Quantum metrology for nonlinear phase shifts with entangled coherent states

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We investigate the phase enhancement of quantum states subject to nonlinear phase shifts. The optimal phase estimation of even entangled coherent states (ECSs) is shown to be better than that of NOON states with the same average particle number $\langle n\rangle$ and nonlinearity exponent k. We investigate the creation of an approximate entangled coherent state (AECS) from a photon-subtracted squeezed vacuum with current optical technology methods and show that a pure AECS is even better than an even ECS for large $\langle n\rangle$. Finally, we examine the simple, but physically relevant, cases of loss in the nonlinear interferometer for a fixed average photon number $\langle n\rangle$.

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I. INTRODUCTION

Quantum metrology is a research field that examines the characteristic fundamental properties of measurements under the laws of quantum mechanics [1,2]. The ultimate goal of this is to achieve measurements at the information theoretical bounds allowed by the laws of quantum mechanics, far beyond their classical counterparts. For optical systems the classic and extremely well studied examples are NOON states [1–6], whose performance allows them to measure linear phase shifts at the Heisenberg limit. Several theoretical studies have recently investigated the role of nonlinearity to help improve the limits of phase enhancement in linear systems [7–11], and the first demonstration of this so-called super-Heisenberg scaling has been shown [12]. The particle-loss and decoherence mechanisms are, however, not fully explored in combined linear and nonlinear interferometers, even theoretically [13–20].

It is not only NOON states that allow linear phase measurements at the Heisenberg limit. Entangled coherent states (ECSs) [21–28] are also able to do this [29–31] and can outperform NOON states in the region of very modest particle numbers with a linear phase operation [32,33]. An important case of entangled coherent states is the two-mode path-entangled state, a state analogous to a NOON state but with one of two modes containing a coherent state rather than a Fock state [30]. This particular ECS can be represented as a superposition of NOON states with different photon numbers [29]. Using linear optical elements, the phase sensitivity of ECSs outperforms that of NOON and states that are created by passing a number-squeezed state through a beam splitter [34], both without [29–31] and with losses [32], because coherent states maintain their properties in the presence of loss. Given all this recent work, a natural question arises regarding the comparison of the phase enhancement for ECSs to NOON and other states in the case of nonlinear phase shifts [7].

There is significant motivation for being able to optimally measure the strength of uncalibrated nonlinearities. As already mentioned, known and calibrated nonlinearities have metrology potential for enhancing the estimation of the linear phase [7–11]. Furthermore, calibrated nonlinearities have potential for nonabsorbing single-photon detection [35–37] and optical quantum computing [38].

In this paper we are therefore going to investigate the nonlinear phase enhancement resulting from a generalized nonlinearity characterized by an exponent k ($k>0$) on four quantum states: NOON, even and odd ECSs, and an approximated ECS (AECS). The AECS is created from a photon-subtracted squeezed state and is experimentally feasible to realize [39,40]. Potential enhancements will be quantified with the quantum Fisher information [41,42]. To begin we will consider a two-mode pure state $|\psi\rangle_{12}$ and a generalized nonlinear phase shifter $U(\phi,k)$ applied to one mode given by

$$U(\phi,k) = e^{i\phi(a_2^\dagger a_2)^k}, \tag{1}$$

where $a_i^\dagger$ (a_i) is a creation (annihilation) operator in spatial mode i [43] (see the details in Sec. II). The exponent k represents the order of the nonlinearity. For example, $k=1$ corresponds to a linear phase shift on the state, $k=2$ is a Kerr phase shift, and $k\neq 2$ gives a more general nonlinear effect in a phase operation. With optical interferometric scenarios in mind, we consider the case of applying the phase operation to just one mode, although it should be noted that other cases can be envisaged [44]. When the generalized phase operation $U(\phi,k)$ is applied to mode 2 of $|\psi\rangle_{12}$, the resultant state is equal to

$$|\psi^k(\phi)\rangle_{12} = [\mathbb{1}\otimes U(\phi,k)]|\psi\rangle_{12}. \tag{2}$$

