

Production of entanglement with highly mixed states

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We study production of entanglement with highly mixed states. We find that entanglement between highly mixed states can be generated via a direct unitary interaction even when both states have purities arbitrarily close to zero. This indicates that purity of a subsystem is not required for entanglement generation. This result is in contrast to previous studies where the importance of the subsystem purity was emphasized.

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Entanglement is considered a genuine quantum correlation that cannot be described by any classical means. Meanwhile, thermal states, particularly when they are in heavy mixtures, are regarded as classical states. In general, generating entanglement using a classical state is much more difficult than when using a nonclassical state. For example, it was shown that nonclassicality is a prerequisite of generating entanglement of light fields using a beam splitter [1].

On the other hand, it is possible to generate entanglement with highly mixed thermal states under certain conditions [2–6]. Bose *et al.* showed [3] that entanglement always arises between a two-level atom and a thermal field inside a cavity irrespective of the temperature of the thermal state as long as the atom was initially in a pure excited state. In their work, the Jaynes-Cummings (JC) interaction was used to model the cavity interaction. This indicates that the purity of the atom forces the atom and the thermal state to become entangled even when the thermal state is extremely mixed. Kim *et al.* showed [3] that two atoms can become entangled through their interactions with a thermal field when the two atoms are initially in their pure states. Jeong and Ralph also pointed out [6] that entanglement between thermal states at arbitrarily high temperatures can be generated using a cross-Kerr nonlinear interaction if an ancillary microscopic superposition is used with a conditioning measurement.

Indeed, these are not the end of the investigations. For example, another interesting question would be whether a thermal state at an arbitrarily high temperature can ever be entangled with a mixed atomic state by a direct unitary interaction. It also remains unanswered whether entanglement may be generated between thermal states at arbitrarily high temperatures by a direct unitary interaction. In this paper, we study several examples to answer these questions. We conclude that entanglement between two highly mixed states can be generated solely by a unitary interaction even when both states have purities arbitrarily close to zero. Our results show that subsystem purity is not a necessary condition for generating entanglement by a direct unitary interaction. Thus the importance of the initial purity in entanglement generation depends on the interaction model.

Reference [3] shows that entanglement is always generated from a pure excited atom and a thermal field through a JC interaction irrespective of the temperature of the field. If both the atomic state and the cavity field are in thermal states with a same temperature, entanglement disappears in the infinite-

temperature limit [3]. The reason for this is that the total state becomes proportional to the identity as the temperature goes to infinity [3]. However, it is possible to prepare the initial atomic state in an independent manner from the temperature of the field. Here we examine such examples where both parties are in mixed states but their degrees of purities are independently varied.

We first consider an atomic state $p|e\rangle\langle e| + (1-p)|g\rangle\langle g|$ with $0 \leq p \leq 1$ and a thermal-field state, $\rho^{\text{th}} = (1-\lambda) \sum_n \lambda^n |n\rangle\langle n|$, where $|g\rangle$ ($|e\rangle$) is the ground (excited) state of the atom and $|n\rangle$ is the photon-number state of the field. We also note that $\lambda = \exp[-\hbar\omega/(k_B T)]$, where k_B is the Boltzmann constant, T is the temperature, \hbar is the Planck constant, and ω is the frequency of the optical field. In our analysis, the purity of state ρ is quantified by the linear entropy $\text{Tr}[\rho^2]$. The purities of the atomic and field states are then $\mathcal{P}_{\text{atom}} = 2(p-1/2)^2 + 1/2$ and $\mathcal{P}_{\text{field}} = (1-\lambda)/(1+\lambda)$, respectively. We take p and λ as independent control parameters of purities of the atom and the field. The purity of the atom is 1 when $p = 1$ while it shows the minimum value $1/2$ when $p = 1/2$, and the purity of the field can be characterized by $0 \leq \lambda \leq 1$. The initial states evolve through a JC interaction, $H_{\text{JC}} = g(|e\rangle\langle g|a + |g\rangle\langle e|a^\dagger)$, where g is the coupling strength and a (a^\dagger) is the annihilation (creation) operator of the field mode. After the interaction with time t , we make a local projection on the field mode into a subspace spanned by $|n\rangle$ and $|n+1\rangle$, then the total state becomes

