Quantifying the Coherence between Coherent States

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In this Letter, we detail an orthogonalization procedure that allows for the quantification of the amount of coherence present in an arbitrary superposition of coherent states. The present construction is based on the quantum coherence resource theory introduced by Baumgratz, Cramer, and Plenio and the coherence resource monotone that we identify is found to characterize the nonclassicality traditionally analyzed via the Glauber-Sudarshan P distribution. This suggests that identical quantum resources underlie both quantum coherence in the discrete finite dimensional case and the nonclassicality of quantum light. We show that our construction belongs to a family of resource monotones within the framework of a resource theory of linear optics, thus establishing deeper connections between the class of incoherent operations in the finite dimensional regime and linear optical operations in the continuous variable regime.

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Introduction.—The differences between the classical physical theories and quantum theories are known to be useful for a variety of informational tasks [1]. As such, the identification and quantification of nonclassical quantum properties such as quantum entanglement [2], nonlocality [3], and quantum discord [4] remain intense areas of research today [5–9].

A recent development in the quantum resources arena is the resource theory of quantum coherence by Baumgratz, Cramer, and Plenio [10], which draws its primary inspiration from the study of entanglement [11]. The theory necessarily assumes some natural, orthonormal basis set $\{|i\rangle\}$, where states $|i\rangle$ are typically considered to be "classical." Since then, variations of such resource theories have also been explored [12]. Recent applications span a diverse range of topics, such as quantum correlations [13,14], interferometric experiments [15], error correction [16], and quantum estimation [17]. See Ref. [18] for an overview. There are also attempts to quantify the coherence in infinite dimensional systems, mainly focusing on the Fock basis $\{|n\rangle\}$, implicitly assuming that Fock diagonal states are the free classical resource in the infinite dimensional regime [19,20].

This approach is, however, in diametric opposition to the traditional notions of classical light based on the Glauber-Sudarshan P representation of the state of the electromagnetic field. Indeed, Fock states are decidedly nonclassical [21,22]. The most general notion of classical light has already been extensively studied since the 1960s [23–26], and it is well established that the quantum states of light that most closely resemble classical light fields, both in photon statistics and dynamics, are the so-called coherent states [27]. Unfortunately, the set of coherent states is overcomplete; in particular, the coherent states are not mutually orthonormal, and therefore do not permit the direct application of the resource theoretical approach outlined in Ref. [10].

In a newer development, a generalization of the theory of coherence called the resource theory of superposition was also proposed [28]. This approach allows one to quantify the amount of superposition among states with support within the vector space spanned by some finite set of normalized, linearly independent, but not necessarily orthogonal, vectors. However, the set of coherent states not only spans an infinite dimensional Hilbert space, their overcompleteness also implies they are not linearly independent. As such, a direct application of this theory does not bridge the gap between coherence or superposition and nonclassical P representations. This remains true even if one were to consider a countably infinite but linearly independent set of states, such as the von Neumann lattice [29], as there are always coherent states outside of this subset that exist as a superposition.

In this Letter, we propose an approach which demonstrates that the quantum resource identified by Baumgratz, Cramer, and Plenio [10] is essentially the same as the notion of nonclassicality identified by Glauber [27]. This nonclassical resource is also closely related to what we refer to as a resource theory of linear optics. We note that linear optical operations form a strict subset of the most general classicality preserving operations, which were previously considered for a resource theory in the continuous variable regime [30]. For readability, all technical proofs will be deferred to the Supplemental Material [31].

Preliminaries.—We will adopt the axiomatic approach for coherence measures as shown in Ref. [10]. The essential ingredients are as follows.

