Faithful test of nonlocal realism with entangled coherent states

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We investigate the violation of Leggett's inequality for nonlocal realism using entangled coherent states and various types of local measurements. We prove mathematically the relation between the violation of the Clauser-Horne-Shimony-Holt form of Bell's inequality and Leggett's one when tested by the same resources. For Leggett inequalities, we generalize the nonlocal realistic bound to systems in Hilbert spaces larger than bidimensional ones and introduce an optimization technique that allows one to achieve larger degrees of violation by adjusting the local measurement settings. Our work describes the steps that should be performed to produce a self-consistent generalization of Leggett's original arguments to continuous-variable states.

I. INTRODUCTION

The concepts of locality and realism are at the core of Bell's celebrated inequality [1] and the consequences of the apparent failure of such intuitively reasonable assumptions in the quantum mechanical description of nature have been at the focus of a very intense theoretical and experimental activity [2]. Yet, it remains unclear whether the departure of a quantum mechanical entangled state from classicality as signaled by the violation of a Bell inequality is the result of the failure of locality, realism, or both of them. In 2003, Leggett attempted to shed some light into this point by formulating an inequality that, by allowing for a degree of nonlocality, tests the breakdown of realism in an entangled resource state [3]. This work has generated a wealth of experimental and theoretical studies directed toward the falsification of nonlocal realism with only weak assumptions on the properties of the resource state to use for the test and thus increasingly more experimental-friendly setups [4–6]. Yet the investigation so far has been limited almost entirely to the case of discrete-variable states.

However, continuous variable (CV) states are endowed of interesting fundamental properties that, frequently, go beyond the mere extension to infinite dimensionality of those characterizing discrete-variable ones [7]. Nonlocality tests have been designed for resources belonging to this realm of quantum states [7–10], and the central role played by CV systems in photon-based quantum technology is now well appreciated. In particular, the class of entangled coherent states (ECSs) [11] has emerged as a genuinely useful set of entangled states having a prominent role, for instance, in quantum teleportation and quantum computing [12–21]. It is thus desirable to extend the formal apparatus designed so far for nonlocal realistic tests to the CV scenario. A few steps in this direction have been performed [22,23], although a more systematic approach is greatly needed. This is the main objective of our work, which aims at stepping into a self-consistent formulation of nonlocal realistic bounds for CV states embodied by ECSs taking into account the inherent differences that such states have with respect to their discrete-variable counterparts. This

is not exempt from difficulties, due to the sort of constraints imposed by Leggett's arguments to the local properties of the resource states to use and which should be reformulated in the CV case. As we show in our work, this leads to the necessity of re-deriving the bound for nonlocal realistic theories so as to introduce a (weak) dependence on the tested state itself. We illustrate this finding by considering various local operators and using the different versions of Leggett's original inequality put forward in Refs. [5,6]. Finally, we thoroughly discuss the relation between violation of Bell-like inequalities and the corresponding falsification of nonlocal realistic theories by the same resource state. This nicely complements the suggestions given in Refs. [22,23] and allows us to highlight an inherent *universality* of the behavior of the Leggett functions associated with ECSs under the formulation of the inequality given in [5,6].

It is important to spell out here the intrinsic significance of our work. While Ref. [23] marked an important step forward in the direction of extending Leggett's argument to the CV realm, the approach used there was only case specific. The proposal put forward in this paper, on the other hand, provides a much more structured scenario. Starting from first principles, not only do we unveil a previously overlooked feature of nonlocal realistic tests run with CV states (namely that the nonlocal realistic bound used for two-level systems may turn out to be rather inappropriate when CV states are in order) but, more crucially, we design a systematic procedure to generalize Leggett's arguments and calculate the nonlocal realistic constraints appropriate to given physical situations. Furthermore, with such a systematic approach, we are able to provide optimized Leggett functions which, for the case of entangled coherent states, require lower parameter thresholds to violate nonlocal realistic theories, therefore improving the results of Ref. [23]. Although our contribution here is the very first step toward the goal declared earlier, it paves the way to a full formalization of Leggett's inequality to infinite dimensional Hilbert spaces, an objective that we aim at pursuing in our current endeavors in this respect.