$$p \begin{pmatrix} P_{n-1} S_{n-1}^2 & 0 & 0 & 0 \\ 0 & P_n C_n^2 & i P_n C_n S_n & 0 \\ 0 & -i P_n C_n S_n & P_n S_n^2 & 0 \\ 0 & 0 & 0 & P_{n+1} C_{n+1}^2 \end{pmatrix} + q \begin{pmatrix} P_n C_{n-1}^2 & 0 & 0 & 0 \\ 0 & P_{n+1} S_{n+1}^2 & -i P_{n+1} C_n S_n & 0 \\ 0 & i P_{n+1} C_n S_n & P_{n+1} C_n^2 & 0 \\ 0 & 0 & 0 & P_{n+2} S_{n+1}^2 \end{pmatrix}, \quad (1)$$

where $C_n = \cos(gt\sqrt{n+1})$, $S_n = \sin(gt\sqrt{n+1})$, $P_n = (1-\lambda)\lambda^n$, and $q = 1-p$. We use negativity of partial transpose (NPT) as an entanglement measure [7–9], defined by $-2\min(0, \epsilon)$, where ϵ is the minimum eigenvalue of the partial transpose of the density matrix with respect to one of its parties. Since the projection is a local operation, observing nonzero

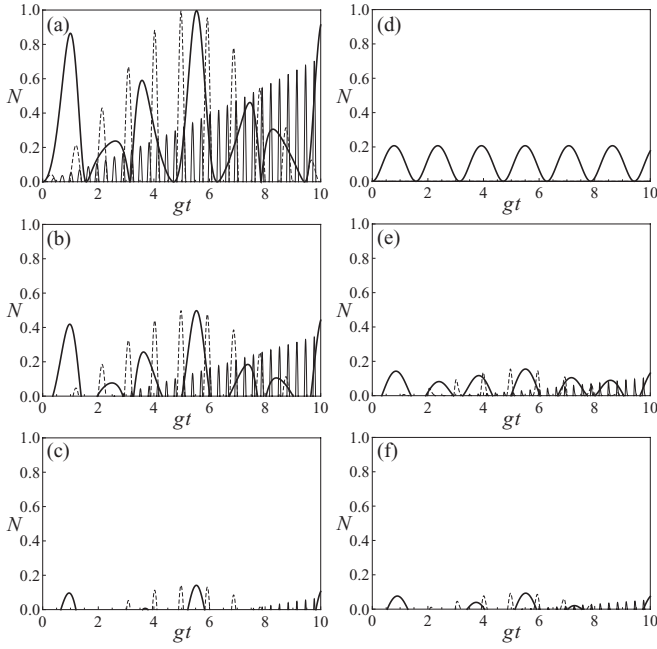


FIG. 1. Negativity of partial transpose (NPT) of the projected density matrix in Eq. (1) versus (interaction strength g) \times (interaction time t) for entanglement between an atom and a thermal state discussed in the text. The projection basis on the field mode is $\{|n\rangle\langle n|, |n+1\rangle\langle n+1|\}$ with $n=0$ (thick), $n=10$ (dashed), and $n=100$ (thin). The temperature of the field is assumed to be infinite for panels (a)–(c) and the purity p of the atomic state is (a) 1, (b) 0.9, and (c) 0.8. The atom is assumed to be maximally mixed ($p=1/2$) for panels (d)–(f) and λ of the field is (d) 0, (e) 0.1, and (f) 0.2 in order of decreasing purity.

NPT of the projected density matrix indicates that the original total state has entanglement.

A nonzero value of NPT for Eq. (1) with any value of n is evidence of the atom-field entanglement. Values of the NPT for several choices of n with normalizations of Eq. (1) are presented in Fig. 1. If the atom was initially in a pure excited state ($p=1$), entanglement of the projected density matrix with $n=0$ shows entanglement except when $gt=n\pi/2$. As shown in Fig. 1, the cases with the other projections ($n=10$ and 100) of the density matrix lead to the conclusion that the entanglement always exists for $t>0$. This is in agreement with the result of Ref. [3].