Now, according to phase estimation theory [41,42], the phase uncertainty is bounded by the quantum Fisher information,

$$\delta\phi \geqslant \frac{1}{\sqrt{\mu F}} \geqslant \frac{1}{\sqrt{\mu F^Q}}, \tag{3}$$

where F and μ denote classical Fisher information and the number of measurements, respectively, and the value of F^Q for pure states is simply given by

$$F^Q = 4[\langle\tilde{\psi}^k|\tilde{\psi}^k\rangle - |\langle\tilde{\psi}^k|\psi^k(\phi)\rangle|^2] = 4(\Delta n^k)^2, \tag{4}$$

with $|\tilde{\psi}^k\rangle = \partial|\psi^k(\phi)\rangle/\partial\phi$ and $(\Delta n^k)^2 = \langle(n^k)^2\rangle - \langle n^k\rangle^2$ ($\langle n^k\rangle = {}_{12}\langle\psi|(a_2^\dagger a_2)^k|\psi\rangle_{12}$). It is important to note that $\langle n^1\rangle$ denotes an average (or mean) photon number. For a specific measurement scenario, the number of measurements μ plays a key role in reaching optimal phase estimation [45–47], and particularly, parity measurement using the maximum-likelihood method shows a good approximation of optimal phase estimation for pure states [32] (see the details in the Appendix). However, we here focused on the optimal measurement setup, which provides a saturation lower bound with $F = F^Q$ assumed by $\mu = 1$ [32,48].

To allow a fair comparison of the phase sensitivity among the various different quantum states under consideration, we will use the same average photon number in one of two modes as the physical resource count for the states [29,43,49,50]. For pure states, we shall demonstrate an inequality for the sensitivity among the three states: NOON (least sensitive), odd ECS, and even ECS (most sensitive). Furthermore we will show that in the limit of large $\langle n^1\rangle$ the AECS is slightly better than the other three states. We shall also consider a small amount of loss for the dispersive and nonlinear interferometer arm because the nonlinear medium providing the phase operation will generally also provide a scattering effect (e.g., particle losses) [51]. We shall demonstrate that, analogous to the linear case ($k = 1$) [32,33], the phase enhancement of ECSs still outperforms that of NOON states, even for nonlinear cases ($k \neq 1$).

For a physical realization, it is known that the ideal ECS can be generated by mixing through a 50:50 beam splitter (BS) a coherent state and a coherent state superposition (CSS) [28], given by

$$|\mathrm{CSS}_\pm(\alpha)\rangle = N_\alpha^\pm(|\alpha\rangle \pm |-\alpha\rangle), \tag{5}$$

where $|\alpha\rangle$ is a coherent state with amplitude α and $N_\alpha^\pm = 1/\sqrt{2(1 \pm e^{-2|\alpha|^2})}$. Since the CSS with small α has been already demonstrated in experiments [52–55], the scheme of building AECSs with a modest photon number may become experimentally feasible with very high fidelity in the near future [56]. Several experiments have demonstrated that nonlinear phase operations can be realized in various setups. For example, self-Kerr phase modulation ($k = 2$) has been measured as a function of electric-field amplitudes in water, fibers, nitrobenzene, Rydberg states, etc. [51,57,58]. Superconducting systems with strong interaction could be a possible candidate of nonlinear phase shifters for $k = 3$ [59]. Notably, the phase shift dependent upon the applied field clearly follows theoretical predictions in the case of a Rydberg electromagnetically induced transparency medium (see Fig. 3 in Ref. [51]).

The remainder of this paper is organized as follows. In Sec. II, the mathematical formalism for generalized phase shifters is described. The phase enhancements of pure ECSs is discussed in the case of ideal preparation in Sec. II A. A feasible approach to implementing the AECS is given in Sec. II B. In Sec. III we also investigate the effects of loss on the phase enhancement behavior. Finally, in Sec. IV we summarize and discuss our results.

II. OPTIMAL PHASE ESTIMATION USING NONLINEARITY IN PURE STATES

Let us first discuss the validity of Eq. (1) for general k. The generalized phase operation is formed by

$$U(\phi) = \exp[i\hat{H}(\phi)t/\hbar], \tag{6}$$

where the total Hamiltonian is equal to

$$\hat{H}(\phi) = \hat{H}_0 + \hat{H}_{\mathrm{int}}(\phi), \tag{7}$$

consisting of the unperturbed Hamiltonian represented by

$$\hat{H}_0 = \int d^3r\left[\frac{1}{2\mu_0}|\hat{B}|^2 + \frac{\epsilon_0}{2}|\hat{E}|^2\right] = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{8}$$

for mode frequency ω and an interaction Hamiltonian given by expanding the polarization of the nonlinear medium,

$$\hat{H}_{\mathrm{int}}(\phi) = \int d^3r\,[\hat{E}\cdot\hat{P}] = \int d^3r\,\hat{E}\left[\sum_{j=1}^{\infty}\frac{\chi^{(j)}}{j+1}(\hat{E})^j\right], \tag{9}$$