The remainder of this paper is organized as follows. In Sec. II, we briefly review the Leggett inequalities derived so far and prove the relation between the violation of the Clauser-Horne-Shimony-Holt (CHSH) form of Bell's inequality and Leggett's one. In Sec. III A, we prove that Leggett's function

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can be optimized over the measurement settings required by one of the forms of inequality introduced in Sec. II so as to get a larger degree of violation. In Sec. III B, we find that a paradoxical phenomenon arises when testing Leggett's inequality test with an ECS and naively using the very same nonlocal realistic bound valid for two-dimensional system. We show a procedure that generalizes such bound to systems other than spin-1/2 ones. In Sec. IV, nonlocal realism is tested for ECS with pseudospin measurement operators in terms of recently derived Leggett inequalities. Finally, in Sec. V, we summarize our findings highlighting the necessity of a more general test tool for nonlocal realism.

II. REVIEW OF LEGGETT INEQUALITIES

Leggett derived his inequality bearing in mind bidimensional systems such as polarization states of light or spin-1/2 particles. Although he also proposed a restricted-ensemble model needing just a single parameter, in analogy with standard Bell's inequality tests for the previously mentioned particles, Leggett's general model exploits the full Poincaré (for polarization states) or Bloch (for spin-1/2 states) sphere. A restricted state ensemble exists on a (circular) cross section of the corresponding sphere. Referring to a local operation as transforming a state into another on the identical sphere, a restricted ensemble model does not necessarily require more than one parameter for such an operation whereas a general one requires at least two parameters. Any Leggett-type inequality starts from the following basic relation,

$$\overline{AB} \leqslant 1 - |\overline{A} - \overline{B}|, \tag{1}$$

involving the arbitrary dichotomic variables A and B taking values ± 1. The overlines indicate statistical averages over an appropriate subensemble. For instance, in an optical system this might be made out of photon states with definite polarization. Leggett assumed that even when the subsystems are nonlocally correlated, the local average of each subsystem should fulfill Malus' law,

$$\begin{aligned}\overline{A}(\mathbf{u},\mathbf{v};\mathbf{a},\mathbf{b}) &= \mathbf{u} \cdot \mathbf{a} = \overline{A}(\mathbf{u};\mathbf{a}), \\ \overline{B}(\mathbf{u},\mathbf{v};\mathbf{a},\mathbf{b}) &= \mathbf{v} \cdot \mathbf{b} = \overline{B}(\mathbf{v};\mathbf{b}),\end{aligned} \tag{2}$$

where $\mathbf{u}$ ($\mathbf{v}$) and $\mathbf{a}$ ($\mathbf{b}$) denote the polarization of a photon and the measurement setting of a polarizer at site A (B), respectively. This relation implies that even if a nonlocal interaction is allowed between subsystems A and B, each local expectation value must depend only on the respective local parameters. Since $\mathbf{u}$ and $\mathbf{a}$ ($\mathbf{v}$ and $\mathbf{b}$) can be equally represented as (unit) vectors denoting directions of polarization on the Poincaré sphere, a measurement vector can be transformed into another by the same local operation as for a polarization state. It is important to note that in Leggett's model, the previously mentioned condition imposed on local averages acts as a constraint on the assumed nonlocal correlations [24].

The observation that Leggett inequality is solely based upon Eq. (1) leads one to prove that the violation of Leggett inequality implies violation of the CHSH inequality. We prove this by contraposition, (i.e., we show that if the CHSH

inequality is not violated, Leggett inequality is not violated either). To prove this, we start from CHSH inequality,

$$-2 \leqslant \overline{AB} + \overline{AB'} + \overline{A'B} - \overline{A'B'} \leqslant 2, \tag{3}$$

and if we set $A' = -B' = \pm 1$ [$A' = B' = \pm 1$], then we get Eq. (1) [Eq. (26)], which proves the claim.

A brief construction of the original Leggett inequality is as follows. After averaging Eq. (1) over all (polarization or spin) states, one can obtain a correlation function $E_{ij}(\varphi)$ that is bounded as

$$E_{ij}(\varphi) \leqslant 1 - f_{ij}(\varphi), \tag{4}$$

where $E_{ij} = \langle \overline{A_i B_j} \rangle$ stands for the correlation function associated with measurement settings or directions $\mathbf{a}_i$, $\mathbf{b}_j$ at site A and B, respectively. While the correlation function is defined in the same way as in the standard CHSH inequality, the bound is not outcome independent, as $f_{ij} = \langle |\overline{A_i} - \overline{B_j}| \rangle$. This function can be considered the nonlocal realistic constraint which, even under the presence of nonlocal correlations between A and B, limits the range of variability of the correlation function.

