of motion respects.

Our objectives in studying the symmetries are in both the action and the physical realizations. A symmetry operation is an action on the set of elements in consideration. So we are interested in the set in question

$$S = \{g_1, g_2, g_3, \cdots\}. \tag{1}$$

For an infinite number of elements, one may introduce a continuous parameter distinguishing the elements. This kind of symmetry operation is studied in the name of group theory which has blossomed in quantum field theory. The quantum field theory unifies the internal symmetries considered in your experience in quantum mechanics and the space-time symmetry discussed in relativity. So group theory is most useful to particle physicists using quantum field theory as their tool.

II. FINITE GROUPS

Read Ref. [1].

III. MATRIX REPRESENTATION

Read Ref. [1].

IV. CONTINUOUS GROUPS

A very short and easily readable book on continuous group or Lie group is Ref. [3].

In this very short chapter, let us introduce the continuous group in terms of the matrix representation. The content is the same as the original group since the matrix representation represents the group. The matrices $\{M_i(i = 1, 2, \cdots)\}$ representing a group is $N \times N$ square matrices.

Without any restriction on M_i, the elements being complex numbers, M_i consists of $2N^2$ elements. Since these matrices are acting on a column vector in the vector space, by this matrices the components of the vector changes to new ones. This change is linear, and the group of these general linear transformation, satisfying the group axiom, is called $GL(N, c)$, N being the dimensionality of the representation and c representing the entries of complex numbers. But $GL(N, c)$ should not contain singular matrices since these singular ones do not have inverses. Among $GL(N, c)$, we can consider a subgroup of matrices, satisfying $Det\ M_i = 1$. This subgroup is a special linear group $SL(N, c)$.

A particularly interesting subgroup of $GL(N, c)$ is the unitary group $U(N)$. The unitary group consists of $N \times N$ unitary matrices, $\{U_i\}$,

$$U_i^\dagger = U^{-1}. \tag{2}$$

The unitarity is maintained by a product of two unitary matrices, due to Eq. (2)

$$(U_iU_j)^\dagger(U_iU_j) = U_j^\dagger U_i^\dagger U_iU_j = I. \tag{3}$$

Thus, the unitarity condition can be a good property for the group. This unitary transformation is particularly interesting in quantum mechanics. In quantum mechanics, a vector or wave function ψ can be represented as an $N \times 1$ column vector, with complex elements. Any meaningful transformation by U must conserve probability density $|\psi|^2$, implying

$$|\psi'|^2 = |U\psi|^2 = \psi^\dagger U^\dagger U\psi = \psi^\dagger\psi. \tag{4}$$

Thus, the probability conserving transformation must be unitary. Sometimes, it is simpler to restrict to a subgroup of $U(N)$, consisting of unitary matrices with determinant 1. This subgroup is called

$$SU(N) = \{M|M^\dagger = M^{-1}, Det\ M = 1\}. \tag{5}$$

Any $N \times N$ unitary matrix U can be written as

$$U = e^{iM_i\alpha_i}, \quad i = 1, 2, \cdots, N_A \tag{6}$$

where M_i are Hermitian matrices $M^\dagger = M$ and the parameters α_i are real numbers. We can say that the above group is an N_A parameter group. Since these parameters are continuous, the group is called a continuous group and has an infinite number of elements. If the transformation is just nothing, i.e. $\alpha_i = 0$ for all i, then $U = I$. If the transformation is infinitesimally near to identity, we can Taylor-expand U,

$$U \simeq I + i\sum_i^{N_A} H_i\delta\alpha_i. \tag{7}$$

A finite transformation is obtained by infinitely many consecutive transformations (n times) of the above infinitesimal transformation with $\delta\alpha_i = \alpha_i/n$. For a specific i,

$$U(\alpha_i) = \lim_{n\to\infty}\left(I + iM_i\frac{\alpha_i}{n}\right)^n = e^{iM_i\alpha_i} \tag{8}$$

Counting the number of independent elements is important in the matrix representation. Since the infinitesimal transformation can be used to generate all the group elements, it is easier to study the infinitesimal transformation first. The infinitesimal transformation is represented by $N \times N$ Hermitian matrices H_i. The number of independent elements of complex and Hermitian H is $2N^2 - N^2$, the subtraction representing the N conditions for the Hermiticity. Thus, $N \times N$ Hermitian matrices have N^2 independent elements. So if we have N^2 independent Hermitian matrices, any unitary matrix having N^2 parameters can be constructed in terms of these independent matrices. These N^2 independent Hermitian matrices are the basic units representing the unitary transformation. If we consider special unitary matrices, we must have one more condition $Det = 1$, implying the traceless Hermitian matrices. Thus, $SU(N)$ needs $N^2 - 1$ independent Hermitian matrices:

$$SU(N)\ algebra : \{F_i | i = 1, 2, \cdots, N^2 - 1; F_i = F_i^\dagger; TrF_i = 0\}. \tag{9}$$

We say that F_i forms a group algebra. F_i being matrices, in general two such matrices do not commute, but

$$[F_i, F_j] = if_{ijk}F_k \tag{10}$$

where repeated indices imply summation. Here, the f_{ijk} are called the structure constants of $SU(N)$ and $f_{ijk} = -f_{jik}$. Thus, the study of a group can be easily done by studying the algebra. But if the group elements for finite rotations cannot be connected to the identity, the story is different. The difference is the topological property of the group space, which we do not discuss here.

