Additivity relations in quantum correlations

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Does the sum of correlations in subsystems constitute the correlation in the total system? Such a concept can be expressed by an *additivity relationship* of correlations. From a *strong subadditivity* condition of von Neumann entropy, four different additivity relations in *total correlation* are derived and quantified. Based upon the classification of the additivity in total correlation, we identify the corresponding additive relationships in *entanglement*. It is also discussed that similar relationships are satisfied for quantum discord of pure states, but it is not always true for mixed states.

I. INTRODUCTION

Quantifying correlations in quantum systems is an important issue in quantum information theory. From a hierarchical picture, measures of correlations can be classified into three parts: total correlations, quantum correlations, and classical correlations [1]. Roughly speaking, *total correlation* encapsulates all the correlations in a composite system, including quantum and classical correlations. Contrarily, *entanglement* is a well-known measure of quantum correlation which does not take into account the classical counterpart [2]. It has been identified that entanglement is a necessary condition for providing exponential speedup in quantum computation [3] and is taken as a useful resource for various quantum information processing. Recently, another class of quantum correlation, *quantum discord*, is proposed [4,5]. It has been argued that discord characterizes quantum correlation better in terms of computational advantage and the operational meanings of discord have recently begun to be understood [6–9].

From the spirit of multipartite correlation which quantifies the overall amount of correlation, it is natural to think about the relationships between the multipartite correlation in the total system and the bipartite correlations of subsystems. A basic example is that, for a tripartite system *A*-*B*-*C*, tripartite correlation is required to be equal to or greater than bipartite correlation in any subsystem. Also, a measure of tripartite correlations should be reduced to a measure of bipartite correlations in a subsystem when a party has completely no correlation with the other parties. For the case of total correlation, such relationships are well defined by additivity relations. We are questioning here whether such relations are generally satisfied for different measures of *quantum correlations*, e.g., entanglement and discord.

Formal definition of total correlation is given as an optimal distance of entropy between a given state and a state without any correlation, *product state*. A multipartite state is called a product state if the state of an *n*-party quantum system $\mathcal{A}_1 \cdots \mathcal{A}_n$ is written as a product, $\pi = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_n$, where π_i is a positive operator in the Hilbert space $\mathcal{A}_i$. The total information can be represented as

$$\mathcal{T}(\rho) = \min_{\sigma \in \pi} S(\rho \| \sigma), \qquad (1)$$

where $S(\rho \| \sigma) = \mathrm{Tr}[\rho \, \log \rho - \rho \, \log \sigma]$ is the relative entropy and "min" denotes the minimization of $S(\rho \| \sigma)$ over σ within a

product state π. After a simple algebra, it can be proved that the closest product state to a state $\rho \in \mathcal{A}_1 \cdots \mathcal{A}_n$ is the product of reduced states, $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$ [10], where $\rho_i = \mathrm{Tr}_{i|\mathrm{All}}[\rho]$ and $\mathrm{Tr}_{i|\mathrm{All}}$ means trace out of all the states except the state in the *i*th Hilbert space. Therefore, for an arbitrary multipartite state, it can be written in a different way as $\mathcal{T}(\rho) = S(\rho \| \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n) = \sum_i S(\rho_i) - S(\rho)$, where $S(\rho) = -\mathrm{Tr}[\rho \, \log \rho]$ is the von Neumann entropy [10]. The quantity is sometimes called total mutual information since the total information becomes mutual information when the system is bipartite, as $I(\rho_{A,B}) = S(\rho_A) + S(\rho_B) - S(\rho_{A,B})$ defined in a bipartite system $A - B$ [1,11,12]. In terms of the quantum relative entropy, $I(\rho_{A,B}) \equiv S(\rho_{A,B} \| \rho_A \otimes \rho_B)$.

A physical state $\mathcal{S}$ of *n*-party quantum system $\mathcal{A}_1 \cdots \mathcal{A}_n$ is called separable if the state can be written as $\mathcal{S} = \sum_i p_i s_1^i \otimes \cdots \otimes s_n^i$, where s_m^i is a state in $\mathcal{A}_m$. By definition, the state is considered to have no entanglement and it is a type of multipartite state which can be prepared in classical methods, i.e., local operation and classical communication. Under the circumstance, a measure of entanglement can be characterized using a distance measure between a given state and the closest separable state, called the relative entropy of entanglement [13–15]. The relative entropy of entanglement is defined as

$$\mathcal{E}(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho \| \sigma), \qquad (2)$$

where $\mathcal{S}$ is the set of separable states. The measure of entanglement is naturally applied to multipartite systems as the criterion of separable states has been well defined for a multipartite system.

In addition, the notion of quantum correlation can be greatly extended once the notion of state distance from classically correlated states instead of separable states is considered [10,16–20]. A physical state ρ of an *n*-party quantum system $\mathcal{A}_1 \cdots \mathcal{A}_n$ is called classically correlated if the state can be written as $\mathcal{C} = \sum_{\vec{k}} p_{\vec{k}} |\vec{k}\rangle \langle \vec{k}|$, where $|\vec{k}\rangle = |k_1\rangle \otimes |k_2\rangle \cdots \otimes |k_n\rangle$ and $|k_l\rangle$ is a local basis at *l*th site. The states are considered to have no entanglement since they are a convex sum of product states. The relative entropy of discord [5,10,16–20] is defined to be

$$\mathcal{D}(\rho) = \min_{\sigma \in \mathcal{C}} S(\rho \| \sigma), \qquad (3)$$

where $\mathcal{C}$ is the set of classically correlated states. Due to its definition, the measure of discord is also naturally applied to multipartite systems.