where $\chi^{(j)}$ is the jth order susceptibility of the medium [60]. A single-mode electric field is given by

$$\hat{E} = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}[au(r) - a^\dagger u^*(r)], \tag{10}$$

where $u(r)$ is the mode function. Due to the lack of phase matching, the single-mode assumption, and the rotating wave approximation, we may neglect the terms in $\chi^{(2x)}$ (x is a positive integer) [60], and then,

$$U(\phi,k) = \exp\left[i\omega t\left(a^\dagger a + \frac{1}{2}\right)\right]\prod_{k=1}^{\infty}\exp[i\phi^{(k)}(a^\dagger a)^k], \tag{11}$$

where the phase parameter of the nonlinearity k is

$$\phi^{(k)} = t\int d^3r\sum_{x=1}^{\infty}\mathcal{F}(\chi^{(2x-1)}). \tag{12}$$

We note that $\mathcal{F}(\chi^{(2x-1)})$ is a function of $\chi^{(2x-1)}$. Therefore, the expression of the nonlinear phase operation in Eq. (1) is appropriate for fixed k.

A well-known example of a nonlinear phase operation is given by the Kerr interaction for $k = 2$ [29,30,61]. In an interaction picture that removes the linear dynamical phase, the nonlinear component is

$$U(\phi,2) = \exp[i\phi^{(2)}(a^\dagger a)^2], \tag{13}$$

with

$$\phi^{(2)} = t\int d^3r\left(\frac{3}{2}\chi^{(3)} + 5\chi^{(5)}\right), \tag{14}$$

where the interaction Hamiltonian is truncated after the fifth-order susceptibility $\chi^{(5)}$.

A. Ideal (theoretical) cases

We now need to review and calculate the phase enhancements using the quantum Fisher information for pure (no loss) cases of the NOON state and ECSs. The NOON state is defined as [4]

$$|\psi_N\rangle_{12} = \frac{1}{\sqrt{2}}(|N\rangle_1|0\rangle_2 + |0\rangle_1|N\rangle_2), \tag{15}$$

where $|N\rangle$ is a number state with photon number N. After a generalized phase shifter $U(\phi,k)$ is applied in mode 2, the resulting state is given by $|\psi_N^k\rangle_{12} = [\mathbb{1} \otimes U(\phi,k)]|\psi_N\rangle_{12}$. From Eq. (4), the quantum Fisher information of the pure NOON states with a nonlinearity of order k is given by $F^Q_{N^k} = N^{2k}$ and

$$\delta\phi_{N^k} > \frac{1}{N^k}. \tag{16}$$

Similarly, the even and odd ECSs are defined by

$$|\mathrm{ECS}_\pm(\alpha_\pm)\rangle = \mathcal{N}^\pm_{\alpha_\pm}[|\alpha_\pm\rangle_1|0\rangle_2 \pm |0\rangle_1|\alpha_\pm\rangle_2], \tag{17}$$

with amplitude α_+ (α_-) for an even (odd) ECS and $\mathcal{N}^\pm_{\alpha_\pm} = 1/\sqrt{2(1 \pm e^{-|\alpha_\pm|^2})}$. After the phase shifter $U(\phi,k)$ is performed in mode 2, we find that the resulting state is given by

$$|\mathrm{ECS}^k_\pm(\alpha_\pm,\phi)\rangle_{12} = [\mathbb{1} \otimes U(\phi,k)]|\mathrm{ECS}_\pm(\alpha_\pm)\rangle_{12}. \tag{18}$$

The quantum Fisher information is then given by

$$F^Q_{E^k_\pm} = 4 f_{\alpha_\pm}\left[\sum_{n=0}^{\infty}\frac{n^{2k}(\alpha_\pm)^{2n}}{n!} - f_{\alpha_\pm}\left(\sum_{n=0}^{\infty}\frac{n^k(\alpha_\pm)^{2n}}{n!}\right)^2\right] \tag{19}$$

for $f_{\alpha_\pm} = e^{-|\alpha_\pm|^2}(\mathcal{N}^\pm_{\alpha_\pm})^2$ and

$$\delta\phi_{E^k_\pm} > \frac{1}{\sqrt{F^Q_{E^k_\pm}}}. \tag{20}$$

In order to compare the phase sensitivity of the different states, we take into account the same *average* particle number of the states [29,43,49,50] in an arm,

$$\langle n^1_N\rangle = \langle n^1_{E_\pm}\rangle = \frac{N}{2} = (\mathcal{N}^\pm_{\alpha_\pm})^2|\alpha_\pm|^2, \tag{21}$$