Figure 1 shows that the NPT tends to disappear as the purity of the atom decreases. In Fig. 1, we also observe similar behavior when the initial atom is in a maximally mixed state $(|g\rangle\langle g| + |e\rangle\langle e|)/2$ and the temperature of the field increases from 0 to infinity. It seems that a certain degree of purity is required to generate entanglement. However, the method used here is merely to find a sufficient condition of the presence of entanglement, and we cannot confirm that there is no entanglement in the original state when the NPT is zero for the projected density matrix [3].

We now consider an entanglement generation scheme between two harmonic oscillators via a cross-Kerr nonlinear interaction. This type of interaction has been explored to study entanglement involving thermal states [6,10–12]. It is analogous to the scheme in the previous example: the

atomic state is replaced by a superposition of the vacuum and the single photon, and the JC interaction is replaced by a cross-Kerr nonlinear interaction. A limitation of the cross-Kerr interaction using fibers in implementing certain quantum gates [13] has been pointed out [14–16]. However, we are here interested in possibility of entanglement generation using possible physical interactions in any systems. It was pointed out that a nonzero conditional phase shift with a high fidelity is possible in a cross-Kerr interaction between pulses with unequal group velocities [17]. Hosseini *et al.* demonstrated the feasibility of achieving a large cross-Kerr interaction at the single-photon level based on a memory-based approach [18].

The cross-Kerr interaction between modes a and b is described by an interaction Hamiltonian $H_{\text{Kerr}} = \chi a^\dagger a b^\dagger b$ where χ is the nonlinear interaction strength. One mode is prepared as a superposition of the vacuum and the single-photon state, generally in a mixture

$$\rho_m = \frac{1}{2}(|0\rangle\langle 0| + r|0\rangle\langle 1| + r|1\rangle\langle 0| + |1\rangle\langle 1|), \quad (2)$$

where the purity, $\text{Tr}[\rho_m^2] = (1+r^2)/2$, is characterized by a real value r . The other mode is prepared as a displaced thermal state, $\rho^{\text{th}} = \int d^2\alpha P_\alpha^{\text{th}}(V,d)|\alpha\rangle\langle\alpha|$, where $P_\alpha^{\text{th}}(V,d) = 2[\pi(V-1)]^{-1} \exp(-\frac{2|\alpha-d|^2}{V-1})$ and $|\alpha\rangle$ is a coherent state of amplitude α . The variance V is related to the average photon number \bar{n} as $V = 2(\bar{n} - d^2) + 1$, and d corresponds to the displacement of the state from the origin in phase space. Note that the purity of ρ^{th} is independent of d since the displacement is a unitary operation. After a cross-Kerr interaction with an interaction time $t = \pi/\lambda$, the total state becomes

$$\frac{1}{2} \{ |0\rangle\langle 0| \otimes \rho^{\text{th}}(V,d) + |1\rangle\langle 1| \otimes \rho^{\text{th}}(V,-d) + r|0\rangle\langle 1| \otimes \sigma(V,d) + r|1\rangle\langle 0| \otimes \sigma(V,-d) \}, \quad (3)$$

where $\sigma(V,d) = \int d^2\alpha P_\alpha^{\text{th}}|\alpha\rangle\langle -\alpha|$. We make local projections on the total state into a subspace spanned by sets $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle_\gamma, |-\rangle_\gamma\}$ where $|\pm\rangle_\gamma \equiv \mathcal{N}_\pm(|\gamma\rangle \pm |-\gamma\rangle)$ with $\mathcal{N}_\pm = \{2(1 \pm \exp[-2\gamma^2])\}^{-1/2}$. The projected state ρ_γ is then