As seen from their definition, the total mutual information, the relative entropy of entanglement, and the relative entropy of discord are multipartite measures of correlations in a unified view of correlations. Those measures have the clear relationship $\mathcal{T} \geqslant \mathcal{D} \geqslant \mathcal{E}$ because $\pi \subset \mathcal{C} \subset \mathcal{S}$, and they also enable us to compare the three criteria of correlation, product states, classically correlated states, and separable states.

In the following section, we discuss the additivity relation in the total correlation, entanglement, and discord. We derive four different additivity relationships using the total correlation (1) and show whether a similar type of correlation can be derived for the case of entanglement and discord. In Sec. III, it is shown that the additivity relationship is satisfied by a specific class of states. As specific examples, we will deal with the generalized GHZ states, a variant of the generalized GHZ states, the generalized W states, and a few mixed states. In Sec. IV, we summarize our result and present the conclusion on the additivity relationship in quantum correlations.

II. ADDITIVE RELATIONS IN MULTIPARTITE CORRELATION

In this section, we discuss the additivity relation of multipartite correlations. At first, we discuss how a multipartite correlation in general can be related with sets of bipartite correlations. We will show that the relationship leads to three different types of inequalities. It is our main question whether such inequalities are satisfied for the case of quantum correlations so that we will study the relationships between multipartite and bipartite quantum correlations using entanglement as well as quantum discord.

A. Bounds on the total correlation

The total information counts all the correlations existing in a multipartite system. The structure of multipartite correlations is quite complicated in general, although there are rules by which they can be decomposed into the correlations of subsystems. In fact, the total mutual information of a tripartite *A-B-C* system can in principle be decomposed by two different types of bipartite correlations as such,

$$\mathcal{T}(A : B : C) = \mathcal{T}(A : B) + \mathcal{T}(AB : C), \qquad (4)$$

and the equality holds for any permutation of *A*, *B*, and *C*. The equation shows how the tripartite correlation is quantified in terms of bipartite correlations. If we consider only the subsystem *A-B* among the total system, there is only correlation between *A* and *B*. In addition, if a system *C* is joined to the subsystem *A-B*, it is natural to understand that the total amount of correlation is increased and the amount corresponds to the correlation existing between the subsystem *A-B* together with the correlation between *A-B* and the additional system *C*. The additional correlation is taken by the term $\mathcal{T}(AB : C)$, while it is still characterized by the bipartite correlation between two different subsystems, *A-B* and *C*.

The relation between the multipartite correlation and the bipartite correlations has been well summarized in the famous strong subadditivity relation of Shannon entropy,

$$H(AB) + H(BC) \geqslant H(B) + H(ABC),$$

where $H(p) = -\sum_i p_i \log p_i$ [21]. The inequality is easy to prove in general, while the quantum version of the strong subadditivity with von Neumann entropy is difficult. The strong subadditivity of von Neumann entropy $S(\rho) = -\text{Tr}[\rho \log \rho]$ [22] refers to the following inequality for tripartite quantum state ρ_{ABC}:

$$S(AB) + S(BC) \geqslant S(ABC) + S(B),$$

where $\rho_{AB} = \text{tr}_C(\rho_{ABC})$, $\rho_{BC} = \text{tr}_A(\rho_{ABC})$, and $\rho_B = \text{tr}_{A,B}(\rho_{ABC})$. The strong subadditivity is widely used in quantum information theory [23]. The proof of the inequality and its operational meaning can be found in [22,24,25].

Using the strong subadditivity of von Neumann entropy, the bounds on the total mutual information are obtained as

$$\mathcal{T}(AB : C) \geqslant \mathcal{T}(A : C), \qquad (5)$$

$$\mathcal{T}(A : B : C) \geqslant \mathcal{T}(A : B) + \mathcal{T}(A : C), \qquad (6)$$

$$\mathcal{T}(A : B : C) \leqslant \mathcal{T}(BC : A) + \mathcal{T}(AC : B). \qquad (7)$$

From the definition, it is straightforward to show that the three inequalities above are equivalent to the strong subadditivity of von Neumann entropy. The strong subadditivity frequently has been represented in the form of Eq. (5) [23], but it can also be represented in the forms of Eqs. (6) and (7) by using the tripartite total mutual information. We can give intuitive interpretations of the three inequalities. The first inequality states that the correlation between *A-B* and *C* contains the correlation between *A* and *C*. The correlation between *A* and *B* is missing in the right-hand side. The second inequality is also intuitively clear because the left-hand side of the inequality counts all the correlations among *A*, *B*, and *C*, so it is greater than or equal to the correlations between *A-B* and *A-C* only. In that case, it can be said that the correlation between *B* and *C* has not been counted. In the third inequality, both two terms of the right-hand side contain the correlation between *A* and *C* (i.e., double counting), so it is less than or equal to the tripartite correlation. Those relations between the correlations in a tripartite system *A-B-C* are represented in the conceptual diagrams in Fig. 1.

B. Bounds on the relative entropy of entanglement

The total information covers all the correlations in both a classical part and a quantum part. One may wonder that there exist the consistent inequalities as (4), (5), (6), and (7) for the measures of quantum correlations. A natural candidate for quantum correlation is quantum entanglement. There are several measures of entanglement which have been studied extensively so far [2]. However, our question requires a measure which can be naturally generalized to multipartite systems and which is consistent to the quantum part of total information in Eq. (1). A proper measure in that purpose is the relative entropy of entanglement which is defined in Eq. (2). The relative entropy of entanglement is always less than or equal to the total information and is naturally generalized to multipartite systems.