where $\langle n^1_{E_\pm}\rangle = \langle \mathrm{ECS}_\pm(\alpha_\pm)|a_2^\dagger a_2|\mathrm{ECS}_\pm(\alpha_\pm)\rangle$ and, in general, $\alpha_+ \neq \alpha_-$. In Fig. 1, the values of the optimal phase estimation are plotted for the three quantum states and are satisfied with the inequality

$$\delta\phi_{N^k} \geqslant \delta\phi_{E^k_-} \geqslant \delta\phi_{E^k_+} \tag{22}$$

for any N and k. The first inequality $\delta\phi_{N^k} \geqslant \delta\phi_{E^k_-}$ shows the pattern of the difference $\delta\phi_{N^k} - \delta\phi_{E^k_-}$ for $k = 1,2,3$ in Fig. 2. Note that $\delta\phi_{E^k_\pm}$ approaches $\delta\phi_{N^k}$ because $|\mathrm{ECS}_\pm(\alpha)\rangle \approx |\psi_N\rangle$ for larger N and that $\delta\phi_{E^k_\pm}$ is a continuous value, while $\delta\phi_{N^k}$ exists discretely due to integer N.

B. Preparation of an approximate ECS

We now present an optical setup to create an AECS with a very high fidelity to an odd ECS, based on current optical technology methods. We also compare the phase enhancement of the AECS with the other states. As shown in Fig. 3, two steps are required for AECS preparation. First, we create a photon-subtracted quantum state, with high fidelity to an odd

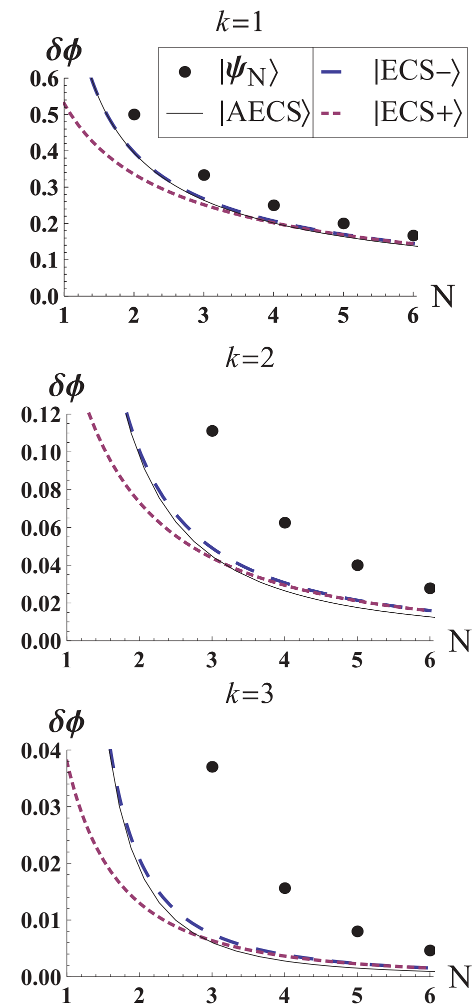

FIG. 1. (Color online) The inequality $\delta\phi_{N^k} \geqslant \delta\phi_{E^k_-} \geqslant \delta\phi_{E^k_+}$ with respect to $N = 2\langle n^1_N\rangle = \langle n^1_{E_\pm}\rangle$ ($k = 1,2,3$).

CSS, given by the scheme in Ref. [56]. The generation of squeezed vacuum states can be given by degenerate parametric down conversion utilizing nonlinearity [28], and a series of experimental results shows that modest strengths of squeezing through second-harmonic generation are achievable with the current technology [62–64]. The scheme of single-photon subtraction through an unbalanced BS from a squeezed vacuum has been already demonstrated [39,40]. Finally, generation of

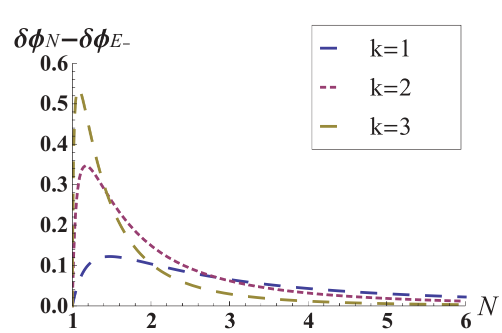

FIG. 2. (Color online) Difference of the optimal phase estimation between $|\psi_N\rangle$ and $|\mathrm{ECS}^k_-(\alpha_-)\rangle$, such as $\delta\phi_{N^k} - \delta\phi_{E^k_-}$ for $k = 1,2,3$.