$$\begin{pmatrix} \mathcal{N}_+^2 C_\gamma & \mathcal{N}_+ \mathcal{N}_- S_\gamma & r \mathcal{N}_+^2 C_\gamma & -r \mathcal{N}_+ \mathcal{N}_- S_\gamma \\ \mathcal{N}_+ \mathcal{N}_- S_\gamma & \mathcal{N}_-^2 C_\gamma & r \mathcal{N}_+ \mathcal{N}_- S_\gamma & -r \mathcal{N}_-^2 C_\gamma \\ r \mathcal{N}_+^2 C_\gamma & r \mathcal{N}_+ \mathcal{N}_- S_\gamma & \mathcal{N}_+^2 C_\gamma & -\mathcal{N}_+ \mathcal{N}_- S_\gamma \\ -r \mathcal{N}_+ \mathcal{N}_- S_\gamma & -r \mathcal{N}_-^2 C_\gamma & -\mathcal{N}_+ \mathcal{N}_- S_\gamma & \mathcal{N}_-^2 C_\gamma \end{pmatrix} + R_\gamma \begin{pmatrix} \mathcal{N}_+^2 & \mathcal{N}_+ \mathcal{N}_- & r \mathcal{N}_+^2 & -r \mathcal{N}_+ \mathcal{N}_- \\ \mathcal{N}_+ \mathcal{N}_- & -\mathcal{N}_-^2 & -r \mathcal{N}_+ \mathcal{N}_- & r \mathcal{N}_-^2 \\ r \mathcal{N}_+^2 & -r \mathcal{N}_+ \mathcal{N}_- & \mathcal{N}_+^2 & \mathcal{N}_+ \mathcal{N}_- \\ -r \mathcal{N}_+ \mathcal{N}_- & r \mathcal{N}_-^2 & \mathcal{N}_+ \mathcal{N}_- & -\mathcal{N}_-^2 \end{pmatrix}, \quad (4)$$

where

$$C_\gamma = \frac{4}{V+1} \exp\left[-\frac{2}{V+1}(\gamma^2 + d^2)\right] \cosh\left[\frac{4\gamma d}{(V+1)}\right],$$

$$S_\gamma = \frac{4}{V+1} \exp\left[-\frac{2}{V+1}(\gamma^2 + d^2)\right] \sinh\left[\frac{4\gamma d}{(V+1)}\right],$$

$$R_\gamma = \frac{4}{V+1} \exp\left[-\frac{2}{V+1}(V\gamma^2 + d^2)\right].$$

We assumed that γ and d are real without loss of generality. Notice that $\text{Tr}[\rho_\gamma(r)] \neq 1$ because the local projection is not

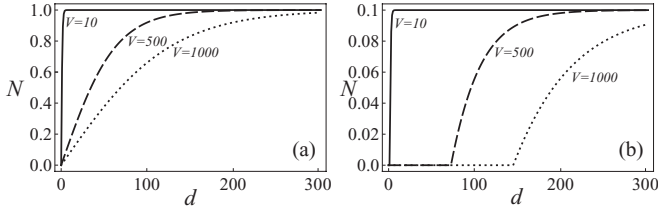


FIG. 2. NPT of the projected density matrix for Eq. (4) against displacement d of the displaced thermal field. We choose the amplitude $\gamma = 2$ for the projection basis $|\pm\rangle_\gamma$ with several values of V , and the normalized purity r is (a) 1 and (b) 0.1. Unless the microscopic superposition is totally mixed as $r = 0$, entanglement is observed for $d \gg \sqrt{V}$ for any values of V .

an unitary operation. The absolute scale of the NPT is not important for the purpose of our study, because any nonzero value is meaningful enough.

We numerically calculated the NPT of $\rho_\gamma(r)/\text{Tr}[\rho_\gamma(r)]$ for some values of V , d , and r , and present the results in Fig. 2. When $d = 0$, the NPT is zero regardless of the value of V . In this case, the original state is separable because $\sigma(V,0)^T = \sigma(V,0)$ in Eq. (3): the density matrix is then invariant under a partial transpose on one side. However, we can observe entanglement for any values of V with $d \gg \sqrt{V}$ as far as r is nonzero in Eq. (2).

