Unified view of quantum correlations and quantum coherence

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In this paper, we argue that quantum coherence in a bipartite system can be contained either locally or in the correlations between the subsystems. The portion of quantum coherence contained within correlations can be viewed as a kind of quantum correlation which we call correlated coherence. We demonstrate that the framework provided by correlated coherence allows us to retrieve the same concepts of quantum correlations as defined by the asymmetric and symmetric versions of quantum discord as well as quantum entanglement, providing a unified view of these correlations. We also prove that correlated coherence can be formulated as an entanglement monotone, thus demonstrating that entanglement may be viewed as a specialized form of coherence.

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I. INTRODUCTION

A fascinating property of quantum mechanics is that it admits superpositions between different physical states [1]. A superposition of quantum states is typically described by pure states, which are completely different in nature to a classical stochastic mixture of states, otherwise called mixed states.

Following the birth of quantum theory, physical demonstrations of quantum coherence arising from superpositions of many different quantum systems such as electrons, photons, atoms, mechanical modes, and hybrid systems have been achieved [2–4].

Recent developments in our understanding of quantum coherence have come from the burgeoning field of quantum information science. One important pillar of the field is the study of quantum correlations. It turns out that in a multipartite setting, quantum mechanical effects allow remote laboratories to collaborate and perform tasks that are otherwise impossible using classical physics [5]. Historically, the most well-studied quantum correlation is perhaps quantum entanglement [6–8]. Subsequent developments of the idea led to the formulation of quantum discord [9,10], and its symmetrized version [11–13] as more generalized forms of quantum correlations. The development of such ideas of the quantumness of correlations has led to a plethora of quantum protocols such as quantum cryptography [14], quantum teleportation [15], quantum superdense coding [16], quantum random access codes [17], remote state preparation [18], random number generation [19], and quantum computing [20,21], among others. Quantum correlations have also proven useful in the study of macroscopic quantum objects [22].

Meanwhile, quantitative theories of entanglement [23,24] have been formulated by characterizing and quantifying entanglement as a resource to achieve certain tasks that are otherwise impossible classically. Building upon this, Baumgratz *et al.* [25] recently proposed a resource theory of quantum coherence. Recent developments have since uncovered interesting connections between quantum coherence and correlation, such as their interconversion with each other [26,27] as well as trade-off relations [28].

In this paper, we demonstrate that quantum correlations can be understood in terms of the coherence contained entirely between subsystems. In contrast to previous studies which established more indirect relationships between quantum correlation and coherence [26–28], our study establishes a direct connection between the two and provides a unified view of quantum correlations which includes quantum discord and entanglement using the framework of quantum coherence.

II. PRELIMINARIES

We will frequently refer to a bipartite quantum state which we denote ρ_{AB}, where A and B refer to local subsystems held by different laboratories. Following convention, we say the subsystems A and B are held by Alice and Bob, respectively. The local state of Alice is obtained by performing a partial trace on ρ_{AB}, and is denoted by $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, and $\{|i\rangle_A\}$ is a complete local basis of Alice's system. Bob's local state and local basis are also similarly defined. In general, the systems Alice and Bob hold may be composite, such that $A = A_1A_2\cdots A_N$ and $B = B_1B_2\cdots B_M$, so the total state may identically be denoted by $\rho_{A_1A_2\cdots A_NB_1B_2\cdots B_M}$.

We will adopt the axiomatic approach for coherence measures as shown in Ref. [25]. For a fixed basis set $\{|i\rangle\}$, the set of incoherent states $\mathcal{I}$ is the set of quantum states with diagonal density matrices with respect to this basis. Then a reasonable measure of quantum coherence $\mathcal{C}$ should satisfy the following properties: (C1) $C(\rho) \geqslant 0$ for any quantum state ρ and equality holds if and only if $\rho \in \mathcal{I}$. (C2a) The measure is nonincreasing under incoherent completely positive and trace-preserving maps (ICPTP) Φ, i.e., $C(\rho) \geqslant C(\Phi(\rho))$. (C2b) Monotonicity for average coherence under selective outcomes of ICPTP: $C(\rho) \geqslant \sum_n p_n C(\rho_n)$, where $\rho_n = \hat{K}_n \rho \hat{K}_n^\dagger / p_n$ and $p_n = \mathrm{Tr}[\hat{K}_n \rho \hat{K}_n^\dagger]$ for all $\hat{K}_n$ with $\sum_n \hat{K}_n^\dagger \hat{K}_n = \mathbb{1}$ and $\hat{K}_n \mathcal{I} \hat{K}_n^\dagger \subseteq \mathcal{I}$. (C3) Convexity, i.e., $\lambda C(\rho) + (1-\lambda)C(\sigma) \geqslant C(\lambda\rho + (1-\lambda)\sigma)$, for any density matrix ρ and σ with $0 \leqslant \lambda \leqslant 1$. Here, we will employ the l_1-norm of coherence, which is defined by $\mathcal{C}(\rho) := \sum_{i \neq j} |\langle i|\rho|j\rangle|$, for any given basis set $\{|i\rangle\}$ (otherwise called the reference basis). It can be shown that this definition satisfies all the properties mentioned [25].

In addition, we will also reference local operations and classical communication (LOCC) protocols in the context of

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the resource theory of entanglement. LOCC protocols allow for two different types of operation. First, Alice and Bob are allowed to perform quantum operations, but only locally on their respective subsystems. Second, they are also allowed classical, but otherwise unrestricted, communication between them. LOCC operations are especially important in the characterization of quantum entanglement, which typically does not increase under such operations. Measures of entanglement that do not increase under LOCC-type operations are referred to as LOCC monotones [29].

III. MAXIMAL COHERENCE LOSS

Before establishing the connection between quantum correlation and coherence, we first consider the measurement that leads to the maximal coherence lost in the system of interest. For a monopartite system, the solution to this is trivial. For any quantum state $\rho = \sum_{i,j} \rho_{i,j} |i\rangle\langle j|$ with a reference basis $\{|i\rangle\}$, it is clear that the measurement that maximally removes coherence from the system is the projective measurement $\Pi(\rho) = \sum_i |i\rangle\langle i|\rho|i\rangle\langle i|$. This measurement leaves behind only the diagonal terms of ρ, so $\mathcal{C}(\Pi(\rho)) = 0$, which is the minimum coherence any state can have.

A less obvious result for a bipartite state is the following statement (proof in Appendix A).

Proposition 1. For any N-partite state $\rho_{A_1A_2\ldots A_N}$ where the coherence is measured with respect to the local reference bases $\{|i\rangle_{A_k}\}$ and $k = 1,2,\ldots,N$, the projective measurement on subsystem A_k that induces maximal coherence loss is the projective measurement onto the local basis $\{|i\rangle_{A_k}\}$.