For a fixed basis $\{|i\rangle\}$, the set of incoherent states $\mathcal{I}$ is the set of quantum states with diagonal density matrices with respect to this basis. Given this, we say that $\mathcal{C}$ is a measure of quantum coherence if it satisfies following properties: (C1) $\mathcal{C}(\rho) \geq 0$ for any quantum state ρ and equality holds if and only if $\rho \in \mathcal{I}$. (C2a) The measure is nonincreasing under incoherent completely positive and trace preserving maps (ICPTP) Φ, i.e., $C(\rho) \geq C[\Phi(\rho)]$. (C2b) Monotonicity for average coherence under selective outcomes of ICPTP: $C(\rho) \geq \sum_n p_n C(\rho_n)$, where $\rho_n = K_n \rho K_n^\dagger / p_n$ and $p_n = \mathrm{Tr}[K_n \rho K_n^\dagger]$ for all K_n with $\sum_n K_n K_n^\dagger = \mathbb{1}$ and $K_n \mathcal{I} K_n^\dagger \subseteq \mathcal{I}$. (C3) Convexity, i.e., $\lambda C(\rho) + (1-\lambda) C(\sigma) \geq C[\lambda\rho + (1-\lambda)\sigma]$, for any density matrix ρ and σ with $0 \leq \lambda \leq 1$.

The set of coherent states will be denoted by $\{|\alpha\rangle\}$. (For an overview, see, for instance, Ref. [34]). It is known that every quantum state of light ρ permits a representation that is diagonal with respect to coherent states, i.e.,

$$\rho = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha|,$$

where the coefficient $P(\alpha)$ is called the Glauber-Sudarshan P distribution [35], or just the P distribution. The P distribution always sums to 1 but may display negativities, in which case it is considered nonclassical.

Finally, we also reference linear optical operations. We specifically take this term to refer to the set of passive unitary optical operations that can be performed using basic building blocks of beam splitters, phase shifters, half and quarter wave plates as described in Ref [36] supplemented with displacement operations, defined by $D(\alpha) \coloneqq e^{(\alpha a^\dagger - \alpha^* a)}$. In contrast, the most general linear canonical transformation includes operations such as squeezing operations, that can give rise to highly nonclassical light. In our context, the defining property of such a linear optical operation is that if the input quantum state is given by pure, classical light, which can be written in the form $|\vec{\alpha}\rangle = |\alpha^1\rangle \ldots |\alpha^k\rangle$ [37], then the output state is also pure and classical, i.e., if U is a unitary linear optical operation, then $U|\vec{\alpha}\rangle = |\vec{\beta}\rangle = |\beta^1\rangle \ldots |\beta^k\rangle$.

Example for pure states.—Consider some orthogonal basis $\{|i\rangle\}$ with $i = 1, \ldots, N$ in some N dimensional Hilbert space and some arbitrary quantum state $|\psi\rangle = \sum_i c_i |i\rangle$. Without any loss in generality, we assume that the coefficients are in decreasing order, so $|c_i| \geq |c_{i+1}|$. Define a CNOT type operation performing the operation $U_i|i\rangle|0\rangle = |i\rangle|i\rangle$.

Suppose we perform a series of such CNOT type operations starting from the basis state with the largest overlap with $|\psi\rangle$, so $U = U_N \ldots U_1$. Applying this unitary, the final result is the state $U|\psi\rangle = \sum_i c_i |i\rangle|i\rangle$. We note that the coherence of $U|\psi\rangle$ in the basis $\{|i\rangle|i\rangle\}$ is the same as the coherence of $|\psi\rangle$ in the basis $\{|i\rangle\}$.

We now define a similar series of CNOT type operations as before, with the exception that the control states are drawn from the nonorthonormal set $\{|\alpha\rangle\}$. The resulting state will have the form $U|\psi\rangle|0\rangle = \sum_i c_i' |\alpha_i'\rangle|\beta_i'\rangle$, where the set of states $\{|\alpha_i'\rangle|\beta_i'\rangle\}$ will be orthonormal so long as $\langle\beta_i|\beta_j\rangle = \delta_{ij}$. We note that this orthogonality condition can always be strictly enforced by an encoding across multiple spatial or polarization modes, but for notational simplicity, we will instead use some set of sufficiently well separated coherent states within a single mode, $\{|\beta_i\rangle\}$, which can be chosen to be arbitrarily close to orthonormal.