In order to construct the original Leggett inequality, we choose two and three measurement settings for party A and B, respectively, as illustrated in Fig. 1(a). When written in terms
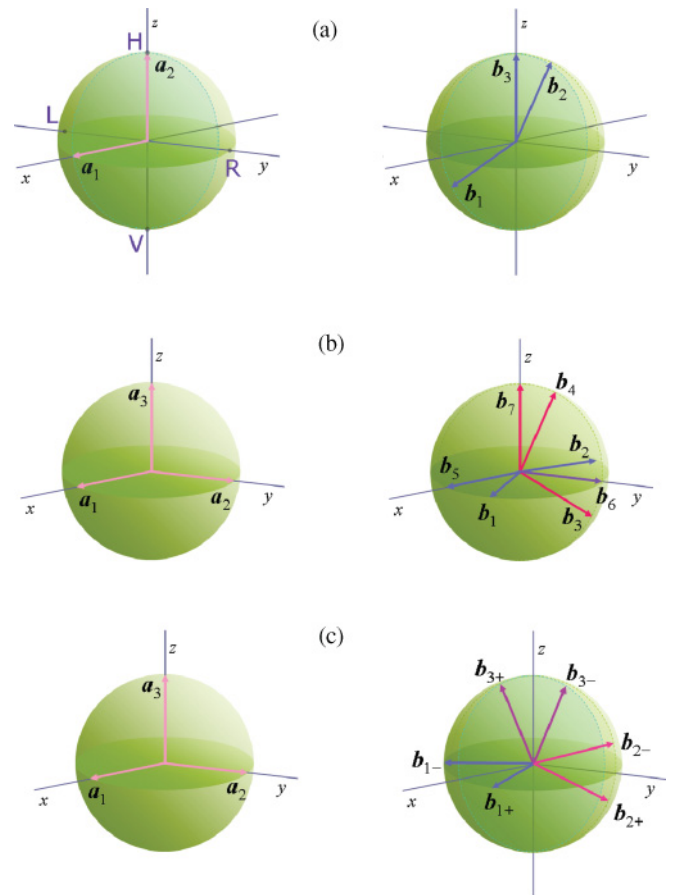

FIG. 1. (Color online) Measurement settings for party A (left) and B (right) represented on the Poincaré sphere for (a) the original Leggett inequality, (b) the "3 + 7 settings" Leggett inequality [5], and (c) the optimal "3 + 6 settings" Leggett inequality [6]. In panel (a), H, V, R, and L denote a horizontal, vertical, right- and left-circularly polarization, respectively.

of spherical polar coordinates, such measurement settings are given by the vectors,

$$\mathbf{a}_1 = \left(\frac{\pi}{2}, 0\right), \quad \mathbf{a}_2 = (0,0),$$

$$\mathbf{b}_1 = \left(\frac{\pi}{2} + \varphi, 0\right), \quad \mathbf{b}_2 = \left(\varphi, \frac{\pi}{2}\right), \quad \mathbf{b}_3 = \mathbf{a}_2,$$

where φ is the parameter discriminating the two settings. With these values, using Eq. (4) and taking $i = 1,2$, $j = 1,2,3$, one obtains the inequality,

$$|E_{11}(\varphi) + E_{23}(0)| + |E_{22}(\varphi) + E_{23}(0)| \leqslant 4 - f_{\min}(\varphi), \quad (5)$$

where $f_{\min}(\varphi) = \min[f_{11}(\varphi) + f_{22}(\varphi) + 2f_{23}(0)]$ and the minimization is performed over the hidden-variable model [25]. Analytically, one gets

$$f_{\min}(\varphi) = \frac{4}{\pi} \left| \sin\frac{\varphi}{2} \right|, \quad (6)$$

independently of the specific hidden-variable model assumed, so that it can be applied to any bidimensional discrete system. We have already commented on the fact that the nonlocal realistic bound is no longer a measurement-setting independent quantity, as it is very clearly exemplified by Eq. (6). Moreover, due to the way the bound has been obtained, the unit vector defining each local subsystem in the corresponding configuration sphere remains well defined. Gröblacher *et al.* provided the first experimental falsification of nonlocal realism based on the previously mentioned Leggett inequality [4] using the polarization-entangled state (PES),