As we shall see in the subsequent sections, the maximal coherence loss will play a role in defining the set of quantum correlations from the point of view of quantum discord and quantum entanglement.

IV. LOCAL AND CORRELATED COHERENCE

Now consider a bipartite state ρ_{AB}, with total coherence $\mathcal{C}(\rho_{AB})$ with respect to local reference bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$. Then $\mathcal{C}(\rho_A)$ and $\mathcal{C}(\rho_B)$ can be interpreted as the coherence that is local to A and B, respectively. In general, the sum of the total local coherences is not necessarily the same as the total coherence in the system. It is therefore reasonable to suppose that a portion of the quantum coherences are not stored locally, but within the correlations of the system itself. This motivates the following definition.

Definition 1 (correlated coherence). With respect to local reference bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$, the correlated coherence for a bipartite quantum system is the local coherences subtracted from the total coherence,

$$\mathcal{C}_{cc}(\rho_{AB}) := \mathcal{C}(\rho_{AB}) - \mathcal{C}(\rho_A) - \mathcal{C}(\rho_B),$$

where ρ_A and ρ_B are the reduced density matrices of A and B, respectively.

Further reinforcing the idea that the local coherences form only a portion of the total coherence present in a quantum system, we have the following property (proof in Appendix B).

Theorem 1. For any bipartite quantum state ρ_{AB}, $\mathcal{C}_{cc}(\rho_{AB}) \geqslant 0$ (i.e., correlated coherence is always nonnegative).

V. CORRELATED COHERENCE AND QUANTUM DISCORD

Of particular interest to the study of quantum correlations is the idea that certain correlations are quantum and certain correlations are classical. Here, we will demonstrate that correlated coherence is able to unify many of these concepts of quantumness under the same framework.

First, note that in our definition of correlated coherence, the choice of reference bases is not unique, while most definitions of quantum correlations are independent of specific basis choices. However, we can retrieve basis independence via a very natural choice of local bases. For every bipartite state ρ_{AB}, the reduced density matrices ρ_A and ρ_B have eigenbases $\{|\alpha_i\rangle\}$ and $\{|\beta_i\rangle\}$, respectively. By choosing these local bases, ρ_A and ρ_B are both diagonal so the local coherences are zero. The implication of this is that for such a choice, *the coherence in the system is stored entirely within the correlations*. Since this can be done for any ρ_{AB}, correlated coherence with respect to these bases becomes a state-dependent property. In the case where there may be degenerate eigenbases available, we will choose the eigenbasis that minimizes correlated coherence so the quantity remains basis independent. For the rest of this paper, unless otherwise stated, we will assume that the choice of local bases for the calculation of correlated coherence will always be the local eigenbases of Alice and Bob.

We first consider the definition of a quantum correlation in the symmetrized version of quantum discord. Under the framework of symmetric discord, a state contains quantum correlations when it cannot be expressed in the form $\rho_{AB} = \sum_{i,j} p_{i,j} |i\rangle_A\langle i| \otimes |j\rangle_B\langle j|$, where $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ are sets of orthonormal vectors. Any such state has zero symmetric discord by definition.

We prove the following theorem.

Theorem 2 (correlated coherence and symmetric quantum discord). For a given state ρ_{AB}, $\mathcal{C}_{cc}(\rho_{AB}) = 0$ if and only if $\rho_{AB} = \sum_{i,j} p_{i,j} |i\rangle_A\langle i| \otimes |j\rangle_B\langle j|$.

Proof. If $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ are the the eigenbases of ρ_A and ρ_B, then $\mathcal{C}_{cc}(\rho_{AB}) = 0$ implies $\mathcal{C}(\rho_{AB}) = 0$, which implies ρ_{AB} only has diagonal terms, so $\rho_{AB} = \sum_{i,j} p_{i,j} |i\rangle_A\langle i| \otimes |j\rangle_B\langle j|$. Therefore, $\mathcal{C}_{cc}(\rho_{AB}) = 0 \Rightarrow \rho_{AB} = \sum_{i,j} p_{i,j} |i\rangle_A\langle i| \otimes |j\rangle_B\langle j|$.

Conversely, if $\rho_{AB} = \sum_{i,j} p_{i,j} |i\rangle_A\langle i| \otimes |j\rangle_B\langle j|$, then the state clearly has zero coherence, which implies $\mathcal{C}_{cc}(\rho_{AB}) = 0$, so the converse is also true. This proves the theorem. ■

This establishes a relationship between correlated coherence and symmetric discord. We now consider the asymmetric version of quantum discord. Under this framework, a state contains quantum correlations when it cannot be expressed in the form $\rho_{AB} = \sum_i p_i |i\rangle_A\langle i| \otimes \rho_B^i$, where ρ_B^j is some normalized density matrix and $\{|i\rangle_A\}$ is some set of orthonormal vectors.

We prove the following.

Theorem 3 (correlated coherence and asymmetric quantum discord). For a given state ρ_{AB}, let $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ be the the eigenbases of ρ_A and ρ_B, respectively. Define the measurement on A onto the local basis as $\Pi_A(\rho_{AB}) := \sum_i (|i\rangle_A\langle i| \otimes \mathbb{1}_B)\rho_{AB}(|i\rangle_A\langle i| \otimes \mathbb{1}_B)$. Then, with respect to these local bases, $\mathcal{C}_{cc}(\rho_{AB}) - \mathcal{C}_{cc}(\Pi_A(\rho_{AB})) = 0$ if and only if $\rho_{AB} = \sum_i p_i |i\rangle_A\langle i| \otimes \rho_B^i$, where ρ_B^i is some normalized density matrix and $\{|i\rangle_A\}$ is some set of orthonormal vectors.

Proof. First, we write the state in the form $\rho_{AB} = \sum_{i,j,k,l} \rho_{ijkl}|i,j\rangle_{AB}\langle k,l|$. We can always write the state in block matrix form such that $\rho_{AB} = \sum_{i,k} |i\rangle_A\langle k| \otimes \rho_B^{i,k}$, where $\rho_B^{i,k} := \sum_{j,l} \rho_{ijkl}|j\rangle_B\langle l|$. If $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ are the the eigenbases of ρ_A and ρ_B, then $\mathcal{C}_{cc}(\rho_{AB}) - \mathcal{C}_{cc}(\Pi_A(\rho_{AB})) = 0$ implies that when $i \neq k$, $\rho_B^{i,k} = 0$. This implies $\rho_{AB} = \sum_i |i\rangle_A\langle i| \otimes \rho_B^{i,i}$. By defining $\rho_B^i = \rho_B^{i,i}/p_i$ where $p_i := \mathrm{Tr}\,\rho_B^{i,i}$, we get $\rho_{AB} = \sum_i p_i|i\rangle_A\langle i| \otimes \rho_B^i$. Therefore, $\mathcal{C}_{cc}(\rho_{AB}) - \mathcal{C}_{cc}(\Pi_A(\rho_{AB})) = 0 \Rightarrow \rho_{AB} = \sum_i p_i|i\rangle_A\langle i| \otimes \rho_B^i$.