Following the previous argument, define $|\psi_i\rangle$ through the recursion relation $|\psi_i\rangle = |\psi_{i-1}\rangle - |\alpha_{i-1}\rangle\langle\alpha_{i-1}|\psi_{i-1}\rangle$, where the coherent state $|\alpha_i\rangle$ satisfies $\langle\alpha_i|\psi_i\rangle = \max_{\alpha'}\langle\alpha'|\psi_i\rangle$ and the initial state $|\psi_1\rangle = |\psi\rangle$ is some given pure quantum state of interest.

Given some finite series of vectors $\{|\alpha_i\rangle\}$ where $i = 1, \ldots, N$, we may consider the CNOT type unitary performing $U_{\alpha_i}|\alpha_i\rangle|0\rangle = |\alpha_i\rangle|\beta_i\rangle$. From this, we construct the unitary map as before: $U_{\mathrm{GS}} = U_{\alpha_N} \ldots U_{\alpha_1}$.

We will call U_{GS} the Gram-Schmidt unitary, since it performs an orthogonalization process. The end result is some orthogonal subspace spanned by $\{|\alpha_i\rangle|\beta_i\rangle\}$ where $i = 1, \ldots, N$. Within this N dimensional subspace, the discrete finite dimensional formulation of coherence will then apply.

Generalization to mixed states.—We now formally define U_{GS}, generalized for mixed states:

Definition 1.—(Gram-Schmidt unitary) For a given density matrix ρ_A, let $\rho_{AB}^{(0)} = \rho_A \otimes |0\rangle_B\langle 0|$. Define $|\alpha^{(i)}\rangle_A$, $i = 1, 2, \ldots$, to be a coherent state achieving the optimal value $\mathrm{Tr}(|\alpha^{(i)}\rangle_A\langle\alpha^{(i)}| \otimes |0\rangle_B\langle 0|\rho_{AB}^{(i-1)}) = \max_\alpha \mathrm{Tr}(|\alpha\rangle_A\langle\alpha| \otimes |0\rangle_B\langle 0|\rho_{AB}^{(i-1)})$, where $\rho_{AB}^{(i)} \coloneqq U_{\alpha^{(i)}}\rho_{AB}^{(i-1)}U_{\alpha^{(i)}}^\dagger$ and $U_{\alpha^{(i)}} \coloneqq \mathbb{1} \otimes \mathbb{1} + |\alpha^{(i)}\rangle_A\langle\alpha^{(i)}| \otimes (|\beta_i\rangle_B\langle 0| + |0\rangle_B\langle\beta_i| - |0\rangle_B\langle 0| - |\beta_i\rangle_B\langle\beta_i|)$ is a CNOT type unitary. We assume that $\{|0\rangle_B, |\beta_i\rangle_B\}$ forms a set of mutually orthonormal vectors.

Let $N \geq 1$ be some integer. Then the following unitary,

$$U_{\mathrm{GS}}^{(N)} = U_{\alpha^{(N)}} \ldots U_{\alpha^{(1)}},$$

is called the Nth Gram-Schmidt unitary. Note that in general, $U_{\mathrm{GS}}^{(N)}$ depends on the state ρ_A.

In the case of degeneracy, where more than one coherent state may achieve $\max_\alpha \mathrm{Tr}(|\alpha\rangle_A\langle\alpha| \otimes |0\rangle_B\langle 0|\rho_{AB}^{(i)})$, the choice of unitaries above is not necessarily unique. To accommodate this, we will also define the set of all possible choices of such unitaries $\mathcal{S}^{(N)}$. We can also generalize to the case of multimode states by considering $|\vec{\alpha}^{(i)}\rangle_{\vec{A}} \coloneqq |\alpha_1^{(i)}\rangle_{A_1} \ldots |\alpha_k^{(i)}\rangle_{A_N}$ in place of $|\alpha^{(i)}\rangle_A$, so that our treatment here can be made as general as possible.