For the relative entropy of entanglement, there exists an inequality which corresponds to Eq. (4) of the total information. The authors of [26] first showed that the following
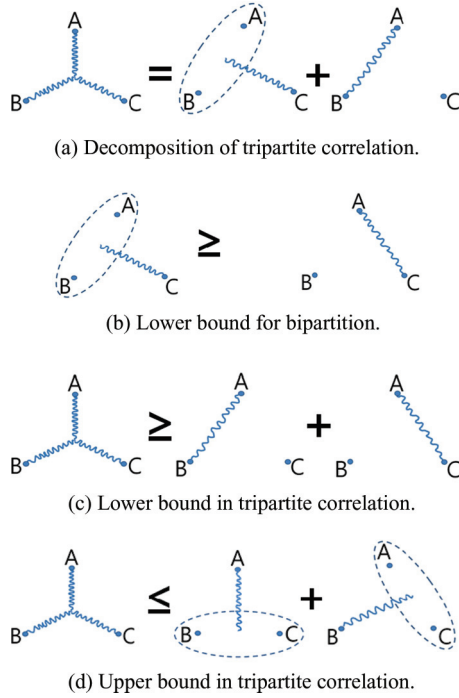
(a) Decomposition of tripartite correlation.

(b) Lower bound for bipartition.

(c) Lower bound in tripartite correlation.

(d) Upper bound in tripartite correlation.

FIG. 1. (Color online) Conceptual diagrams for the relations between the correlations in a three-party system *A-B-C*. Subfigures (a) to (d) correspond to Eqs. (4) to (7), respectively.

inequality holds for any tripartite pure system *A-B-C*:

$$\mathcal{E}(A:B:C) \geqslant \mathcal{E}(A:B) + \mathcal{E}(AB:C). \qquad (8)$$

They derived it by using the inequality $S(\sigma_{ABC}\|\rho_{ABC}) - S(\sigma_{AB}\|\rho_{AB}) \geqslant S(\sigma_{AB}) - S(\sigma_{ABC})$ [27], which holds for any state ρ_{ABC} and for any triseparable state σ_{ABC}. Unlike the equation (4) of the total mutual information, Eq. (8) is an inequality. It means that the tripartite entanglement contains entanglement between *A*, *B* and *AB*, *C*, but they are not exactly same.

Also, the following inequality holds for any tripartite system *A-B-C*:

$$\mathcal{E}(AB:C) \geqslant \mathcal{E}(A:C). \qquad (9)$$

It can be derived directly from the monotonicity of the relative entropy under partial trace, $S(\rho^A\|\sigma^A) \leqslant S(\rho^{AB}\|\sigma^{AB})$ [23].

Combining Eqs. (8) and (9) gives the following inequalities, which hold for any tripartite pure system *A-B-C*:

$$\mathcal{E}(A:B:C) \geqslant \mathcal{E}(A:B) + \mathcal{E}(A:C). \qquad (10)$$

At the same time, a symmetrized version of the inequality can be found as $\mathcal{E}(A:B:C) \geqslant \frac{2}{3}[\mathcal{E}(A:B) + \mathcal{E}(B:C) + \mathcal{E}(A:C)]$ by taking the average of Eq. (10) over all permutations of *A*, *B*, and *C*.

The inequality (10) corresponds to Eq. (6) of the total mutual information. It is a weaker inequality than the monogamy inequality of entanglement $\mathcal{E}(A:BC) \geqslant \mathcal{E}(A:B) + \mathcal{E}(A:C)$, which is not known if it holds for the entanglement measure. It is not a unique property of quantum correlations. We can see the analogy in the classical correlations between classical random variables. For three classical random variables *X*, *Y*, and *Z*, $H(X:Y:Z) \geqslant H(X:Y) + H(X:Z)$,

where $H(X:Y:Z) = H(X) + H(Y) + H(Z) - H(X,Y,Z)$ is a multivariate mutual information.

There is also the same form of upper bound for the regularized relative entropy of entanglement $\mathcal{E}^{\infty}(\rho) = \lim_{n\to\infty}\frac{1}{n}\mathcal{E}(\rho^{\otimes n})$,

$$\mathcal{E}^{\infty}(A:B:C) \leqslant \mathcal{E}^{\infty}(BC:A) + \mathcal{E}^{\infty}(AC:B),$$

which holds for any tripartite pure system. The inequality is also proven in [26], and it follows from the fact that $\mathcal{E}$ is an entanglement monotone.

Therefore, it is possible to conclude that similar kinds of additivity relationship with total correlation are satisfied for the case of entanglement. Although it is found that they do not coincide sharply, they asymptotically behave in the same way.

C. Bounds on the relative entropy of discord

Now we consider the quantum discord in Eq. (3) as another candidate of quantum correlations. It is well known that the criterion of nonzero quantum discord is not identical to the criterion of nonzero entanglement. Unlike entanglement, quantum discord is defined to be nonzero even when a state is separable. The difference makes it nontrivial to consider whether there exist the consistent additivity relations as Eqs. (4) to (7) for quantum discord in multipartite systems. We take the relative entropy of discord in Eq. (3) as quantum discord measure. The relative entropy of discord differs from the relative entropy of entanglement in the set of states whose minimization should be taken. It enables us to genuinely compare the difference in the separability criterion and the classicality criterion.