For the converse, if $\rho_{AB} = \sum_i p_i|i\rangle_A\langle i| \otimes \rho_B^i$, then clearly, $\Pi_A(\rho_{AB}) = \rho_{AB}$, so $\mathcal{C}_{cc}(\rho_{AB}) - \mathcal{C}_{cc}(\Pi_A(\rho_{AB})) = 0$. ■

Note that the above relationship with asymmetric quantum discord is expressed as a *difference* between the correlated coherence of ρ_{AB} and the postmeasurement state $\Pi_A(\rho_{AB})$. While this characterization of quantum correlations may at first appear to diverge from the one given in Theorem 2, they are actually similar since $\mathcal{C}_{cc}(\Pi_A\Pi_B(\rho_{AB})) = 0$ so $\mathcal{C}_{cc}(\rho_{AB}) = \mathcal{C}_{cc}(\rho_{AB}) - \mathcal{C}_{cc}(\Pi_A\Pi_B(\rho_{AB}))$. It is therefore possible to interpret quantum discord as the correlated coherence loss when either party performs a maximally coherence destroying measurement only their subsystems (see Appendix A). When the projective measurement is performed only on one side, one retrieves the asymmetric version of quantum discord, and the symmetrized version is obtained when the coherence destroying measurement is performed by both parties. An immediate implication of Theorems 2 and 3 can be found in, for instance, the no-broadcasting theorem (see Refs. [30–32]), where it is known that zero discord (and hence zero correlated coherence) is necessary and sufficient for quantum broadcasting. From our results, we now gain an additional interpretation that local broadcasting is impossible because of the presence of coherence that cannot be locally destroyed through measurements.

Furthermore, it is interesting to note the formal similarity of correlated coherence in Definition 1 with the definition of quantum mutual information, which is given by $S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, where $S(\cdot)$ is the von Neumann entropy of a quantum state. Indeed, in Ref. [13], it is demonstrated that the total correlation, as measured by mutual information, minus the locally observable correlations, defines quantum discord. Despite starting from very different initial assumptions, both approaches (correlated coherence and mutual information) are able to identify similar sets of classical quantum states.

This similarity notwithstanding, correlated coherence and mutual information are nonetheless very different measures. In Ref. [33], it was shown that the computation of quantum discord in the sense of Refs. [9,13] is a NP-complete problem, so direct computation of quantum discord even for moderately sized systems is impossible. In comparison, the computation of correlated coherence requires only the eigenbasis for the reduced systems, which is likely much more efficiently computable. We therefore propose correlated coherence as a more natural candidate for the study of quantum correlations in the macroscopic quantum regime.

VI. CORRELATED COHERENCE AND ENTANGLEMENT

Under the framework of entangled correlations, a state contains quantum correlations when it cannot be expressed as a convex combination of product states $\sum_i p_i|\alpha_i\rangle_A\langle\alpha_i| \otimes |\beta_i\rangle_B\langle\beta_i|$, where $|\alpha_i\rangle$ and $|\beta_i\rangle$ are normalized but not necessarily orthogonal vectors that can repeat. It is also possible to extend our methodology to entangled quantum states. In order to do this, we consider extensions of the quantum state ρ_{AB}, which were also considered in Ref. [34]. We say that a state ρ_{ABC} is an extension of ρ_{AB} if $\mathrm{Tr}_C(\rho_{ABC}) = \rho_{AB}$. For our purpose, we will consider extensions of the form $\rho_{AA'BB'}$.

Theorem 4. Let $\rho_{AA'BB'}$ be some extension of a bipartite state ρ_{AB} and choose the local bases to be the eigenbases of $\rho_{AA'}$ and $\rho_{BB'}$, respectively. Then with respect to these local bases, $\min \mathcal{C}_{cc}(\rho_{AA'BB'}) = 0$ if and only if $\rho_{AB} = \sum_i p_i|\alpha_i\rangle_A\langle\alpha_i| \otimes |\beta_i\rangle_B\langle\beta_i|$ for some set of normalized vectors $|\alpha_i\rangle$ and $|\beta_i\rangle$ that are not necessarily orthogonal and may repeat. The minimization is over all possible extensions of ρ_{AB} of the form $\rho_{AA'BB'}$.

Proof. If $\min \mathcal{C}_{cc}(\rho_{AA'BB'}) = 0$, then $\rho_{AA'BB'}$ must have the form $\sum_{i,j} p_{i,j}|\mu_i\rangle_{AA'}\langle\mu_i| \otimes |\nu_j\rangle_{BB'}\langle\nu_j|$ (see Theorem 2). Since $\rho_{AA'BB'}$ is an extension, $\mathrm{Tr}_{A'}\mathrm{Tr}_{B'}(\rho_{AA'BB'}) = \sum_{i,j} p_{i,j}\mathrm{Tr}_{A'}(|\mu_i\rangle_{AA'}\langle\mu_i|) \otimes \mathrm{Tr}_{B'}(|\nu_j\rangle_{BB'}\langle\nu_j|) = \rho_{AB}$. Let $\rho_A^i := \mathrm{Tr}_{A'}(|\mu_i\rangle_{AA'}\langle\mu_i|)$ and $\rho_B^j := \mathrm{Tr}_{B'}(|\nu_j\rangle_{BB'}\langle\nu_j|)$. Then, $\rho_{AB} = \sum_{i,j} p_{i,j}\rho_A^i \otimes \rho_B^j$. This is equivalent to saying $\rho_{AB} = \sum_i p_i|\alpha_i\rangle_A\langle\alpha_i| \otimes |\beta_i\rangle_B\langle\beta_i|$, for some set of (non-necessarily orthogonal) vectors $\{|\alpha_i\rangle\}$ and $\{|\beta_i\rangle\}$. This proves $\min \mathcal{C}_{cc}(\rho_{AA'BB'}) = 0 \Rightarrow \rho_{AB} = \sum_i p_i|\alpha_i\rangle_A\langle\alpha_i| \otimes |\beta_i\rangle_B\langle\beta_i|$.