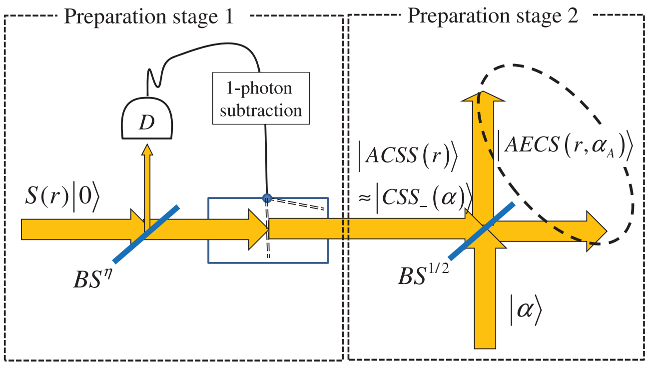

FIG. 3. (Color online) Schematics of generating $|\text{AECS}(r,\alpha_A)\rangle$ from a squeezed vacuum $S(r)|0\rangle$ and a coherent state $|\alpha\rangle$. After single-photon subtraction at stage 1, state $|\text{ACSS}(r)\rangle$ is very similar to $|\text{CSS}_{-}(\alpha)\rangle$. The coherent state in stage 2 has the same amplitude of the old CSS (BS^{η} and D are an unbalanced BS with a transmission rate η and a single-photon detector).

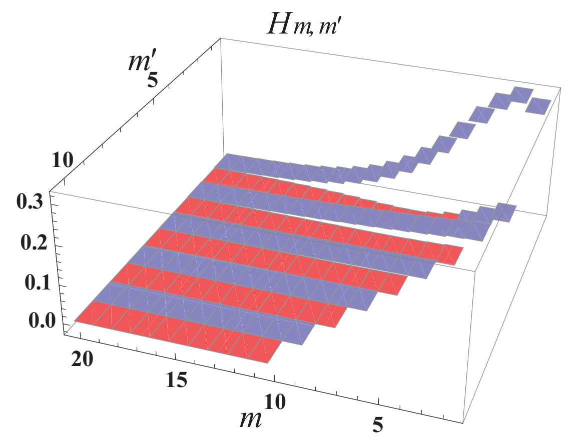

FIG. 4. (Color online) The coefficient $H(m,m')$ of $|\text{AECS}(r_0,\alpha_A)\rangle$ ($\alpha_A \approx 1.9807$). The major contribution of the NOON state and the minor of m and m' states is clear. Blue (dark gray) indicates a positive value, and red (light gray) indicates a negative value.

the odd ECS follows from the well-known technique of mixing a traveling CSS with a controlled coherent state through a 50:50 BS.

In Fig. 3, the schematics shows how to generate $|\text{AECS}(r,\alpha_A)\rangle$ from a squeezed vacuum $S(r)|0\rangle$ and a coherent state $|\alpha\rangle$ ($S(r) = \exp\{-\frac{r}{2}[a^2 - (a^{\dagger})^2]\}$ and r is a squeezing parameter). It was shown that the fidelity between a squeezed single-photon state $S(r)|1\rangle$ and an odd CSS $|\text{CSS}_{-}(\alpha)\rangle$ with small α is extremely high [56] and that a photon-subtracted squeezed-vacuum state $aS(r)|0\rangle$ is identical to $S(r)|1\rangle$ [65]. We begin by preparing a squeezed vacuum $S(r)|0\rangle$ and then performing single-photon subtraction by using BS^{η} (η is the transmission rate) and a single-photon detector. The resultant state is called an ACSS and possesses a very high fidelity compared with the ideal odd CSS. In detail, when a single photon is detected, the resultant state $|\text{ACSS}(r_0)\rangle = aS(r_0)|0\rangle$ ($r_0 = \text{arcsinh}[2\alpha_0/3]$) is given by

$$|\text{ACSS}(r_0)\rangle = f_r \sum_{k=0}^{\infty} \frac{\sqrt{(2k+1)!}}{2^k k!} (\tanh r_0)^k |2k+1\rangle \qquad (23)$$

for $f_r = (1 - \tanh^2 r_0)^{3/4}$ with the maximized fidelity between $|\text{ACSS}(r_0)\rangle$ and $|\text{CSS}_{-}(\alpha_0)\rangle$.