After the orthogonalization process, a pure state will have the form $U|\psi\rangle_A|0\rangle_B = c_0|\epsilon\rangle_A|0\rangle_B + \sum_{i=1}^N c_i|\alpha^{(i)}\rangle_A|\beta_i\rangle_B$, where the set of states $\{|\alpha^{(i)}\rangle_A|\beta_i\rangle_B\}$ will be orthogonal. The vector $|\epsilon\rangle_A|0\rangle_B$ represents the portion of the vector space that is not orthogonalized by the Nth Gram-Schmidt unitary, which we can always remove by projecting onto the subspace spanned by $\{|\alpha^{(i)}\rangle_A|\beta_i\rangle_B\}$. We introduce the following quantity:

Definition 2.—(N coherence) For some discrete finite dimensional coherence measure $\mathcal{C}$, we define the N coherence $\mathcal{C}_\alpha$ for a pure state $|\psi\rangle_A$ to be

$$\mathcal{C}_\alpha(|\psi\rangle_A, N) = \inf_{U_{\rm GS}^{(N)}\in\mathcal{S}^{(N)}} \mathcal{C}[\Phi_{\rm GS}^{(N)}(|\psi\rangle_A\langle\psi|)],$$

where $\Phi_{\rm GS}^{(N)}(\rho_A) = \Pi_{\rm GS}^{(N)}[U_{\rm GS}^{(N)}(\rho_A\otimes|0\rangle_B\langle 0|)U_{GS}^{(N)\dagger}]\Pi_{\rm GS}^{(N)}/\mathrm{Tr}\{\Pi_{\rm GS}^{(N)}[U_{\rm GS}^{(N)}(\rho_A\otimes|0\rangle_B\langle 0|)U_{\rm GS}^{(N)\dagger}]\Pi_{\rm GS}^{(N)}\}$ is called the Nth Gram-Schmidt map. The projector $\Pi_{\rm GS}^{(N)} \coloneqq \sum_{i=1}^N |\alpha^{(i)}\rangle_A\langle\alpha^{(i)}|\otimes|\beta_i\rangle_B\langle\beta_i|$ is the projection onto the N dimensional subspace spanned by $\{|\alpha^{(i)}\rangle_A|\beta_i\rangle_B\}$ where the vectors $\{|\alpha_A^{(i)}\rangle\}$ and $\{|\beta_i\rangle_B\}$ are the same vectors previously defined in Definition 1. More generally, for any mixed quantum state ρ_A, we employ the following definition:

$$\mathcal{C}_\alpha(\rho_A, N) = \inf_{(\rho_{AE}, U_{\rm GS})\in(\mathcal{E},\mathcal{S}^{(N)})} \mathcal{C}[\Phi_{\rm GS}^{(N)}(\rho_{AE})],$$

where $\mathcal{E} \coloneqq \{\rho_{AE}|\mathrm{Tr}_E\rho_{AE} = \rho_A\}$ is the set of extensions of ρ_A. The coherence $\mathcal{C}$ is measured with respect to the set of orthogonal vectors $\{|\vec{\alpha}^{(i)}\rangle_{AE}|\beta_i\rangle_B\}$ specified by $U_{\rm GS}^{(N)}$.

In general, we allow the coherence measure $\mathcal{C}$ to be any finite dimensional coherence measure satisfying the axioms listed in Ref. [10], with only one additional requirement. The coherence measure $\mathcal{C}$ should be asymptotically continuous in the sense that if some state ρ has infinitesimally small coherence, then it is infinitesimally close to some incoherent state σ. That is, if we have some sequence of states ρ^n such that $\lim_{n\to\infty}\mathcal{C}(\rho^n) = 0$ then for every $\epsilon > 0$, there is some $n_{\max}$ such that for every $n > n_{\max}$, there exists some incoherent state σ^n such that $\frac{1}{2}\|\rho^n - \sigma^n\|_{tr} < \epsilon$. This is satisfied, for instance, by both coherence measures introduced in Ref. [10]. This is because both the l_1 norm [38] and the relative entropy [39] are lower bounded by the trace norm.