For the converse, suppose $\rho_{AB} = \sum_i p_i|\alpha_i\rangle_A\langle\alpha_i| \otimes |\beta_i\rangle_B\langle\beta_i|$, and consider the purification of ρ_{AB} of the form $|\psi\rangle_{ABA'B'C'} = \sum_i \sqrt{p_i}|\alpha_i\rangle_A|\beta_i\rangle_B|i\rangle_{A'}|i\rangle_{B'}|i\rangle_{C'}$. $\{|i\rangle_{A'}\}$ is a set of orthonormal vectors, similarly for B' and C'. Since this is a purification, $\rho_{AA'BB'} = \mathrm{Tr}_{C'}(|\psi\rangle_{ABA'B'C'}\langle\psi|)$ is clearly an extension of ρ_{AB}. Furthermore, the eigenbases of $\rho_{AA'}$ and $\rho_{BB'}$ are $\{|\alpha_i\rangle_A|i\rangle_{A'}\}$ and $\{|\beta_i\rangle_B|i\rangle_{B'}\}$, respectively. Since $\rho_{AA'BB'} = \sum_i p_i|\alpha_i\rangle_A|i\rangle_{A'}\langle\alpha_i|_A\langle i|_{A'} \otimes |\beta_i\rangle_B|i\rangle_{B'}\langle\beta_i|_B\langle i|_{B'}$, $\mathcal{C}_{cc}(\rho_{AA'BB'}) = 0$ with respect to the eigenbases of $\rho_{AA'}$ and $\rho_{BB'}$. Therefore, $\min \mathcal{C}_{cc}(\rho_{AA'BB'}) = 0$, which completes the proof. ■

Theorem 4 is sufficient to characterize the set of separable (and hence also entangled) quantum states through correlated coherence. In the following, we will demonstrate that the relationship between entanglement and coherence can be pushed further still. We now construct an entanglement monotone using the correlated coherence of a quantum state. In order to do this, we first define the symmetric extensions of a given quantum state.

Definition 2 (unitarily symmetric extensions). Let $\rho_{AA'BB'}$ be an extension of a bipartite state ρ_{AB}. The extension $\rho_{AA'BB'}$ is said to be unitarily symmetric if it remains invariant up to local unitary operations on AA' and BB' under a system swap between Alice and Bob.

More formally, let $\{|i\rangle_{AA'}\}$ and $\{|j\rangle_{BB'}\}$ be complete local bases on AA' and BB', respectively. Define the SWAP operator $U_{\text{SWAP}}|i,j\rangle_{AA'BB'} := |j,i\rangle_{AA'BB'}$. Then $\rho_{AA'BB'}$ is unitarily symmetric if there exists local unitary operations $U_{AA'}$ and $U_{BB'}$ such that $U_{AA'} \otimes U_{BB'}(U_{\text{SWAP}}\rho_{AA'BB'}U_{\text{SWAP}}^\dagger)U_{AA'}^\dagger \otimes U_{BB'}^\dagger = \rho_{AA'BB'}$.

Following from the observation that the minimization of coherence over all extensions is closely related to the the separability of a quantum state, we define the following.

Definition 3. Let $\rho_{AA'BB'}$ be some unitarily symmetric extension of a bipartite state ρ_{AB} and choose the local bases to be the eigenbases of $\rho_{AA'}$ and $\rho_{BB'}$, respectively. Then the entanglement of coherence is defined to be

$$E_{cc}(\rho_{AB}) := \min \mathcal{C}_{cc}(\rho_{AA'BB'}).$$

The minimization is over all possible unitarily symmetric extensions of ρ_{AB} of the form $\rho_{AA'BB'}$.

It can be verified that E_{cc} satisfies the following elementary properties (proofs in Appendix C):

1. A bipartite quantum state ρ_{AB} is separable if and only if $E_{cc}(\rho_{AB}) = 0$.
2. For a bipartite quantum state ρ_{AB}, $E_{cc}(\rho_{AB})$ is invariant under local unitary operations on A and B.
3. $E_{cc}(\rho_{AB})$ is convex and nonincreasing under mixing, i.e., $\lambda E_{cc}(\rho_{AB}) + (1-\lambda)E_{cc}(\sigma_{AB}) \geqslant E_{cc}(\lambda\rho_{AB} + (1-\lambda)\sigma_{AB})$ for any two bipartite quantum states ρ_{AB} and σ_{AB}, and $\lambda \in [0,1]$.
4. Consider the bipartite state ρ_{AB} where $A = A_1A_2$ is a composite system. Then the entanglement of coherence is nonincreasing under a partial trace, i.e., $E_{cc}(\rho_{A_1A_2B}) \geqslant E_{cc}(\mathrm{Tr}_{A_1}(\rho_{A_1A_2B}))$.
5. $E_{cc}(\rho_{AB})$ is nonincreasing under local projective operations by Alice and Bob..
6. $E_{cc}(\rho_{AB})$ is invariant under classical communication between Alice and Bob.
7. $E_{cc}(\rho_{AB})$ is nonincreasing under LOCC-type operations.

The following theorem says that $E_{cc}(\rho_{AB})$ is a reasonable measure of entanglement (i.e., it is an entanglement monotone).

Theorem 5 (entanglement monotone). The entanglement of coherence E_{cc} is an entanglement monotone in the sense that it satisfies:

i. $E_{cc}(\rho_{AB}) = 0$ if and only if $E_{cc}(\rho_{AB})$ is separable.

ii. $E_{cc}(\rho_{AB})$ is invariant under local unitaries on A and B.

iii. $E_{cc}(\rho_{AB}) \geqslant E_{cc}(\Lambda_{\mathrm{LOCC}}(\rho_{AB}))$ for any LOCC procedure Λ_{LOCC}.

Proof. It follows directly from properties 1, 2, and 7 above.■

VII. CONCLUSION

To conclude, we defined a quantity we call correlated coherence of quantum states which can be interpreted as the portion of the total coherence that is shared between two subsystems. The framework of the correlated coherence identifies the same nonclassical correlations as those of (both symmetric and asymmetric) quantum discord and quantum entanglement. We also provide a direct proof that entanglement can be viewed as a type of coherence by constructing an entanglement monotone through correlated coherence.

We stress that correlated coherence is not the definition of a new type of quantum correlation. Instead, it operates entirely within the existing framework of coherence where the coherence is in this case measured using some "proper" reference frame. The results presented demonstrate that through the proper choice of reference frames, we are able to retrieve other well-understood quantum correlated sets, such quantum discord and quantum entanglement, thus strongly suggesting that the quantum properties of correlations originate from the quantum properties of coherence. It also suggest that multiple, separate approaches may not be necessary in the study of quantum correlated sets, and that the theory of coherence is sufficiently rich enough to encompass the study of both single systems as well as correlated, multipartite systems. The relationship between coherence and other nonclassical measures of correlations, such as nonlocality, is an area that warrants further investigation in the future.

The implications of this are of fundamental significance to our current understanding of quantum correlations. This connection may eventually allow for the development of a set of common tools in the treatment of various forms of quantum correlations and quantum coherence.