In the second stage, the odd AECS can be built with the generated ACSS $|\text{ACSS}(r_0)\rangle_1$ mixed with an extra coherent state $|\alpha_0\rangle_2$ through a 50:50 BS. The state is written as

$$\begin{aligned} |\text{AECS}(r_0,\alpha_A)\rangle &= \sum_{m=1}^{\infty} \sum_{m'=0}^{m-1} [H_{m,m'}(|m\rangle|m'\rangle - |m'\rangle|m\rangle)], \\ &\approx |\text{ECS}_{-}(\alpha_{-})\rangle, \end{aligned} \qquad (24)$$

where $H_{m,m'}$ is the coefficient of the state in a Fock basis ($\alpha_A = \sqrt{2}\alpha_0$). As shown in Fig. 4, the resultant state $|\text{AECS}(r_0,\alpha_A)\rangle$ is approximately equivalent to the desired odd ECS $|\text{ECS}_{-}(\alpha_{-})\rangle$ with high fidelity ($\approx$0.975) if $\alpha_{-} = \alpha_A \approx 1.9807$. The state $|\text{AECS}(r_0,\alpha_A)\rangle$ consists of the ideal ECS ($m' = 0$) and the unbalanced photon states, called m and m' states ($m' \neq 0$) [66]. In other words, the outcome state contains a superposition of NOON states for $m' = 0$, and it also includes the states possessing unbalanced photon numbers in both modes for $m' \neq 0$.

Then, after the generalized phase shifter $U(\phi,k)$ in mode 2, we can estimate the phase enhancement of the final state given by $|\text{AECS}^k(r_0,\alpha_A,\phi)\rangle = [\mathbb{1} \otimes U(\phi,k)]|\text{AECS}(r_0,\alpha_A)\rangle$.

For the phase enhancement of the AECS compared with ideal ECSs, the value of $\langle n^1_{E_A}\rangle = \langle \text{AECS}\, |a_2^{\dagger}a_2|\text{AECS}\rangle$ should be $N/2$ in Eq. (21). As shown in Fig. 1, it is true that the results from AECS are very close to those from ideal odd ECSs for $k = 1,2$ but are slightly better than those of the other states at $k = 3$ for a modest number of N (see thin black lines in Fig. 1). This is because the detailed shape of the AECS is slightly different from the ideal odd ECS. In Fig. 5, this advantage of the phase enhancement can be explained by the fact that the distribution of $H(m,0)$ is narrower but has a longer tail in the AECS compared to a coherent state $|2.0\rangle$, indicating a photon distribution of $|\text{ECS}_{\pm}(\alpha_{\pm})\rangle$, for the same average photon number. The m and m' states might provide a minor contribution of phase enhancement in the AECS.

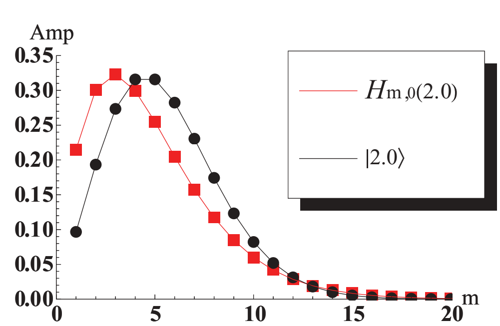

FIG. 5. (Color online) The amplitude of $H_{m,0}(\alpha_A)$ (Amp) for $\alpha_A = 2.0$ and that of coherent state $|2.0\rangle$ with respect to photon number m. Both states contain the same average photon number. This implies that the AECS reaches the peak amplitude in smaller m and has a longer tail than the coherent state.

III. SMALL LOSS CASES IN A NONLINEAR PHASE OPERATION

The phenomenon of imperfect phase operations may lead to small particle losses in the arm. In particular, losses of particles could occur in the nonlinear phase operation itself. However, for the metrology, detection, and quantum information applications mentioned in the Introduction, it is important that losses, while realistically nonzero, are only small. The focus of our interest here is thus this low-loss-application regime. For the comparison of phase enhancement in lossy states, we choose the fixed average photon number of the states, such as

$$\langle n_N^1\rangle = \langle n_{E_+}^1\rangle = \langle n_{E_-}^1\rangle = \langle n_{E_A}^1\rangle = 2.0, \tag{25}$$

which implies $N = 4$. For example, $|\mathrm{AECS}(r_0,2.0)\rangle$ and $|\mathrm{ECS}_-(1.9807)\rangle$ have the same average photon number, such as $\langle n_{E_-}^1\rangle = \langle n_{E_A}^1\rangle = 2$.