Next, we define the ϵ smoothed version of the above quantity so as to consider states in the immediate vicinity of the state of interest.

Definition 3.—(ϵ-smoothed N coherence) The ϵ-smoothed N coherence for some $\epsilon > 0$ is the quantity

$$\mathcal{C}_\alpha(\rho_A, N, \epsilon) \coloneqq \inf_{\rho'_A\in\mathcal{B}(\rho_A,\epsilon)} \mathcal{C}_\alpha(\rho'_A, N),$$

where $\mathcal{B}(\rho_A,\epsilon) = \{\rho'_A|\frac{1}{2}\|\rho'_A - \rho_A\|_{tr}\le\epsilon\}$ is the ϵ ball centred at ρ_A with respect to the trace norm.

Finally, the main figure of merit that we consider is the following:

Definition 4.—(α coherence) The α coherence is the limiting value of the ϵ smoothed N coherence:

$$\mathcal{C}_\alpha(\rho_A) \coloneqq \lim_{\epsilon\to 0}\lim_{N\to\infty}\mathcal{C}_\alpha(\rho_A, N, \epsilon).$$

In Definition 4, we have combined the finite dimensional formulation of coherence with that of nonclassical systems of light. The α coherence may therefore be interpreted as the limiting case of the coherence identified by Baumgratz, Cramer, and Plenio [10], optimized over state extensions and all degenerate cases, if any. Coherence effects are typically signs of nonclassicality if an appropriate basis is chosen but it remains to be shown what kind of nonclassicality the above quantity measures.

Main results.—It can be shown that, for a given state ρ_A, a vanishing value of the α coherence is equivalent to the existence of a Glauber-Sudarshan P distribution (referred to hereafter simply as the P distribution) for ρ_A which is a probability density on the complex plane. A nonzero value of the α coherence is, therefore, an indicator of nonclassicality.

Theorem 1: The α coherence $\mathcal{C}_\alpha(\rho_A) = 0$ iff ρ_A is a classical state.

In quantum optics, nonclassicality is usually manifest in the measurement statistics of moments of the quadrature or number operators. Specifically, a classical P distribution constrains these correlation functions to satisfy linear or nonlinear inequalities, depending on the nonclassical features of interest [22,40]. Theorem 1 extends the general operational content of the fact that a quantum state associated with a P distribution that is a *bona fide* probability distribution fails to exhibit nonclassical characteristics.

We now consider a possible resource theory where the "free" operations are linear optical operations including displacement operations. We also allow additional free resources in the form of classical ancillas, where classicality means classical P distributions.

Definition 5.—(Linear optical maps) A quantum map Φ_L is called a linear optical map or operation if

$$\Phi_L(\rho_A) = \mathrm{Tr}_E(U_L\rho_A\otimes\sigma_E U_L^\dagger),$$

where U_L is some linear optical unitary operation. σ_E is some classical, possibly multimode ancillary system.

A set of Kraus operators $\{K_i\}$ satisfying $\sum_i K_i^\dagger K_i = \mathbb{1}$ with corresponding POVM elements $K_i^\dagger K_i$ representing classical measurement outcomes i is called a linear optical measurement if there exists linear optical unitary U_L and classical ancilla $\sigma_{EE'}$ and some set of orthogonal vectors $\{|\alpha'_i\rangle_{E'}\}$ such that

$$\mathrm{Tr}_E(U_L\rho_A\otimes\sigma_{EE'}U_L^\dagger) = \sum_i p_i\rho_A^i\otimes|\alpha'_i\rangle_{E'}\langle\alpha'_i|$$

for some density matrices ρ_A^i, where $p_i\rho_A^i \coloneqq K_i\rho_A K_i^\dagger$ and $p_i \coloneqq \mathrm{Tr}(K_i\rho_A K_i^\dagger)$.