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APPENDIX A: MAXIMAL COHERENCE LOSS

In this Appendix, we prove that the local projective measurement that induces maximal coherence loss is the projection onto the local basis that defines the coherence of the system. Formally:

Proposition. For any bipartite state $\rho_{AB} = \sum_{i,j,k,l} \rho_{i,j,k,l}|i,j\rangle_{AB}\langle k,l|$ where the coherence is measured with respect to the local reference bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$, the projective measurement on subsystem B that induces maximal coherence loss is $\Pi_B(\rho_{AB}) = \sum_j (\mathbb{1}_A \otimes |j\rangle_B\langle j|)\rho_{AB}(\mathbb{1}_A \otimes |j\rangle_B\langle j|)$.

Proof. We begin by using the spectral decomposition of a general bipartite quantum state $\rho_{AB} = \sum_n p_n|\psi^n\rangle_{AB}\langle\psi^n|$. Assume that the subsystems have local reference bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ such that $\rho_{AB} = \sum_n \sum_{i,j,k,l} p_n\psi^n_{i,j}(\psi^n_{k,l})^*$ $|i,j\rangle_{AB}\langle k,l|$. The coherence of the system is measured with respect to these bases. To reduce clutter, we remove the subscripts pertaining to the subsystems AB for the remainder of the proof. Unless otherwise stated, it should be clear from the context which subsystem every operator belongs to.

Consider some complete basis on B, $\{|\lambda_m\rangle\}$, and corresponding projective measurement $\Pi_B(\rho) = \sum_m(\mathbb{1} \otimes |\lambda_m\rangle\langle\lambda_m|)\,\rho\,(\mathbb{1} \otimes |\lambda_m\rangle\langle\lambda_m|)$. Computing the matrix elements, we get

$$\begin{aligned}\langle i,j|\Pi_B(\rho)|k,l\rangle : &= [\Pi_B(\rho)]_{i,j,k,l}\\ &= \sum_n \sum_{p,q} p_n\psi^n_{i,p}\big(\psi^n_{kq}\big)^*\\ &\quad\times \sum_m \langle j|\lambda_m\rangle\langle\lambda_m|l\rangle\langle q|\lambda_m\rangle\langle\lambda_m|p\rangle.\end{aligned}$$

Note that minimizing the absolute sum of all the matrix elements will also minimize the coherence, since the diagonal elements of any density matrix always sum to 1 and are non-negative. Consider the absolute sum of all the matrix elements of $\Pi_B(\rho)$:

$$\begin{aligned}\sum_{i,j,k,l}\left|[\Pi_B(\rho)]_{i,j,k,l}\right| &= \sum_{i,j,l,k}\left|\sum_n\sum_{p,q}p_n\psi^n_{i,p}\left(\psi^n_{k,q}\right)^*\sum_m\langle j|\lambda_m\rangle\langle\lambda_m|l\rangle\langle q|\lambda_m\rangle\langle\lambda_m|p\rangle\right|\\
&\geqslant \sum_{\substack{i,k\\ j=l}}\left|\sum_n\sum_{p,q}p_n\psi^n_{i,p}\left(\psi^n_{k,q}\right)^*\sum_m\langle j|\lambda_m\rangle\langle\lambda_m|l\rangle\langle q|\lambda_m\rangle\langle\lambda_m|p\rangle\right|\\
&= \sum_{i,k,j}\left|\sum_n\sum_{p,q}p_n\psi^n_{i,p}\left(\psi^n_{k,q}\right)^*\sum_m\langle j|\lambda_m\rangle\langle\lambda_m|j\rangle\langle q|\lambda_m\rangle\langle\lambda_m|p\rangle\right|\\
&\geqslant \sum_{i,k}\left|\sum_n\sum_{p,q}p_n\psi^n_{i,p}\left(\psi^n_{k,q}\right)^*\sum_m\sum_j\langle j|\lambda_m\rangle\langle\lambda_m|j\rangle\langle q|\lambda_m\rangle\langle\lambda_m|p\rangle\right|\\
&= \sum_{i,k}\left|\sum_n\sum_{p,q}p_n\psi^n_{i,p}\left(\psi^n_{k,q}\right)^*\sum_m\langle q|\lambda_m\rangle\langle\lambda_m|p\rangle\right| = \sum_{i,k}\left|\sum_n\sum_{p,q}p_n\psi^n_{i,p}\left(\psi^n_{k,q}\right)^*\delta_{q,p}\right|\\
&= \sum_{i,k}\left|\sum_{n,p}p_n\psi^n_{i,p}(\psi^n_{k,p})^*\right|.\end{aligned}$$

The first inequality comes from omitting non-negative terms in the sum, while the second inequality comes from moving a summation inside the absolute value function. Note that the final equality is exactly the absolute sum of the elements when $|\lambda_j\rangle = |j\rangle$ since

$$\sum_j(\mathbb{1}_A\otimes|j\rangle_B\langle j|)\rho_{AB}(\mathbb{1}_A\otimes|j\rangle_B\langle j|) = \sum_{i,j,k}\sum_n p_n\psi^n_{i,j}\left(\psi^n_{k,j}\right)^*|i,j\rangle\langle k,j|.$$

This proves the proposition. ∎

Since any N-partite state $\rho_{A_1A_2\cdots A_N}$, is allowed to perform a bipartition such that $\rho_{A_1A_2\cdots A_N} = \rho_{A'A_N}$ where $A' = A_1\cdots A_{N-1}$, we also get the following as a straightforward corollary.

Corollary. For any N-partite state $\rho_{A_1A_2\cdots A_N}$ where the coherence is measured with respect to the local reference bases $\{|i\rangle_{A_k}\}$ and $k = 1,2,\ldots,N$, then the projective measurement on subsystem A_k that induces maximal coherence loss is the projective measurement onto the local basis $\{|i\rangle_{A_k}\}$.

APPENDIX B: NON-NEGATIVITY OF CORRELATED COHERENCE

In this Appendix, we prove the non-negativity of correlated coherence mentioned in the main text.

Theorem 6. For any bipartite quantum state ρ_{AB}, $\mathcal{C}_{cc}(\rho_{AB}) \geqslant 0$ (i.e., correlated coherence is always non-negative).