$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B), \quad (7)$$

as a resource. Here $|H\rangle$ ($|V\rangle$) denotes a single-photon state with horizontal (vertical) polarization. They also constructed a theoretical nonlocal realistic model, based on the assumption of rotational invariance of the correlations arising from the use of a PES, simulating the quantum mechanical predictions (including the violation of the CHSH inequality). Paterek *et al.* and Branciard *et al.* [5] modified the original argument and elaborated new classes of inequalities that do not build on the previously mentioned rotational invariance and require only a finite number of measurement settings, thus resulting in much more experimental friendly criteria [5]. Of those inequalities, the simplest one reads

$$\begin{aligned} L = &\frac{1}{2}|E_{11}(\varphi) + E_{22}(\varphi) + E_{15}(0) + E_{26}(0)| \\ &+ \frac{1}{2}|E_{23}(\varphi) + E_{34}(\varphi) + E_{26}(0) + E_{37}(0)| \\ &\leqslant 4 - f_{\min}(\varphi), \end{aligned}$$

where $f_{\min}(\varphi) = |\sin(\varphi/2)|$ and the measurement vectors shown in Fig. 1(b) are given by

$$\mathbf{a}_1 = \left(\frac{\pi}{2}, 0\right), \quad \mathbf{a}_2 = \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \mathbf{a}_3 = (0,0),$$

$$\mathbf{b}_1 = \left(\frac{\pi}{2}, \varphi\right), \quad \mathbf{b}_2 = \left(\frac{\pi}{2}, \frac{\pi}{2} + \varphi\right), \quad \mathbf{b}_3 = \left(\varphi, \frac{\pi}{2}\right), \quad (8)$$

$$\mathbf{b}_4 = \left(\frac{\pi}{2} + \varphi, \frac{\pi}{2}\right), \quad \mathbf{b}_5 = \mathbf{a}_1, \quad \mathbf{b}_6 = \mathbf{a}_2, \quad \mathbf{b}_7 = \mathbf{a}_3.$$

We will refer to the previous inequality as "3 + 7 settings" Leggett inequality. It is worth noticing that, by adopting finite measurement settings instead of infinite ones, the realistic constraint decreases and, accordingly, the bound increases by a small amount.

In Ref. [6], Branciard *et al.* proposed and experimentally demonstrated the optimal inequality,

$$L = \frac{2}{3} \sum_{i=1}^{3} \left| E_{i+}\left(\frac{\varphi}{2}\right) + E_{i-}\left(\frac{\varphi}{2}\right) \right| \leqslant 4 - f_{\min}(\varphi), \quad (9)$$

where $f_{\min}(\varphi) = (4/3)|\sin(\varphi/2)|$ and the correlation function $E_{i\pm}$ is evaluated for the measurement vectors $\mathbf{a}_i$ and $\mathbf{b}_{i\pm}$ shown in Fig. 1(c). Vectors $\mathbf{a}_i$'s ($i = 1, 2, 3$) are the same as in Eq. (8) while the $\mathbf{b}_{i\pm}$'s are given by

$$\mathbf{b}_{1\pm} = \left(\frac{\pi}{2}, \pm\frac{\varphi}{2}\right), \quad \mathbf{b}_{2\pm} = \left(\frac{\pi}{2} \mp \frac{\varphi}{2}, \frac{\pi}{2}\right),$$

$$\mathbf{b}_{3\pm} = \left(\frac{\varphi}{2}, \frac{\pi}{2} \mp \frac{\pi}{2}\right). \quad (10)$$

As the inequality requires three and six settings at A and B sites, respectively, we refer to it as the "3 + 6 settings" Leggett inequality. The claimed optimality arises from the fact that Eq. (9) requires fewer settings and has a tighter nonlocal realistic bound than Eqs. (5) and (8).