The following defines a possible resource theory of linear optics.

Definition 6.—We call $\mathcal{Q}$ a nonclassicality measure if the following conditions are satisfied: (1) $\mathcal{Q}(\rho) = 0$ if and only if ρ is classical. (2) (a) (Weak monotonicity) $\mathcal{Q}$ is monotonically decreasing under linear optical operations Φ_L, i.e., $\mathcal{Q}(\rho_A)\ge\mathcal{Q}[\Phi_L(\rho_A)]$. (b) (Strong monotonicity) Let $\{K_i\}$ be a set of Kraus operators corresponding to a linear optical measurement with outcomes i. Then $\mathcal{Q}$ is nonincreasing when averaged over measurement outcomes i, i.e., $\mathcal{Q}(\rho)\ge\sum_i p_i\mathcal{Q}(\rho_i)$, where $p_i \coloneqq \mathrm{Tr}(K_i^\dagger K_i\rho)$ and $\rho_i \coloneqq (1/p_i)K_i\rho K_i^\dagger$.(3) $\mathcal{Q}$ is convex, i.e., $\mathcal{Q}(\sum_i p_i\rho_i)\le\sum_i p_i\mathcal{Q}(\rho_i)$.

Based on the above definition, it can be demonstrated that the α coherence is a linear optical monotone.

Theorem 2: The α coherence is a nonclassicality measure.

Examples.—Here, we present some numerical plots of the α coherence for some important classes of pure states. For pure states in particular, the optimization is much simpler as the only possible extensions are trivial, thus sidestepping part of the optimization involved in Definition 4. We will employ the relative entropy of coherence [10] as our coherence measure.

In Fig 1 we compare the α coherence for the even and odd cat states $|\alpha\rangle \pm |-\alpha\rangle$, Fock states $|n\rangle$, and squeezed states $S(\xi)|0\rangle$ with a real squeezing parameter ξ. For both Fock states and squeezed states, the α coherence monotonically increases with n and ξ, indicating strong nonclassicality. For odd cat states, we see strong nonclassicality around $\alpha \approx 0$. This is because in the limit $\alpha \to 0^+$, the odd cat approaches the single photon state, an archetypical example of nonclassical light. In contrast, for even cat states, as $\alpha \to 0^+$, the state approaches the vacuum, so the α coherence vanishes. It is interesting to note that nonclassicality peaks most strongly in the region $\alpha = 1$. We interpret this as a signature of the infinite dimensional nature of the underlying Hilbert space, as the state tends towards a superposition of two orthogonal states as $\alpha \to \infty$. We also note that when $\alpha \to \infty$, the α coherence asymptotically approaches a constant value, in contrast to a measure of quantum macroscopicity [41] which increases with the separation α.

Figure 2 is a numerical plot of the nonclassicality for a given mean particle number. We see that the Fock states are the most nonclassical states on a per particle basis over the region considered, which is again not unexpected due to the granular nature of this form of light.

Other possible measures.—Consider now a nonclassicality measure based on the negative volume of the P distribution. In the most general case, negativities in the P distribution can come in the form of regular continuous

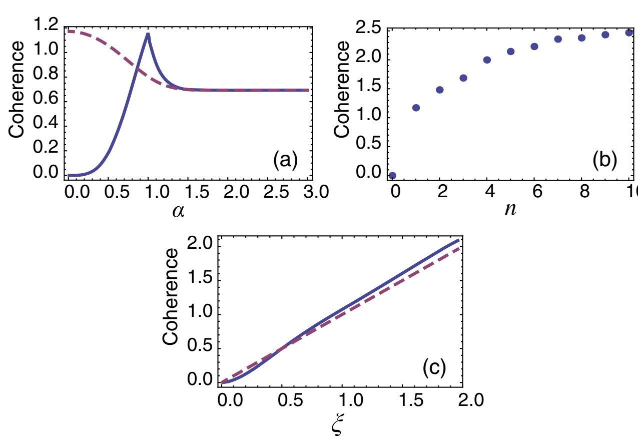

FIG. 1. α coherence for photonic states. (a) Even (solid line) and odd (dotted line) cat states $|\alpha\rangle \pm |-\alpha\rangle$, (b) Fock states $|n\rangle$, and (c) squeezed states $S(\xi)|0\rangle$ are compared.