Proof. Let $\rho_{AB} = \sum_n\sum_{i,j,k,l}p_n\psi^n_{i,j}(\psi^n_{k,l})^*|i,j\rangle_{AB}\langle k,l|$, then

$$\begin{aligned}\mathcal{C}_{cc}(\rho_{AB}) = \mathcal{C}(\rho_{AB}) - \mathcal{C}(\rho_A) - \mathcal{C}(\rho_B) &= \sum_{\substack{(i,j)\\ \neq(k,l)}}\left|\sum_n p_n\psi^n_{i,j}\left(\psi^n_{k,l}\right)^*\right| - \sum_{i\neq k}\left|\sum_n p_n\sum_j\psi^n_{i,j}\left(\psi^n_{k,j}\right)^*\right| - \sum_{j\neq l}\left|\sum_n p_n\sum_i\psi^n_{i,j}\left(\psi^n_{i,l}\right)^*\right|\\
&\geqslant \sum_{\substack{(i,j)\\ \neq(k,l)}}\left|\sum_n p_n\psi^n_{i,j}\left(\psi^n_{k,l}\right)^*\right| - \sum_{\substack{j\\ i\neq k}}\left|\sum_n p_n\psi^n_{i,j}\left(\psi^n_{k,j}\right)^*\right| - \sum_{\substack{i\\ j\neq l}}\left|\sum_n p_n\psi^n_{i,j}\left(\psi^n_{i,l}\right)^*\right|\\
&= \left(\sum_{\substack{(i,j)\\ \neq(k,l)}} - \sum_{\substack{j=l\\ i\neq k}} - \sum_{\substack{i=k\\ j\neq l}}\right)\left|\sum_n p_n\psi^n_{i,j}\left(\psi^n_{k,l}\right)^*\right|.\end{aligned}$$

The inequality comes from moving a summation outside of the absolute value function. Since $\sum_{\substack{(i,j)\\ \neq(k,l)}} = \sum_{\substack{j\neq l\\ i\neq k}} + \sum_{\substack{j=l\\ i\neq k}} + \sum_{\substack{i=k\\ j\neq l}}$, the final equality above is always a sum of non-negative values, which completes the proof. ∎

APPENDIX C: ELEMENTARY PROPERTIES OF THE ENTANGLEMENT OF COHERENCE

In this Appendix, we will prove useful elementary properties of the entanglement of coherence. The proofs are presented in the same order as they are mentioned in the main text.

Property 1 (EOC of separable states). If a bipartite quantum state ρ_{AB} is separable, $E_{cc}(\rho_{AB}) = 0$.

Proof. The proof is nearly identical to Theorem 4 in the main text, with the additional observation that $\rho_{AA'BB'} = \sum_{i,j} p_{i,j}|\mu_i\rangle_{AA'}\langle\mu_i| \otimes |\nu_j\rangle_{BB'}\langle\nu_j|$ is unitarily symmetric. To see this, define $U_{AA'} := \sum_i |\nu_i\rangle_{AA'}\langle\mu_i|$ and $U_{BB'} := \sum_i |\mu_i\rangle_{AA'}\langle\nu_i|$. It is easy to verify that is satisfies

$$U_{AA'} \otimes U_{BB'}(U_{\text{SWAP}}\rho_{AA'BB'}U_{\text{SWAP}}^{\dagger})U_{AA'}^{\dagger} \otimes U_{BB'}^{\dagger} = \rho_{AA'BB'},$$

where U_{SWAP} is the same SWAP operator as in the main text so it is unitarily symmetric. ■

Property 2 (invariance under local unitaries). For a bipartite quantum state ρ_{AB}, $E_{cc}(\rho_{AB})$ is invariant under local unitary operations on A and B

Proof. Without loss in generality, we only need to prove it is invariant under local unitary operations on A.

For some bipartite state ρ_{AB}, let $\rho^*_{AA'BB'}$ be the optimal unitarily symmetric extension such that $E_{cc}(\rho_{AB}) = \mathcal{C}_{cc}(\rho^*_{AA'BB'})$. Let $|i\rangle_{AA'}$ and $|j\rangle_{BB'}$ be the eigenbases of $\rho^*_{AA'}$ and $\rho^*_{BB'}$, respectively. With respect to these bases, $\rho^*_{AA'BB'} = \sum_{ijkl} \rho_{ijkl}|i,j\rangle_{AA'BB'}\langle k,l|$.

Suppose we perform a unitary $U = U_A \otimes \mathbb{1}_{A'BB'}$ on A such that so $U|i,j\rangle = |\alpha_i,j\rangle$, where $\{|\alpha_i\rangle\}$ is an orthonormal set. Since $U\rho^*_{AA'BB'}U^{\dagger} = \sum_{ijkl} \rho_{ijkl}|\alpha_i,j\rangle_{AA'BB'}\langle\alpha_k,l|$, it is clear that the off-diagonal matrix elements are invariant under the new bases $|\alpha_i,j\rangle_{AA'BB'}$ so $E_{cc}(\rho_{AB}) = E_{cc}(U\rho^*_{AA'BB'}U^{\dagger})$, which proves the proposition. ■

Property 3 (convexity). $E_{cc}(\rho_{AB})$ is convex and nonincreasing under mixing:

$$\lambda E_{cc}(\rho_{AB}) + (1-\lambda)E_{cc}(\sigma_{AB}) \geqslant E_{cc}(\lambda\rho_{AB} + (1-\lambda)\sigma_{AB}).$$

For any two bipartite quantum states ρ_{AB} and σ_{AB}, and $\lambda \in [0,1]$.

Proof. Let $\rho^*_{AA'BB'}$ and $\sigma^*_{AA'BB'}$ be the optimal unitarily symmetric extensions for ρ_{AB} and σ_{AB}, respectively, such that $E_{cc}(\rho_{AB}) = \mathcal{C}_{cc}(\rho^*_{AA'BB'})$ and $E_{cc}(\sigma_{AB}) = \mathcal{C}_{cc}(\sigma^*_{AA'BB'})$.

Consider the state $\tau_{AA'A''BB'B''} := \lambda\rho^*_{AA'BB'} \otimes |0,0\rangle_{A''B''}\langle 0,0| + (1-\lambda)\sigma^*_{AA'BB'} \otimes |1,1\rangle_{A''B''}\langle 1,1|$ for $\lambda \in [0,1]$. Direct computation will verify that with respect to the eigenbases of $\tau_{AA'A''}$ and $\tau_{BB'B''}$, $\mathcal{C}_{cc}(\tau_{AA'A''BB'B''}) = \lambda\mathcal{C}_{cc}(\rho^*_{AA'BB'}) + (1-\lambda)\mathcal{C}_{cc}(\sigma^*_{AA'BB'}) = \lambda E_{cc}(\rho_{AB}) + (1-\lambda)$ $E_{cc}(\sigma_{AB})$. However, as $\text{Tr}_{A'A''B'B''}(\tau_{AA'A''BB'B''}) = \lambda\rho_{AB} + (1-\lambda)\sigma_{AB}$, it is an extension of $\lambda\rho^*_{AB} + (1-\lambda)\sigma^*_{AB}$.