Adding a BS with vacuum input can mimic this lossy condition in the dispersive interferometer arm after the nonlinear phase shift (T is the transmission rate of the BS), while injecting a BS before the nonlinear phase operation is reasonable if the mixed state comes from environmental losses in the preparation stage [16]. Here, we examine the phase enhancement of mixed states by generalizing the results of Ref. [16], for example,

$$\delta\phi \geqslant \frac{1}{\sqrt{F^Q}} \geqslant \frac{1}{\sqrt{C_k^Q}}, \tag{26}$$

where $C_k^Q = 4(\langle H_1^k\rangle - \langle H_2^k\rangle^2)$ for any k [67]. In particular, this equation shows an excellent match with the exact value of the quantum Fisher information in the small-loss region ($T \approx 1$) [16]. For $k = 1$, the bound C^Q is given by

$$C_1^Q = 4[T^2(\langle n^2\rangle - \langle n\rangle^2) + T(1-T)\langle n\rangle], \tag{27}$$

and for $k = 2$,

$$\begin{aligned} C_2^Q = 4\{&T^4\langle n^4\rangle + 6T^3(1-T)\langle n^3\rangle \\ &+ T^2(1-T)(3-11T)\langle n^2\rangle + T(1-T) \\ &\times(1-6T+6T^2)\langle n\rangle - [T^4\langle n^2\rangle^2 + 2T^3(1-T)\langle n\rangle\langle n^2\rangle \\ &+ T^2(1-T)^2\langle n\rangle^2]\}. \end{aligned} \tag{28}$$

As shown in Fig. 6, the ECSs, including AECS, significantly outperform NOON states (thick solid lines) for $k = 1,2$ in a small photon-loss window. For NOON, even and odd ECSs, and AECSs, the expectation values in Eqs. (27) and (28) are given by

$$\langle n^k\rangle = N^k/2, \tag{29}$$

$$\langle n_{E_\pm}^k\rangle = f_{\alpha_\pm}\sum_{n=0}^{\infty}\frac{n^k(\alpha_\pm)^{2n}}{n!}, \tag{30}$$

$$\begin{aligned} \langle n_{E_A}^k\rangle &= \langle \mathrm{AECS}(r,\alpha_A)|(a_2^\dagger a_2)^k|\mathrm{AECS}(r,\alpha_A)\rangle. \\ &= \sum_{m=1}^{\infty}\sum_{m'=0}^{m-1}\{(H_{m,m'})^2[m^k + (m')^k]\}. \end{aligned} \tag{31}$$

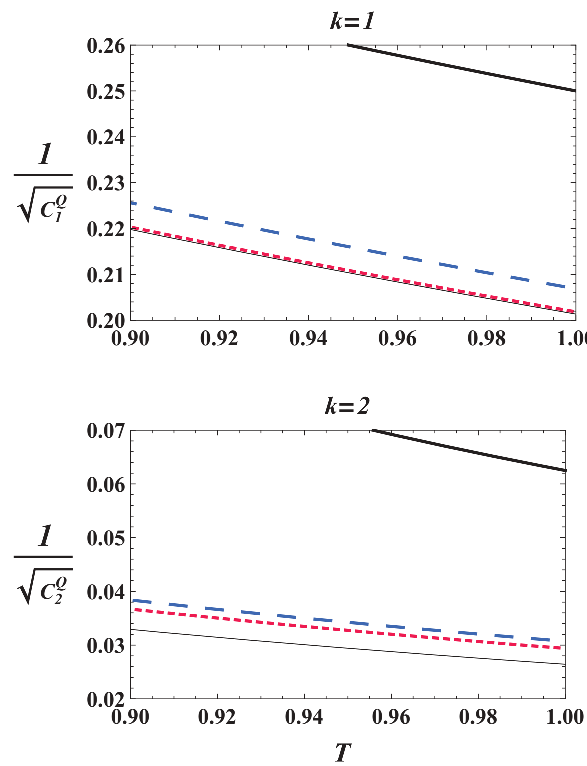

FIG. 6. (Color online) The bound of the optimal phase sensitivity ($1/\sqrt{C_k^Q}$) with small lossy conditions for $k = 1,2$ (T is the transmission rate of the BS mimicking photon losses). The thick solid line is for NOON states, and the thin solid line is for AECS, which is very similar to the dashed lines for even (short-dashed line) and odd (long-dashed line) ECSs. The classical precision limit (uncorrelated states) does not depend on the nonlinearity k, and its value starts above 0.5 for $\langle n^1\rangle = 2.0$.