First, we show that there is the consistent lower bound on the tripartite relative entropy of discord. The minimization in $\mathcal{D}(\rho) = \min_{\sigma\in\mathcal{C}} S(\rho\|\sigma)$ can be reduced to a minimization over a set of local bases $\{|\vec{k}\rangle\}$ [10],

$$\mathcal{D}(\rho) = \min_{\{|\vec{k}\rangle\}} \Lambda_{\rho}(\{|\vec{k}\rangle\}) - S(\rho), \qquad (11)$$

where $\Lambda_{\rho}(\{|\vec{k}\rangle\}) = -\sum_{\vec{k}}\langle\vec{k}|\rho|\vec{k}\rangle \log\langle\vec{k}|\rho|\vec{k}\rangle$. For the matter of convenience, we write $A:B:C$ instead of ρ_{ABC} from now on. $A:B:C$ denotes a tripartition of a system *A-B-C*.

Theorem 1. The following inequality holds for any tripartite pure system *A-B-C*:

$$\mathcal{D}(A:B:C) \geqslant \mathcal{D}(A:B) + \mathcal{D}(AB:C). \qquad (12)$$

Proof. For any pure state, bipartite discord is equivalent to the entropy of entanglement, so $\mathcal{D}(AB:C) = S(AB) = S(C)$. Then the inequality is equivalent to $\min \Lambda_{A:B:C}(\{|\vec{k}\rangle\}) \geqslant \min \Lambda_{A:B}(\{|\vec{k}\rangle\})$. Let $\{|\vec{k}^*\rangle\}$ be a basis which minimizes $\Lambda_{A:B:C}(\{|\vec{k}\rangle\})$, and consider the set of local measurement operators $\{|\vec{k}^*\rangle\langle\vec{k}^*|\}$. The outcomes of the local measurements can be treated as three classical random variables, which are denoted *X*, *Y*, and *Z*. They correspond to the outcomes of the local measurements on *A*, *B*, and *C*, respectively. For any classical random variables *X*, *Y*, and *Z*, Shannon entropies satisfy $H(X,Y,Z) \geqslant H(X,Y)$. Using this inequality, $\min \Lambda_{A:B:C}(\{|\vec{k}\rangle\}) = H(X,Y,Z) \geqslant H(X,Y) \geqslant \min \Lambda_{A:B}(\{|\vec{k}\rangle\})$, so the proof is completed. ∎

We also found the analogous inequality for discord as the inequality (5) of the total mutual information and the inequality (9) of the relative entropy of entanglement.

Theorem 2. The following inequality holds for any tripartite pure system *A-B-C*:

$$\mathcal{D}(AB:C) \geqslant \tfrac{1}{2}[\mathcal{D}(A:C) + \mathcal{D}(B:C)], \qquad (13)$$

and its proof can be given below.

Proof. The inequality (13) can be easily proven from just $\mathcal{T} \geqslant \mathcal{D}$. Given a pure system *A-B-C*, the bipartite discord $\mathcal{D}(AB:C)$ is equivalent to the entropy of entanglement $S(C)$, and $S(AC) = S(B)$ and $S(BC) = S(A)$. Therefore, $S(C) = \tfrac{1}{2}[\mathcal{T}(A:C) + \mathcal{T}(B:C)] \geqslant \tfrac{1}{2}[\mathcal{D}(A:C) + \mathcal{D}(B:C)]$ from $\mathcal{T} \geqslant \mathcal{D}$. ∎

Corollary. Consider a tripartite pure system *A-B-C* whose parties are ordered so $\mathcal{D}(A:B) \geqslant \mathcal{D}(B:C) \geqslant \mathcal{D}(A:C)$. Then the following inequalities hold:

$$\mathcal{D}(AB:C) \geqslant \mathcal{D}(A:C), \qquad (14)$$

$$\mathcal{D}(A:B:C) \geqslant \mathcal{D}(A:B) + \mathcal{D}(A:C), \quad (15)$$

$$\mathcal{D}(BC:A) + \mathcal{D}(AC:B) \geqslant \mathcal{D}(A:B) + \mathcal{D}(A:C). \quad (16)$$

Proof. They are directly followed by Eqs. (12), (13), and the assumption $D(A:B) \geqslant D(B:C) \geqslant D(A:C)$. On the other hand, by taking the average of (15) over all permutations of three parties, one can obtain $\mathcal{D}(A:B:C) \geqslant \tfrac{2}{3}[\mathcal{D}(A:B) + \mathcal{D}(B:C) + \mathcal{D}(A:C)]$.