III. GENERALIZING LEGGETT-TYPE INEQUALITIES FOR ARBITRARY SYSTEMS

A. Optimizing Leggett-type inequalities by Eulerian rotation

As can be noticed by inspecting Fig. 1, the inequalities discussed so far all contain measurement settings whose vectors are parallel to the coordinate axes of the Poincaré sphere. Such choices are due to mere algebraic convenience in the analytical derivation of the various forms of Leggett inequality. Moreover, for the case of PES discussed, the correlation functions upon which the inequalities are built depend only on the angle between two measurement vectors rather than the direction of each of them individually.

Clearly, this might well not be the case for another choice of states. In what follows we show that, for specific resource states, by adopting a rigid-body rotation approach to the research for the measurement settings to be used in a Leggett-type inequality (see Fig. 2), the Leggett functions L's in Eqs. (5) and (8) can be numerically optimized to get a larger
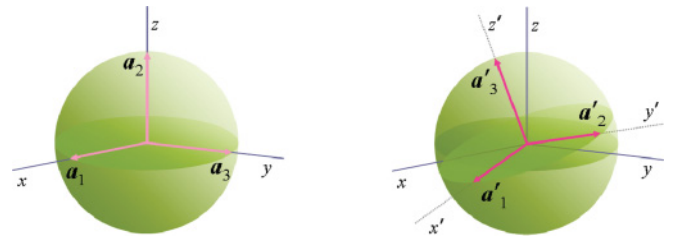

FIG. 2. (Color online) The original measurement vectors (left) on the Poincaré (or Bloch) sphere and their rigid-body rotated version (right). The relative angles between the vectors are maintained during the rotation. This approach is used in our work to optimize the degree of violation of nonlocal realistic models.

degree of violation. In contrast, the function in Eq. (9) hardly increases under such optimization as it is already optimal.

B. Leggett inequality for entangled coherent states

So far, the approach used in both the experimental and theoretical contexts has been focused on two-dimensional systems. However, if the state of each subsystem is well defined, Leggett's arguments could equally well be applied to systems of larger dimensionality, such as qudits or continuous variables (CVs). Among the states belonging to the latter class, ECSs can be regarded as very appealing due to their similarity to discrete states (such as the one in Eq. (7)) and their strong and well-studied nonlocal properties [26–29]. In what follows we use the ECSs

$$|\text{ECS}_\pm\rangle_{AB} = \mathcal{N}_\pm[|\alpha\rangle_A|-\alpha\rangle_B \pm |-\alpha\rangle_A|\alpha\rangle_B], \quad (11)$$

where $\mathcal{N}_\pm$ are normalization factors. For simplicity, we assume hereafter that α is real and omit subscripts A,B denoting the two subsystems. It should be noted that we treat the ECSs in a $2 \otimes 2$ Hilbert space where the basis vectors are $|\alpha\rangle$ and $|-\alpha\rangle$, as in Ref. [13], even though they can be considered CV states.

Very recently, Leggett inequality tests on CV systems have been studied [22,23] adopting homodyne measurements and using the sequence of local unitary operations,

$$\hat{R}(\theta,\phi) = \hat{D}\left(-\frac{i\phi}{4\alpha}\right)\hat{U}_{\text{NL}}\hat{D}\left(\frac{i\theta}{4\alpha}\right)\hat{U}_{\text{NL}}\hat{D}\left(\frac{i\phi}{4\alpha}\right), \quad (12)$$

where $\hat{a}(\hat{a}^\dagger)$ is the bosonic annihilation (creation) operator, $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the displacement operator of amplitude α and $\hat{U}_{\text{NL}} = \exp[-i\pi(\hat{a}^\dagger\hat{a})^2/2]$ is the time-evolution operator for a field propagating in a self-Kerr medium for a dimensionless time $\pi/2$. The local operator $\hat{R}(\theta,\phi)$ transforms a coherent state $|\pm\alpha\rangle$ with $|\alpha| \gg 1$ as

$$\begin{aligned} |\alpha\rangle &\to \sin\frac{\theta}{2}|\alpha\rangle + e^{-i\phi}\cos\frac{\theta}{2}|-\alpha\rangle, \\ |-\alpha\rangle &\to e^{i\phi}\cos\frac{\theta}{2}|\alpha\rangle - \sin\frac{\theta}{2}|-\alpha\rangle. \end{aligned} \quad (13)$$

That is, $\hat{R}(\theta,\phi)$ mimics the effects of a rotation in the bidimensional space spanned by $\{|\pm\alpha\rangle\}$ and $\hat{R}(\theta,\phi)|\alpha\rangle$ can be any state in the corresponding Bloch-like sphere. Reference [23] used an ECS as an entangled resource to show that nearly the same behavior as in the discrete-system case is achieved [23].