Therefore, these results show that ECSs still outperform the phase enhancement achieved by NOON states in the region of small losses after the nonlinear phase operation ($k = 2$).

IV. SUMMARY AND REMARKS

In summary, we have analyzed phase enhancement of ECSs for nonlinear phase shifts using quantum Fisher information to quantify the results. As shown in linear optical elements [32,33], the phase sensitivity of ECSs outperforms that of NOON states for modest average photon numbers, converging to the limit of NOON states for large average photon numbers. We have presented the form of generalized (nonlinear) phase operations in terms of the power of number operators and obtained an inequality for the phase enhancement of NOON states and odd and even ECSs, all with respect to the same average photon number as a physical resource. We have also investigated the feasibility of creating an AECS in optical setups based on current technology and examined its phase sensitivity. Finally, we have shown that the behavior of the phase sensitivity for ECSs significantly outperforms that for NOON states for the $k = 2$ nonlinear example in the presence of small losses.

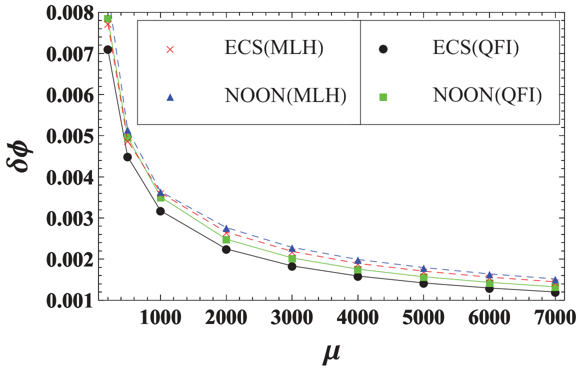

FIG. 7. (Color online) Phase uncertainties of even ECSs and NOON states using the maximum-likelihood method (ML) and quantum Fisher information with $N = 4$ in Eq. (21). The phase sensitivities are indicated as red crosses for ECSs with ML, blue triangles for NOON states with ML, black circles for ECSs given by $1/\sqrt{\mu F^{Q}_{E_+}}$, and green squares for NOON states with $1/\sqrt{\mu F^{Q}_{N}}$. We observe that ECSs always have better phase sensitivity than NOON states in cases of both ML and F^{Q} for the same number of measurements μ.

The fidelity between the odd ECS $|\mathrm{ECS}_-(\alpha_-)\rangle$ and AECS $|\mathrm{AECS}(r_0,\alpha_A)\rangle$ is very high, but in general the AECS has more degrees of freedom with r and α (see Fig. 3). Thus, there is an opportunity to generate other useful quantum states, different from ECSs, by tuning r and α, which could give further improvements. Similarly, one may consider using the approximate even CSS generated in Ref. [40] instead of using the odd CSS obtained by subtracting a single photon [39]. In addition, even though it is known that the self-Kerr-type nonlinear interaction can be performed with small losses in tunable three- or four-level systems [68], it is still an open question as to whether useful higher-order nonlinear phase operations can also be performed minimizing loss mechanisms.

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APPENDIX: OPTIMAL PHASE ESTIMATION OF ECS AND NOON STATES USING THE MAXIMUM-LIKELIHOOD METHOD

We perform numerical simulations of optimal phase estimation in both pure ECSs and NOON states for $k = 1$. This shows that the bounds of phase uncertainty given by quantum Fisher information are smaller than those given by maximum-likelihood analysis using parity measurements. We assume that a small unknown parameter ($\phi = 0.01 \ll 1$) is encrypted in the states with linear phase generator $U(\phi) = \exp[i a^{\dagger} a \phi]$ and repeated photon number parity measurements μ times after the states pass through a 50:50 beam splitter. The maximum-likelihood estimator is adopted from Ref. [13], and the half width at half maximum of the estimator function is calculated as the uncertainty of the estimated phase to be compared with the phase estimation given by quantum Fisher information $\delta\phi \geqslant 1/\sqrt{\mu F^{Q}}$.

In Fig. 7, the phase uncertainties are shown with respect to μ using the maximum-likelihood method (ML) and the quantum Fisher information (QF). The average photon numbers of the ECSs and NOON states are fixed at $N = 4$. As we enlarge μ from 200 to 7000, the phase uncertainties using both ECSs and NOON states monotonously decrease, and the ECSs show a smaller phase uncertainty than the NOON states. Note that the cases of ML with parity measurements approach but do not reach the saturation of the bounds given by QF because the parity measurement is not the optimal measurement setup [32].

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