In order that Eqs. (14), (15), and (16) are satisfied for any permutation of *A*, *B*, and *C*, one needs to show that

$$\mathcal{D}(AB:C) \geqslant \max\{\mathcal{D}(B:C), \mathcal{D}(A:C)\} \qquad (17)$$

for any tripartite pure system *A-B-C*. Its analytical proof is not found. However, we conjecture with numerical evidence that it holds for any three-qubit pure state. For pure states, $\mathcal{D}(AB:C) = S(C)$ and $\mathcal{D}(A:C) = \min \Lambda_{A:C}(|\vec{k}\rangle) - S(B)$. The main task of the numerical computations is the minimization of $\Lambda_{A:C}(|\vec{k}\rangle)$ over all local bases. In general, there are several local minima in $\Lambda(|\vec{k}\rangle)$, so the results of the numerical minimizations cannot guarantee that it always finds the global minimum; it just provides an upper bound. Fortunately, however, the part consisting of the minimization is on the right-hand side of the inequality (17), so we can provide numerical evidence for the inequality by testing numerous samples. We compare both sides in Eq. (17) for 10^6 random samples of pure three-qubit states. No quantum state has been found violating Eq. (17). The result is given in Fig. 2. The random samples are generated from the parametrization [28] of three-qubit states $|\psi\rangle = \lambda_0|000\rangle + e^{i\phi}\lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$. To minimize Eq. (11), we parametrized an arbitrary local basis $\{|0'\rangle, |1'\rangle\}$ as

$$\begin{aligned} |0\rangle &= \sqrt{1-t^2}|0'\rangle + t|1'\rangle, \\ |1\rangle &= e^{i\phi}(t|0'\rangle - \sqrt{1-t^2}|1'\rangle), \end{aligned} \qquad (18)$$

where $0 \leqslant e^{i\phi} \leqslant 2\pi$ and $0 \leqslant t \leqslant 1$. In fact, four real parameters are required to cover all bases, but that can be reduced to two as we only consider $\Lambda_\rho(\{|\vec{k}\rangle\})$. The number of parameters is two per each qubit, so computing $\min \Lambda_{A:C}(\{|\vec{k}\rangle\})$ requires numerical minimization over four parameters. ∎
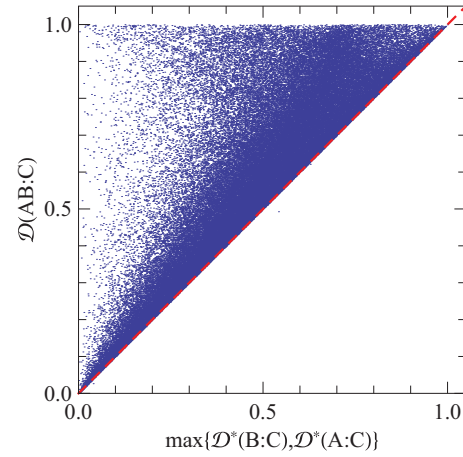

FIG. 2. (Color online) Numerical tests of the inequality (17) for 10^6 random samples of three-qubit pure states. No sample was found violating. $\mathcal{D}^*$ is approximated discord obtained from numerical minimization. Rigorously speaking, $\mathcal{D}^*$ is equal to or greater than $\mathcal{D}$.

The inequality (14) is suggestive of the monogamy inequality [29–31]. We note that they look similar, but are different. The monogamy of entanglement is a unique property of quantum states, and it is not found in correlations of classical variables. In contrast, Eq. (16) also holds for the mutual information of classical variables. The inequality (16) is studied for the global discord in Ref. [32], and it is shown that Eq. (14) is a sufficient condition for Eq. (16).

For general mixed states, we could not find any example violating Eqs. (14), (15), or (16). However, we note that they can be violated by the other permutations of *A*, *B*, and *C*. Bipartite discord of some mixed states can increase after discarding a subsystem, so Eq. (17) does not hold for general mixed states. Consider the state $\rho = p|000\rangle\langle 000| + (1-p)|1+1\rangle\langle 1+1|$, where $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$. For the state, $\mathcal{D}(BC:A) = 0$ and $\mathcal{D}(A:C) > 0$, so $\mathcal{D}(BC:A) \ngeqslant \mathcal{D}(A:C)$ for any measure of discord. In our measure of discord, $\mathcal{D}(BC:A) = 0$ and $\mathcal{D}(A:C) = \mathcal{D}(A:B:C) = \min\{p, (1-p)\}$. We leave the details of the calculation to the Appendix. In contrast, we mentioned that the relative entropy of entanglement always satisfies $\mathcal{E}(AB:C) \geqslant \mathcal{E}(B:C)$. The difference of entanglement and discord occurs from the criteria of classically correlated states and separable states. If ρ_{ABC} is separable in the bipartition *AB* and *C*, then ρ_{AC} and ρ_{BC} are also separable. However, ρ_{AC} and ρ_{BC} can be not classically correlated, although ρ_{ABC} is classically correlated in the bipartition *AB* and *C*.

On the other hand, we leave an unanswered question of whether the inequality $\mathcal{D}(A:B:C) \leqslant \mathcal{D}(BC:A) + \mathcal{D}(AC:B)$ holds for tripartite systems *A-B-C* whose parties are ordered so $\mathcal{D}(A:B) \geqslant \mathcal{D}(B:C) \geqslant \mathcal{D}(A:C)$. There is an example violating it for the other permutations of *A*, *B*, and *C*. For the state $\rho = p|000\rangle\langle 000| + (1-p)|1+1\rangle\langle 1+1|$, $\mathcal{D}(AB:C) + \mathcal{D}(BC:A) = 0$, but $\mathcal{D}(A:B:C) = \min\{p, (1-p)\}$.

We have discussed the consistent additivity relations for the multipartite measures of total correlations and quantum correlations. It would also be worth considering the relations for multipartite measures of classical correlations. In terms of the quantum relative entropy, a measure of classical

correlations is defined [10] as

$$\mathcal{C}(\rho) = \min_{\sigma \in \pi} S(\chi_\rho \| \sigma). \qquad (19)$$

Here, χ_ρ is the closest classically correlated state to ρ. It is not known whether the measure of classical correlations also has the consistent additivity relations as the measures of total correlations and quantum correlations. The part is left as an open question. On the other hand, it is evident that for the classically correlated states, $\rho \in \mathcal{C}$, those measures also follow the relations (4), (5), (6), and (7) of the total mutual information because the classical correlation is just the total correlation.