However, CV systems may have, in general, quite different local behaviors from discrete-variable ones. Therefore, the same nonlocal realistic bound in Eqs. (5), (8), and (9), could not be suitable for the case of CV states, too. As an example, let us consider the general approach developed in Ref. [23]: We retain the same local operations given in Eq. (12), but we replace the homodyne detection with an on or off measurement formally described by the operator,

$$\hat{O} = \sum_{n=1}^{\infty}|n\rangle\langle n| - |0\rangle\langle 0| = \mathbb{1} - 2|0\rangle\langle 0|, \quad (14)$$

where $\mathbb{1}$ is the identity operator and $|n\rangle$ is a Fock state with n excitations. Such measurement has outcome $+1$ (-1) if a state has *any* excitations (is in the vacuum state). The expectation value $\langle\alpha|\hat{O}|\alpha\rangle$ converges to 1 as α grows

since the vacuum contribution of a coherent state diminishes accordingly. This is the reason why the ECSs in Eq. (11) show no violation of the Bell-CHSH inequality if $\alpha \gg 1$ [26,29].

The correlation function for an ECS probed along the directions identified by the measurement vectors $\mathbf{a}_A = (\theta_A,\phi_A)$, $\mathbf{b}_B = (\theta_B,\phi_B)$ is given by

$$E_\pm(\varphi) =_{AB}\langle\text{ECS}_\pm|\hat{\Pi}_A(\theta_A,\phi_A)\otimes\hat{\Pi}_B(\theta_B,\phi_B)|\text{ECS}_\pm\rangle_{AB}, \quad (15)$$

where we have introduced the rotated on or off operators $\hat{\Pi}_i(\theta_i,\phi_i) = \hat{R}_i^\dagger(\theta_i,\phi_i)\hat{O}_i\hat{R}_i(\theta_i,\phi_i)$ $(i = A,B)$. As before, φ is the angle between $\mathbf{a}_A$ and $\mathbf{b}_B$. It turns out that Leggett inequality is violated for almost any value of α, including the case of $\alpha \to \infty$, where the degree of violation grows to the maximum value allowed by the specific inequality being tested. This is due to the fact that $\langle 0|\pm\alpha\rangle \to 0$ as $\alpha \gg 1$, which implies that

$$\begin{aligned} \langle\alpha|\hat{\Pi}_i(\theta_i,\phi_i)|\alpha\rangle &= 1 - 2|\langle 0|\hat{R}(\theta_i,\phi_i)|\alpha\rangle|^2 \to 1, \\ \langle\alpha|\hat{\Pi}_i(\theta_i,\phi_i)|-\alpha\rangle &= e^{-2\alpha^2} - 2\langle\alpha|\hat{R}(\theta_i,\phi_i)|0\rangle \\ &\quad \times \langle 0|\hat{R}^\dagger(\theta_i,\phi_i)|-\alpha\rangle \to 0, \end{aligned} \quad (16)$$

regardless of the values of θ_i,ϕ_i. In turn, this means that the sums in the right-hand sides of Eqs. (5), (8), and (9) all converge to 4, thus saturating the degree of violation of the tested inequalities.