It remains to be proven that the extension above is also unitarily symmetric. Let $\Xi^{\text{SWAP}}_{X\leftrightarrow Y}$ denote the SWAP operation between X and Y. Let the operators $U_{AA'}$, $U_{BB'}$, $V_{AA'}$, $V_{BB'}$ satisfy $\rho^*_{AA'BB'} = U_{AA'} \otimes U_{BB'}\,\Xi^{\text{SWAP}}_{AA'\leftrightarrow BB'}(\rho^*_{AA'BB'})U^{\dagger}_{AA'} \otimes U^{\dagger}_{BB'}$ and $\sigma^*_{AA'BB'} = V_{AA'} \otimes V_{BB'}\,\Xi^{\text{SWAP}}_{AA'\leftrightarrow BB'}(\sigma^*_{AA'BB'})V^{\dagger}_{AA'} \otimes V^{\dagger}_{BB'}$, respectively. It can be verified that the local unitary operators $W_{AA'A''} := U_{AA'} \otimes |0\rangle_{A''}\langle 0| + V_{AA'} \otimes |1\rangle_{A''}\langle 1|$ and $W_{BB'B''} := U_{BB'} \otimes |0\rangle_{B''}\langle 0| + V_{BB'} \otimes |1\rangle_{B''}\langle 1|$ satisfy $\tau^*_{AA'A''BB'B''} = W_{AA'A''} \otimes W_{BB'B''}\,\Xi^{\text{SWAP}}_{AA'A''\leftrightarrow BB'B''}(\tau_{AA'A''BB'B''})W^{\dagger}_{AA'B''} \otimes W^{\dagger}_{BB'B''}$, so it is also unitarily symmetric.

Since E_{cc} is a minimization over all unitarily symmetric extensions, we have $\lambda E_{cc}(\rho_{AB}) + (1-\lambda)E_{cc}(\sigma_{AB}) = \mathcal{C}_{cc}(\tau_{AA'A''BB'B''}) \geqslant E_{cc}(\lambda\rho_{AB} + (1-\lambda)\sigma_{AB})$, which completes the proof. ■

Property 4 (contraction under partial trace). Consider the bipartite state ρ_{AB} where $A = A_1A_2$ is a composite system. Then the entanglement of coherence is nonincreasing under a partial trace:

$$E_{cc}(\rho_{A_1A_2B}) \geqslant E_{cc}(\text{Tr}_{A_1}(\rho_{A_1A_2B})).$$

Proof. Let $\rho^*_{A_1A_2A'BB'}$ be the optimal unitarily symmetric extension of $\rho_{A_1A_2B}$ such that $E_{cc}(\rho_{A_1A_2B}) = \mathcal{C}_{cc}(\rho^*_{A_1A_2A'BB'})$. It is clear that $\text{Tr}_{A_1A'B'}(\rho^*_{A_1A_2A'BB'}) = \text{Tr}_{A_1}(\rho_{A_1A_2B}) = \rho_{A_2B}$ so $\rho^*_{A_1A_2A'BB'}$ is an unitarily symmetric extension of $\text{Tr}_{A_1}(\rho_{A_1A_2B})$. Since E_{cc} is a minimization over all such extensions, $E_{cc}(\rho_{A_1A_2B}) \geqslant E_{cc}(\text{Tr}_{A_1}(\rho_{A_1A_2B}))$. ■

Property 5 (contraction under local projections). Let π^i_A be a complete set of rank 1 projectors on subsystem A such that $\sum_i \pi^i_A = \mathbb{1}_A$, and define the local projection $\Pi_A(\rho_A) := \sum_i \pi^i_A\rho_A\pi^i_A$. The entanglement of coherence is contractive under a local projections,

$$E_{cc}(\rho_{AB}) \geqslant E_{cc}(\Pi_A(\rho_{AB})),$$

or, if A is a composite system, $A = A_1A_2$,

$$E_{cc}(\rho_{AB}) \geqslant E_{cc}(\Pi_{A_1}(\rho_{A_1A_2B})).$$

Proof. First, we observe that any projective measurement can be performed via a controlled-NOT (CNOT) type operation with an ancilla, followed by tracing out the ancilla:

$$\begin{aligned}\text{Tr}_X &\left[U^{\text{CNOT}}_{XY}\left(|0\rangle_X\langle 0| \otimes \sum_{i,j}\rho_{ij}|i\rangle_Y\langle j|\right)\left(U^{\text{CNOT}}_{XY}\right)^{\dagger}\right] \\ &= \sum_{i,i}\rho_{ii}|i\rangle_Y\langle i|. \end{aligned} \tag{C1}$$

The unitary performs the operation $U^{\text{CNOT}}_{XY}|0,i\rangle_{XY} = |i,i\rangle_{XY}$. Since adding an uncorrelated ancilla does not increase E_{cc}, we have $E_{cc}(|0\rangle_{A_3}\langle 0| \otimes \rho_{A_1A_2B}) = E_{cc}(\rho_{A_1A_2B})$. As E_{cc} is invariant under local unitaries (Proposition 2) and contractive under partial trace (Proposition 4), this proves the property. ■

Property 6 (invariance under classical communication). For a bipartite state ρ_{AB}, suppose that on Alice's side, $A = A_1A_2$ is a composite system and A_1 is a classical registry storing classical information. Then E_{eqc} remains invariant if a copy of A_1 is created on Bob's side.

More formally, let $\rho_{A_1A_2B_2} = \sum_i p_i|i\rangle_{A_1}\langle i| \otimes |\psi_i\rangle_{A_2B_2}\langle\psi_i|$ be the initial state, and let $\sigma_{A_1A_2B_1B_2} = \sum_i p_i|i\rangle_{A_1}\langle i| \otimes |\psi_i\rangle_{A_2B_2}\langle\psi_i| \otimes |i\rangle_{B_1}\langle i|$ be the state after Alice communicates a copy of A_1 to Bob, then

$$E_{cc}(\rho_{A_1A_2B_2}) = E_{cc}(\sigma_{A_1A_2B_1B_2}).$$

Proof. Let $\Xi^{\text{SWAP}}_{X\leftrightarrow Y}$ denote the SWAP operation between X and Y. Let $\rho^*_{A_1A_2A'B_1B_2B'}$ be the optimal unitarily symmetric extension of $\rho_{A_1A_2B_2}$ such that $E_{cc}(\rho_{A_1A_2B_2}) = \mathcal{C}_{cc}(\rho^*_{A_1A_2A'B_1B_2B'})$. Note that $\mathcal{C}_{cc}(\rho^*_{A_1A_2A'B_1B_2B'}) = \mathcal{C}_{cc}(|0\rangle_{A''}\langle 0| \otimes \rho^*_{A_1A_2A'B_1B_2B'} \otimes |0\rangle_{B''}\langle 0|)$.