III. SPECIFIC EXAMPLES OF ADDITIVITY RELATIONS

In the preceding section, we have shown that the additivity relations from Eq. (4) to Eq. (7) are also satisfied by entanglement and discord [for discord, Eq. (7) was left unsolved, and only Eq. (16) is proved] with the help of numerical analysis, especially when the total system is in a three-qubit pure state. We note that the equality (4) for the total mutual information becomes an inequality (i.e., "=" is replaced with "⩾") when applied to entanglement and discord. In subsection A, we now present several examples of three-qubit pure states in which all multipartite entanglement, discord, and classical correlations satisfy all the additivity relations from (5) to (7) as well as the inequality (12). In the following section, we have tried to identify the validity of the additivity relationship when the total system is in a mixed state. It was possible to find specific cases where the additivity relationship is not satisfied.

A. Additivity relations of discord in pure states

We first consider a GHZ state in the form of

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Both the tripartite entanglement and discord of the GHZ state are equal to 1, while both the bipartite entanglement and discord are zero for any bipartite subsystem. Accordingly, if we let $\mathcal{Q}$ represent either $\mathcal{E}$ or $\mathcal{D}$, $\mathcal{Q}(A : B : C) = \mathcal{Q}(AB : C) = 1$ and $\mathcal{Q}(A : B) = 0$ for any permutation of A, B, and C, so that all the additivity relations from (4) to (7) are satisfied. In the case of a more general form, $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$, it is straightforward to obtain $\mathcal{Q}(A : B : C) = \mathcal{Q}(AB : C) = -|\alpha|^2 \log |\alpha|^2 - |\beta|^2 \log |\beta|^2$ and $\mathcal{Q}(A : B) = 0$. All the additivity relations are then satisfied for any permutation. In addition, we can calculate classical correlations $\mathcal{C}$ in Eq. (19). Using the fact that the closest product state to a given state is the product of its reduced states [10], we find $\mathcal{C}(A : B : C)/2 = \mathcal{C}(AB : C) = \mathcal{C}(A : B) = -|\alpha|^2 \log |\alpha|^2 - |\beta|^2 \log |\beta|^2$. Again, all the additivity relations are met.

While any bipartite subsystem of the GHZ state has zero discord, one may apply an appropriate nonlocal unitary operation to the GHZ state and obtain
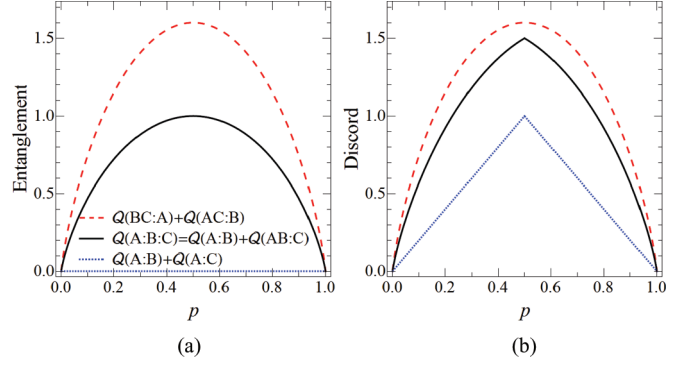
$$\frac{1}{\sqrt{2}}(|000\rangle + |{+}11\rangle)$$

FIG. 3. (Color online) Additivity relations of entanglement (left) and discord (right) for the states $|\phi\rangle = \sqrt{p}|000\rangle + \sqrt{(1-p)}|{\pm}11\rangle$, plotted versus p. Symbol $\mathcal{Q}$ is used to represent $\mathcal{E}$ or $\mathcal{D}$.

with nonzero discord $\mathcal{D}(A : B)$ and $\mathcal{D}(A : C)$. Now we examine the additivity relations for a more general state $|\phi\rangle = \sqrt{p}|000\rangle + \sqrt{1-p}|{+}11\rangle$. There is no entanglement in any bipartite subsystem of $|\phi\rangle$. We can obtain a representation for $\mathcal{E}(A : B : C)$ by using inequality (8). Let us consider a tripartite separable state $\sigma = p|000\rangle\langle 000| + (1-p)|{+}11\rangle\langle {+}11|$, and it is clear that $\mathcal{E}(A : B : C) \leqslant S(|\phi\rangle\langle\phi\|\sigma) = -p \log p - (1-p) \log(1-p)$. However, it is also true that $\mathcal{E}(A : B) + \mathcal{E}(AB : C) = -p \log p - (1-p) \log(1-p)$, so $\mathcal{E}(A : B : C) = \mathcal{E}(A : B) + \mathcal{E}(AB : C) = -p \log p - (1-p) \log(1-p)$ from Eq. (8). Bipartite entanglement is easily found as $\mathcal{E}(AC : B) = -p \log p - (1-p) \log(1-p)$ and $\mathcal{E}(BC : A) = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-$, where $\lambda_\pm = 1/2 \pm \sqrt{1/4 - p(1-p)/2}$. Now one can check that all the additivity relations are satisfied. For the other permutations of A, B, and C, the results are the same except for a change of "=" in (4) to "⩾." We plot $\mathcal{E}(A : B : C)$, $\mathcal{E}(BC : A) + \mathcal{E}(AC : B)$, and $\mathcal{E}(A : B) + \mathcal{E}(A : C)$ in Fig. 3(a).