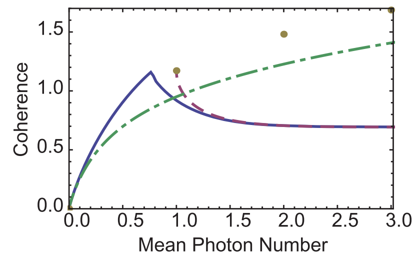

FIG. 2. α coherence plotted against mean photon numbers $\langle n\rangle = \langle a^\dagger a\rangle$. Even (solid line) and odd (dotted line) cat states $|\alpha\rangle \pm |-\alpha\rangle$, Fock states (circular points) $|n\rangle$, and squeezed states (dashed-dotted line) $S(\xi)|0\rangle$ are compared.

functions, which are directly accessible, or singularities. Suppose we restrict ourselves to the case where the P distribution is a regular continuous function. We define the following:

Definition 7.—(Negativity) Let the P distribution of the state ρ be given by $p(\alpha)$, where $p(\alpha)$ is a regular function. Let $\mathcal{N} = \{\alpha | p(\alpha) \ngeq 0\}$, then the quantity

$$\mathcal{C}_-(\rho) = -\int_{\mathcal{N}} d^2\alpha p(\alpha)$$

is called the negativity of the P distribution.

The following result shows that both the α coherence and the negativity of the P distribution belong to similar resource theories, which further supports the argument that the α coherence is closely related to negativities in the P distribution.

Theorem 3: Suppose for some ρ, $\mathcal{C}_-(\rho)$ is finite integrable. Then $\mathcal{C}_-$ is a nonclassicality measure with respect to the set of states with positive P distributions.

Still other possible measures of nonclassicality measures can also be constructed. For instance, we can also consider geometric measures of nonclassicality. Suppose we have some distance measure $D(\rho, \sigma)$ over the Hilbert space that is monotonically decreasing under quantum operations over both its arguments. Then it is immediately clear that the quantity $\inf_{\sigma\in\mathcal{P}^+} D(\rho, \sigma)$, where the optimization is over the set $\mathcal{P}_+$ of all classical states, will satisfy at least the weak monotonicity condition laid out in Definition 6.

Conclusion.—We described a general procedure that allows us to quantify the superposition amongst any complete set of quantum states, whether they are orthogonal or not. The key insight here is that the scheme laid out by Baumgratz, Cramer, and Plenio [10] can be generalized via a reasonably motivated orthogonalization procedure. This orthogonalization procedure is then applied to the set of coherent states as a special case and the resulting coherence measure, the α coherence, is shown to identify incoherent states with nonclassical states in the sense of the Glauber-Sudarshan P distribution. This demonstrates that states with nonclassical P distributions are essentially the limiting case of the same quantum resources identified in

Ref. [10], when the incoherent basis is chosen as the set of coherent states.

The α coherence also belongs to a class of resource theoretic nonclassicality measures that we refer to as a linear optical resource theory. This strongly implies that linear optical monotones are appropriate measures of the nonclassicality of light. The results also suggest deeper connections between incoherent operations and linear optical elements. For instance, it is known that a parallel feature of nonclassical light and coherence is that both quantum resources may be faithfully converted to entanglement [14,42–45], where the relevant operation is the beam splitter for the optical case, and a CNOT gate for coherence. Our results make it clear why this is so. Nonclassical light may in fact be interpreted as a form of coherence, where linear optical operations (beam splitter) are the continuous variable counterparts of the finite dimensional incoherent operations (CNOT). We hope that this relationship will open up potentially interesting new lines of investigation.

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