Define a CNOT type operation between A_1 and B'' such that $U^{\mathrm{CNOT}}_{A_1B''}|0,i\rangle_{A_1B''} = |i,i\rangle_{A_1B''}$. Ordinarily, such an operation cannot be done by Bob locally unless he has access to subsystem A_1 on Alice's side. However, since $\rho^*_{A_1A_2A'BB'}$ is unitarily symmetric, there exist local unitaries $U_{A_1A_2A'}$ and $U_{B_1B_2B'}$ such that $\Xi^{\mathrm{SWAP}}_{A_1A_2A'\leftrightarrow B_1B_2B'}(\rho^*_{A_1A_2A'B_1B_2B'}) = U_{A_1A_2A'} \otimes U_{BB'}\rho^*_{A_1A_2A'BB'}U^\dagger_{A_1A_2A'} \otimes U^\dagger_{BB'}$. This implies that the Bob can effectively perform $U^{\mathrm{CNOT}}_{A_1B''}$ locally by first performing the SWAP operation through local unitaries, gain access to the information in A_1, copy the classical registry to B'' by performing $U^{\mathrm{CNOT}}_{B_1B''}$ locally, and then undo the SWAP operation via another set of local unitary operations.

This means there must exist $V_{A_1A_2A'A''}$ and $V_{B_1B_2B'B''}$ such that

$$V_{A_1A_2A'A''} \otimes V_{B_1B_2B'B''}(|0\rangle_{A''}\langle 0| \otimes \rho^*_{A_1A_2A'B_1B_2B'} \otimes|0\rangle_{B''}\langle 0|)V^\dagger_{A_1A_2A'A''} \otimes V^\dagger_{B_1B_2B'B''} \tag{C2}$$

is a unitarily symmetric extension of $U^{\mathrm{CNOT}}_{A_1B''}(\rho_{A_1A_2B_2} \otimes |0\rangle_{B''}\langle 0|)U^{\mathrm{CNOT}\,\dagger}_{A_1B''}$. However, because this state is equivalent to $\sigma_{A_1A_2B_1B_2}$ as defined previously, it is also a unitarily symmetric extension of $\sigma_{A_1A_2B_1B_2}$. Since $\mathcal{C}_{cc}$ is invariant under local unitary operations, we have

$$\begin{aligned}
E_{cc}(\rho_{A_1A_2B_2}) &= E_{cc}(\rho^*_{A_1A_2A'BB'})\\
&= E_{cc}(|0\rangle_{A''}\langle 0| \otimes \rho^*_{A_1A_2A'BB'} \otimes |0\rangle_{B''}\langle 0|)\\
&= \mathcal{C}_{cc}\big[V_{A_1A_2A'A''} \otimes V_{BB'B''}\big(|0\rangle_{A''}\langle 0| \otimes \rho^*_{A_1A_2A'BB'}\\
&\quad \otimes|0\rangle_{B''}\langle 0|\big)V^\dagger_{A_1A_2A'A''} \otimes V^\dagger_{BB'B''}\big]\\
&\geqslant E_{cc}(\sigma_{A_1A_2B_1B_2}),
\end{aligned}$$

where the last inequality comes from the fact that the entanglement of coherence is a minimization over all unitarily symmetric extensions. On the other hand, a unitarily symmetric extension of $\sigma_{A_1A_2B_1B_2}$ is also an unitarily symmetric extension of $\rho_{A_1A_2B_2}$ so $E_{cc}(\rho_{A_1A_2B_2}) \leqslant E_{cc}(\sigma_{A_1A_2B_1B_2})$. This implies that $E_{cc}(\rho_{A_1A_2B_2}) = E_{cc}(\sigma_{A_1A_2B_1B_2})$, which completes the proof. ∎

Property 7 (contraction under LOCC). For any bipartite state ρ_{AB} and let Λ_{LOCC} be any LOCC protocol performed between A and B. Then E_{cc} is nonincreasing under such operations:

$$E_{cc}(\rho_{AB}) \geqslant E_{cc}(\Lambda_{\mathrm{LOCC}}(\rho_{AB})).$$

Proof. We consider the scenario where Alice performs a positive-operator valued measure (POVM) on her subsystems, communicates classical information of her measurement outcomes to Bob, who then performs a separate operation on his subsystem based on this measurement information.

Suppose Alice and Bob begin with the state $\rho_{A_1B_1}$. By Naimark's theorem, any POVM can be performed through a unitary interaction between the state of interest and an uncorrelated pure state ancilla, followed by a projective measurement on the ancilla and finally tracing out the ancillary systems. In order to facilitate Alice and Bob's performing of such quantum operations, we add uncorrelated ancillas to the state, which does not change the entanglement of coherence so $E_{cc}(\rho_{A_1B_1}) = E_{cc}(|0,0\rangle_{M_AA_2}\langle 0,0| \otimes \rho_{A_1B_1} \otimes |0,0\rangle_{M_BB_2}\langle 0,0|)$. For Alice's procedure, we will assume the projection is performed on M_A, so M_A is a classical register storing classical measurement outcomes.

In the beginning, Alice performs a unitary operation on subsystems $M_AA_1A_2$, followed by a projection on M_A which makes it classical. We represent the composite of these two operations with Ω_A, which represents Alice's local operation. Since E_{cc} is invariant under local unitaries (Property 2) but contractive under a projection (Property 4), Ω_A is a contractive operation.

The next part of the procedure is a communication of classical bits to Bob. This procedure is equivalent to the copying of the state of the classical register M_A to the register M_B. However, E_{cc} is invariant under such communication (Property 6). We represent this operation as $\Gamma_{A\to B}$. The next step requires Bob to perform an operation on his quantum system based on the communicated bits. He can achieve this by performing a unitary operation on subsystems $M_BB_1B_2$. We represent this operation with Ω_B, which does not change E_{cc}. The final step of the procedure requires tracing out the ancillas, $\mathrm{Tr}_{M_AA_2M_BB_2}$, which is again contractive (Property 4).

Since every step is either contractive or invariant, we have the following inequality:

$$\begin{aligned}
E_{cc}(\rho_{A_1B_1}) &= E_{cc}(|0,0\rangle_{M_AA_2}\langle 0,0| \otimes \rho_{A_1B_1} \otimes |0,0\rangle_{M_BB_2}\langle 0,0|)\\
&\geqslant E_{cc}[\mathrm{Tr}_{M_AA_2M_BB_2} \circ \Omega_B \circ \Gamma_{A\to B} \circ \Omega_A\\
&\quad \times (|0,0\rangle_{M_AA_2}\langle 0,0| \otimes \rho_{A_1B_1} \otimes |0,0\rangle_{M_BB_2}\langle 0,0|)].
\end{aligned}$$

Any LOCC protocol is a series of such procedures from Alice to Bob or from Bob to Alice, so we must have $E_{cc}(\rho_{AB}) \geqslant E_{cc}(\Lambda_{\mathrm{LOCC}}(\rho_{AB}))$, which completes the proof. ∎

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