As the next step, we consider two slightly different entanglement generation schemes where conditioning measurements are used in addition to unitary interactions [6]. After the Kerr interaction between a microscopic state in Eq. (2) and a thermal state, we measure out the microscopic part on the basis $(|0\rangle \pm |1\rangle)/\sqrt{2}$ [6]. The resultant state of the remaining mode is highly nonclassical exhibiting singular behavior on its Wigner function [6]. After we transmit this state through a 50 : 50 beam splitter, the state becomes

$$\begin{aligned} \rho^{\text{BS}(\pm)} = & N \int d^2\alpha P_\alpha^{\text{th}}(V,d) \{ |\delta\rangle\langle\delta| \otimes |-\delta\rangle\langle-\delta| + |-\delta\rangle\langle-\delta| \\ & \times |-\delta\rangle\langle\delta| \otimes |\delta\rangle\langle\delta| \pm r |\delta\rangle\langle-\delta| \otimes |-\delta\rangle\langle\delta| \pm r |-\delta\rangle\langle\delta| \\ & \times \langle\delta| \otimes |\delta\rangle\langle-\delta| \}, \end{aligned} \quad (5)$$

where $\delta = \alpha/\sqrt{2}$ and $N = (2 \pm 2r \exp[-2d^2/V]/V)^{-1}$. The NPT of the locally projected density matrix by sets $\{|+\rangle_\gamma, |-\rangle_\gamma\}$ for both parties is plotted in Fig. 3. One can observe that entanglement between thermal states is generated even when they are highly mixed ($V \gg 1$) unless the initial microscopic state was totally mixed with $r = 0$. Interestingly,

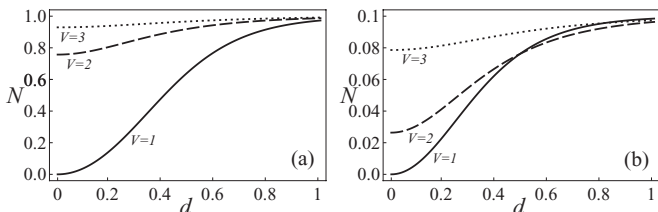


FIG. 3. NPT of locally projected state of $\rho^{\text{BS}(+)}$ with (a) $r = 1$ and (b) $r = 0.1$. We used $\gamma = 2$ for the projection basis. Entanglement is always observed for large values of $V \gg 1$ even when $d = 0$ unless $r = 0$.

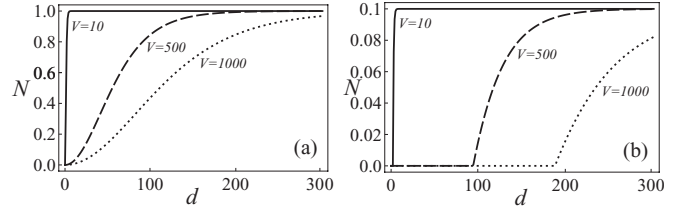


FIG. 4. NPT of the locally projected state of $\rho^{TT(+)}$ versus the displacement d with (a) $r = 1$ and (b) $r = 0.1$. The amplitude of the projection basis is $\gamma = 2$ as in Figs. 2 and 3. Each curve corresponds to $V = 10$ (solid), 500, (dashed), and 1000, (dotted). Nonzero values of d much larger than \sqrt{V} are required to observe entanglement compared to $\rho^{\text{BS}(+)}$ cases.

a high mixture ($V \gg 1$) enables one to observe entanglement even with $d = 0$ and a small value of r .

The other entanglement-generation scheme between two macroscopic thermal states is as follows: The microscopic state successively interacts with two thermal states via the cross-Kerr interactions, and it is measured out along the basis $(|0\rangle \pm |1\rangle)/\sqrt{2}$. The resultant state is

$$\begin{aligned} \rho^{TT(\pm)} = & N \{ \rho_a^{\text{th}}(V,d) \otimes \rho_b^{\text{th}}(V,d) + \rho_a^{\text{th}}(V,-d) \\ & \otimes \rho_b^{\text{th}}(V,-d) \pm r \sigma_a(V,d) \otimes \sigma_b(V,d) \pm r \sigma_a(V,-d) \\ & \otimes \sigma_b(V,-d) \}. \end{aligned} \quad (6)$$

We also project this state into a subspace spanned by sets $\{|+\rangle_\gamma, |-\rangle_\gamma\}$ for each mode, then calculate the NPT which is presented in Fig. 4. Again, an initial microscopic state with any value of $r > 0$ is useful to generate entanglement between thermal states regardless of V . Here, however, a condition of $d \gg \sqrt{V}$ is required to clearly observe entanglement.