In the case of the relative entropy of discord, one can show that $\mathcal{D}(A : B : C) = -p \log p - (1-p) \log(1-p) + \min\{p, (1-p)\}$ and $\mathcal{D}(A : B) = \mathcal{D}(A : C) = \min\{p, (1-p)\}$. Their proofs are straightforward (see the Appendix). We also simply obtain $\mathcal{D}(BC : A) = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-$ and $\mathcal{D}(AC : B) = -p \log p - (1-p) \log(1-p)$, and find that all the additivity relations are satisfied. For the other permutations, again, the same results are obtained except for the change from "=" to "⩾" in Eq. (4). The results are plotted in Fig. 3. In addition, we examine the additivity relations for the measure of classical correlations. The classical correlations are obtained as

$$\mathcal{C}(A : B : C) = -\frac{q}{2} \log \frac{q}{2} - \left(1 - \frac{q}{2}\right) \log \left(1 - \frac{q}{2}\right)$$
$$- q \log q - (1-q) \log(1-q) - q,$$
$$\mathcal{C}(A : B) = -\frac{q}{2} \log \frac{q}{2} - \left(1 - \frac{q}{2}\right) \log \left(1 - \frac{q}{2}\right) - q,$$
$$\mathcal{C}(AB : C) = -q \log q - (1-q) \log(1-q),$$
$$\mathcal{C}(BC : A) = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-,$$

where $q = \min\{p, 1 - p\}$, and all the additivity relations are met.

We now consider a W state

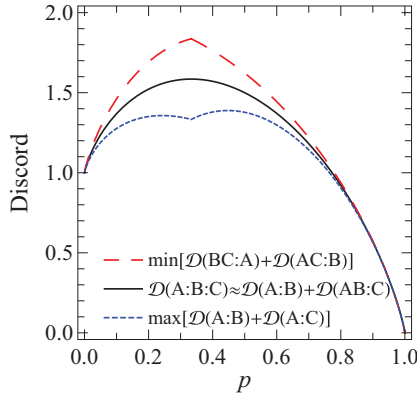$$|\text{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$

FIG. 4. (Color online) Additivity relations of discord for the generalized W states $\sqrt{p}|001\rangle + \sqrt{\frac{1-p}{2}}(|010\rangle + |100\rangle)$, plotted versus p. Here, the min (max) stands for taking the minimum (maximum) over all permutations of A, B, and C.

with which it is known that $\mathcal{E}(BC : A) = \mathcal{E}(AC : B) = \log 3 - 2/3 \approx 0.92$, $\mathcal{E}(A : B : C) = 2 \log 3 - 2 \approx 1.17$ [33], and $\mathcal{E}(A : B) = \mathcal{E}(A : C) = \log 3 - 4/3 \approx 0.25$ [15]. All the additivity relations are then satisfied. In a more general case of

$$\alpha|001\rangle + \beta|010\rangle + \gamma|100\rangle,$$

the additivity relations of discord can be examined as follows. First, quantum discord with bipartition AB and C is easily obtained to $\mathcal{D}(AB : C) = -|\alpha|^2 \log |\alpha|^2 - (|\beta|^2 + |\gamma|^2) \log(|\beta|^2 + |\gamma|^2)$ because the total system is pure. $\mathcal{D}(A : C)$ and $\mathcal{D}(A : B : C)$ are upper bounded respectively by $-|\alpha|^2 \log(|\alpha|^2/|\alpha|^2 + |\gamma|^2) - |\gamma|^2 \log(|\gamma|^2/|\alpha|^2 + |\gamma|^2)$ and $-|\alpha|^2 \log |\alpha|^2 - |\beta|^2 \log |\beta|^2 - |\gamma|^2 \log |\gamma|^2$ as we choose the logical basis in the expression (11). The inequality (5) is then satisfied for discord as

$$\mathcal{D}(AB : C) - \mathcal{D}(A : C) \geqslant -(|\alpha|^2 + |\gamma|^2) \log(|\alpha|^2 + |\gamma|^2)$$
$$- (|\beta|^2 + |\gamma|^2) \log(|\beta|^2 + |\gamma|^2)$$
$$+ |\gamma|^2 \log |\gamma|^2 \geqslant 0.$$

The last inequality holds because for given $|\gamma|$, $-(|\alpha|^2 + |\gamma|^2) \log(|\alpha|^2 + |\gamma|^2) - (|\beta|^2 + |\gamma|^2) \log(|\beta|^2 + |\gamma|^2)$ is minimized at $\alpha = 0$ or $\beta = 0$. The inequality (7) is similarly satisfied as

$$\mathcal{D}(AB : C) + \mathcal{D}(A : BC) - \mathcal{D}(A : B : C)$$
$$\geqslant -(|\alpha|^2 + |\beta|^2) \log(|\alpha|^2 + |\beta|^2)$$
$$- (|\beta|^2 + |\gamma|^2) \log(|\beta|^2 + |\gamma|^2)$$
$$+ |\beta|^2 \log |\beta|^2 \geqslant 0.$$

The inequalities also hold for the other permutations of A-B-C, because the procedure is invariant under permutations. In Fig. 4, we plot the numerical result for the special case $|\alpha|^2 = p$ and $|\beta|^2 = |\gamma|^2 = \frac{1-p}{2}$. It shows that $\mathcal{D}(A : B : C) \approx \mathcal{D}(A : B) + \mathcal{D}(AB : C)$, so Eq. (4) approximately holds.