The algebra specifies the group. It defines the group. The reason is the following. Suppose $\{M_i\}$ and $\{P_i\}$ represent two different representations of the group G. The group multiplication $g_i g_j = g_k$ is represented by matrices M_i and equivalently by P_i. But if the algebras for these two sets of matrices are different, then one cannot obtain the same group multiplication since the exponentiation involves the commutators. Thus, we conclude that every representation of the group must satisfy the same algebra, except for the singlet. For the singlet one needs only one complex number and the unitary matrices multiplying to this is 1×1 and two such multiplications must commute.

Beyond the singlet, two matrices multiplied do not commute in general. This commutation relation, beyond the singlet, defines the group. The fundamental representation corresponds to the smallest square matrices. Thus, we calculated the number of independent Hermitian matrices from this smallest square matrices above. But for a larger square matrices, some chosen matrices may satisfy the same algebra. It forms a different representation from the fundamental one.

Consider the group $SU(2)$. It is obtained by unitary, unimodular(Det=1) 2×2 matrices. Any such matrices are obtained by exponentiating the three independent traceless 2×2 Hermitian matrices. These can be chosen as

$$\{F_i = \frac{\sigma_i}{2}\}, \quad i = 1, 2, 3, \tag{11}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{12}$$

which satisfy

$$[F_i, F_j] = i\epsilon_{ijk}F_k \tag{13}$$

where ϵ_{ijk} are 1 (or -1) if all the indices are different and a cyclic permutation of 123 (or 213) and 0 if any of two indices are the same. The dimension of this fundamental representation is two and it is called the doublet representation denoted as **2**.

If a set of bigger square matrices satisfies the same algebra as Eq. (13), then that set also defines an $SU(2)$ group. But the vector space on which they act is bigger than **2**. For example, for the triplet representation **3** the matrices acting on them are constructed by the three Hermitian matricies

$$F_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{14}$$

which satisfies the same algebra given in Eq. (13). So it defines an $SU(2)$ representation.

For $SU(N)$, the adjoint representation has $N^2 - 1$ components and the fundamental representation has N components.

The 3×3 matrices did not use the full set of Hermitian 3×3 matrices. The $SU(3)$ has eight 3×3 Hermitian matrices satisfying the $SU(3)$ algebra. The three Hermitian given in (14) belongs to them. Anyway, the $SU(2)$ is a subgroup of $SU(3)$. For the **3** of $SU(2)$, it is a subset of **8** of $SU(3)$. For the **2** of $SU(2)$ (remember that it is a column vector), one can interpret the upper two components of **3** of $SU(3)$ as the $SU(2)$ doublet.

There are examples realizing the above $SU(2)$.

The set of angular momentum operators satisfies the $SU(2)$ algebra, and depending on the dimensionality of the matrices the angular momentum representation has

$$\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \cdots, (\mathbf{2l+1}) \tag{15}$$

for angular momenta $0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, l$, respectively.

The isospin concept introduced by Heisenberg in 1932 can be analyzed by $SU(2)$. Since the masses of proton and neutron are very similar, one can assume that strong interactions respect some kind of a symmetry, or a group. Since there are only two such states, one looks for a group having a doublet representation. One example is the $SU(2)$, proton and neutron being components of an $SU(2)$ vector,

$$\begin{pmatrix} p \\ n \end{pmatrix}. \tag{16}$$

Of course, one may worry that how one considers the obviously different p and n as the same thing since by an $SU(2)$ transformation proton changes to neutron and vice versa. It is interpreted that in a world neglecting the electromagnetic interactions proton and neutron behaves in the same way. Indeed, when one considers the strong interactions only, this seems to be the case. The electromagnetic interaction is a small perturbation.

The group $SU(3)$ was used to invent the quark model by M. Gell-Mann and G. Zweig, and called nowadays as the flavor group.

The group $SU(2) \times U(1)$ is the group considered in the standard model of particle physics. The group $SU(3)$ is the group respected by quantum chromodynamics(QCD), the force mediating strong interactions. The fundamental representation of this QCD group is **3** and hence three elements. Thus, quarks come in three kinds(called colors) which have exactly the same property except in the color.

Thus, one can see how useful the groups $SU(N)$ is.

Exercise 1. For the permutation group S_3, represent its elements by 2×2 matrices which represent the group faithfully. List all the subgroups of S_3. Are they invariant subgroups? Construct the multiplication table and show that your matrix representation reproduces the multiplication table.

Exercise 2. Show that Eq. (13) holds interms of F_i given in Eqs. (11) and (14).

Exercise 3. If the rotation around the isospin space is 180^o around the $y-$axis of the isospin space what will be the nucleon doublet given in Eq. (16)?

REFERENCES

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[3] H. J. Lipkin, *Lie Groups for Pedestrians* (North-Holland, Amsterdam, 1966).