We now come to a natural question: Can two highly mixed states, when their purities become arbitrarily small, still be entangled through a direct unitary interaction? Here we simply prepare two identical displaced thermal states and apply the cross-Kerr interaction directly between those states. In order to calculate the effect of the interaction, we first point out that two-mode states where each mode has a definite photon number parity are eigenstates of the cross-Kerr interaction for time $t = \pi/\chi$ so that $U = \exp[i\pi a^\dagger a b^\dagger b]$. It works as a controlled-phase gate on the $|\pm\rangle_\gamma$ basis where $|\pm\rangle_\gamma$ is

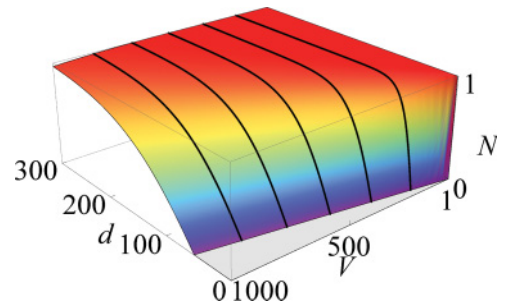


FIG. 5. (Color online) NPT of ρ_γ^ψ with $\gamma = 2$. In this case V is the only parameter that adjusts the purities of both parties simultaneously. We can always obtain the entanglement even in $V \rightarrow \infty$ limit with sufficient displacement $d \gg \sqrt{V}$. This indicates that nonzero purity of the system is sufficient to generate entanglement.

defined in the above Eq. (4) [19]: $U|+\rangle_\gamma|\pm\rangle_\gamma = |+\rangle_\gamma|\pm\rangle_\gamma$ and $U|-\rangle_\gamma|\pm\rangle_\gamma = \pm|-\rangle_\gamma|\pm\rangle_\gamma$. Based on this, we calculate the evolution of the two thermal states under U as

$$U|\alpha\rangle|\beta\rangle = \frac{1}{2}(|\alpha\rangle|\beta\rangle + |\alpha\rangle|-\beta\rangle + |-\alpha\rangle|\beta\rangle - |-\alpha\rangle|-\beta\rangle) \equiv |\psi\rangle \quad (7)$$

and $\rho^\psi = \int d^2\alpha d^2\beta P_\alpha^{\text{th}}(V,d)P_\beta^{\text{th}}(V,d)|\psi\rangle\langle\psi|$. The locally projected density matrix with basis $\{|+\rangle_\gamma, |-\rangle_\gamma\}$ is

$$\rho_\gamma^\psi = \begin{pmatrix} X_\gamma^2 & X_\gamma S_\gamma & S_\gamma X_\gamma & -S_\gamma^2 \\ X_\gamma S_\gamma & X_\gamma Y_\gamma & S_\gamma^2 & -S_\gamma Y_\gamma \\ S_\gamma X_\gamma & S_\gamma^2 & Y_\gamma X_\gamma & -Y_\gamma S_\gamma \\ -S_\gamma^2 & -S_\gamma Y_\gamma & -Y_\gamma S_\gamma & Y_\gamma^2 \end{pmatrix}, \quad (8)$$

where $X_\gamma = C_\gamma + R_\gamma$ and $Y_\gamma = C_\gamma - R_\gamma$ in terms of C_γ , S_γ , and R_γ defined in Eq. (4). We plot the results in Fig. 5, which shows that entanglement is generated even in the high-temperature limit as long as $d \gg \sqrt{V}$. We observe that two thermal states at arbitrarily high temperatures are entangled

through a direct unitary interaction as long as the condition $d \gg \sqrt{V}$ is satisfied.

In summary, we have studied several examples to explore entanglement generation involving highly mixed physical systems. Our results reveal some interesting facts concerning the generation of entanglement involving such highly mixed systems. The purity of initial states is not necessarily a prerequisite for entanglement generation. It rather depends on the model of the interaction between the initial states. In particular, we have shown that entanglement between thermal states can be generated via a direct unitary interaction even when both states have purities arbitrarily close to zero.

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