B. Additivity relations of discord in mixed states

Our arguments about the additivity relations of discord were restricted to pure states. We noted that the additivity relations of discord are violated by some mixed systems whose ordering does not satisfy $\mathcal{D}(A : B) \geqslant \mathcal{D}(B : C) \geqslant \mathcal{D}(A : C)$. In this subsection, we examine a couple of mixed state example states
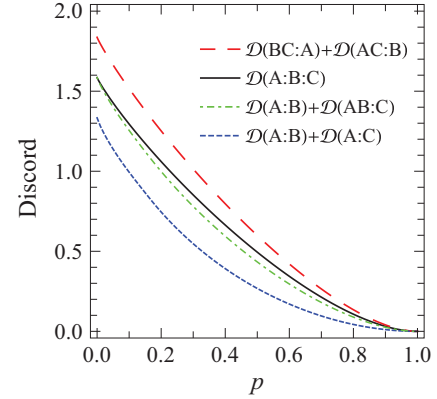

FIG. 5. (Color online) Additivity relations of discord for the mixed state $\rho = (1 - p)|\mathrm{W}\rangle\langle\mathrm{W}| + p\mathcal{I}/8$, plotted versus p.

in order to check whether the additivity relations of discord hold. For the state

$$\rho = (1 - p)|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + p\mathcal{I}/8,$$

where $\mathcal{I}$ is the identity matrix, $\mathcal{D}(A : B : C) = \mathcal{D}(BC : A)$ and $\mathcal{D}(A : B) = 0$, so the additivity relations are trivially met. The state

$$\rho = (1 - p)|\mathrm{W}\rangle\langle\mathrm{W}| + p\mathcal{I}/8$$

also satisfies the additivity relations and it is shown in Fig. 5. The state $\rho = (1 - p)|\mathrm{W}\rangle\langle\mathrm{W}| + \frac{p}{2}(|000\rangle\langle000| + |1+1\rangle\langle1+1|)$ satisfies the ordering required, so Eq. (6) is satisfied for all p, but $\mathcal{D}(A : B : C) \not\geqslant \mathcal{D}(A : B) + \mathcal{D}(B : C)$ for $p > p^*$, where $p^* \approx 0.8$.

IV. CONCLUSION

We have studied four additivity relations of the measures of multipartite correlations which are based on the quantum relative entropy. The relations were motivated by the definition of the total mutual information and the strong subadditivity of von Neumann entropy. They specify multipartite correlations in terms of bipartite correlations in subsystems, so they give insight of to what extent multipartite correlations contain bipartite correlations of subsystems. We have identified the additivity relations of entanglement, and argued that discord has the additivity relations for pure states, but it is not always true for mixed states.

APPENDIX

Let us compute the relative entropy of discord of the state $\rho = p|000\rangle\langle000| + (1 - p)|1+1\rangle\langle1+1|$. First, we argue that taking minimization over all local base of B is enough to obtain $\mathcal{D}(A : B : C)$ of the state. Consider a measure of discord $\mathcal{D}^*(\rho_{ABC}) = \min_{\sigma \in \mathcal{C}^*} S(\rho\|\sigma)$, where $\mathcal{C}^*$ is the set of all quantum states in the form $\sigma = \sum_l p_l|i\rangle_B\langle i| \otimes \sigma_{AC}^i$, where $\{|i\rangle_B\}$ is a local basis of B, and σ_{AC}^i's are arbitrary

quantum states of AC. Apparently, $\mathcal{D}^* \leqslant \mathcal{D}$. It can be shown that the minimum is obtained for a state of the form $\sigma = \sum |i\rangle\langle i| \otimes \langle i|\rho|i\rangle$, as $\mathcal{D}$ can be reduced to the expression (11). On the other hand, in the case of our target state, σ is also a classically correlated state for every choosing of the local basis of B. Therefore, $D^* = D$ in this case. Now, the local base of A can be parametrized as Eq. (18), and simple

differentiations find two local minima of $S(\rho\|\sigma)$ at $\sigma^* = p|000\rangle\langle000| + \frac{(1-p)}{2}(|101\rangle\langle101| + |111\rangle\langle111|)$ and $\sigma^{**} = \frac{p}{2}(|0+0\rangle\langle0+0| + |0-0\rangle\langle0-0|) + (1-p)|1+1\rangle\langle1+1|$. Consequently, $\mathcal{D}(A : B : C) = \mathcal{D}(A : B) = \min\{p,(1-p)\}$, and then discord of the state $|\psi\rangle = \sqrt{p}|000\rangle + \sqrt{1-p}|111\rangle$ is computed to $\mathcal{D}(A : B : C) = -p\log p - (1-p)\log(1-p) + \min\{p,(1-p)\}$ by applying (12).

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[21] p_i can be a probability of an arbitrary system. If it is a probability of system A-B-C, it can be denoted as $p_{ABC}(i,j,k)$. Once the probabilities of the total system are given, the probabilities of the subsystem can be given as its marginal, as, e.g., $p_{AB}(i,j) = \sum_k p_{ABC}(i,j,k)$, $p_{AC}(i,k) = \sum_j p_{ABC}(i,j,k)$, $p_{BC}(j,k) = \sum_i p_{ABC}(i,j,k)$, and so on.

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