However, as explained earlier, the CHSH inequality cannot be violated using on or off measurements and large-amplitude coherent states. Therefore, as proven in the previous section, Leggett inequality cannot be violated either in the same range of α, in striking contrast with what has been observed earlier. This paradoxical situation arises exactly from the reasons highlighted before (i.e., a state living in a Hilbert space that is not bidimensional may deviate from the predictions of Malus' law and, as such, could originate a new nonlocal realistic constraint $f_{\min}(\varphi)$. With this in mind, we have to look for a quantum-mechanical substitute of the classical Eq. (2). As Malus' law deals with the statistical average of the expectation value of an (arbitrary) measurement vector with an (arbitrary) state on the Bloch sphere, a natural yet rigorous way to reformulate it in this context is to consider the local average of the expectation value of a rotated on or off measurement with a rotated coherent state, that is

$$\overline{A}(\mathbf{u};\mathbf{a}) = \langle\alpha|\hat{R}^\dagger(\theta_u,\phi_u)\hat{\Pi}(\theta_a,\phi_a)\hat{R}(\theta_u,\phi_u)|\alpha\rangle, \quad (17)$$

with $\mathbf{u} = (\theta_u,\phi_u)$ and $\mathbf{a} = (\theta_a,\phi_a)$. With this at hand, we could in principle newly evaluate $f_{\min}(\varphi)$ for the on or off-measurement approach. However, as it is difficult to derive a closed-form analytical expression, we have numerically evaluated the nonlocal realistic bound for each value of φ. Unfortunately, no violations of Leggett inequalities can be observed, despite the use of the optimization technique described in the previous section. For on or off measurements, the constraints turn out to be too weak to falsify nonlocal realistic models tested with ECSs: The Leggett function and the bound have values nearly approaching 4 in almost all ranges of α and φ and the former has slightly smaller values than the latter.
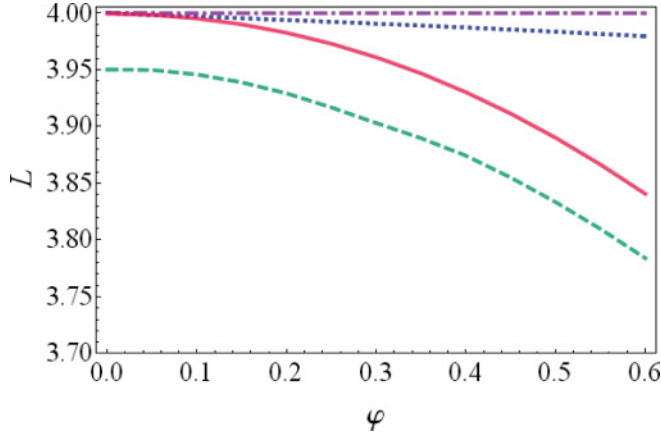
FIG. 3. (Color online) Optimized 3+7 setting Leggett function L for $|ECS_-\rangle$ with parity measurement against φ at $\alpha = 5$ (green dashed) and 50 (red solid). The nonlocal realistic bounds at $\alpha = 5$ (blue dotted) and 50 (violet dot-dashed) are also plotted. The plotting range is the same as Fig. 4 for comparison. Note that the optimized Leggett function turns out not to overcome the bound in any range of α and φ including the above range.

To provide an example of measurement which cannot falsify the Leggett inequality, we use the following parity operator instead,

$$\hat{\mathcal{O}} = \sum_{n=0}^{\infty}[|2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n|], \qquad (18)$$

which gives $+1\,(-1)$ when a state has an odd (even) number of excitations. The CHSH inequality can be tested via phase-space methods and using displaced parity operators and the Wigner function [10,26,29]. Similarly, one can perform Leggett inequality tests in the same way as in the previous sections by replacing the on or off operator in Eq. (14) with the previously mentioned parity operator. However, this would not be sufficient to observe any violation. As partly can be seen in Fig. 3, the Leggett function cannot overcome the bound in any range of α and φ even when adopting the optimization scheme in Sec. III A.

IV. LEGGETT INEQUALITY TEST FOR ECS WITH PSEUDOSPIN MEASUREMENTS

The ECSs (11) are known to show Bell violations for almost any value of α, when pseudospin measurements are used [26]. The pseudospin operators are defined as [30]

$$\hat{s}_z = (-1)^{\hat{a}\hat{a}^\dagger}, \quad \hat{s}_- = \hat{s}_+^\dagger = \sum_{n=0}^{\infty}|2n\rangle\langle 2n+1|, \qquad (19)$$

and satisfy the SU(2) algebra of standard spin-1/2 particles. Here, $\hat{s}_z$ is the parity operator in Eq. (18). As discussed in Ref. [30], we need the combined local operation and measurement observable given by

$$\mathbf{a}\cdot\hat{\mathbf{s}} = \sin\theta(e^{i\phi}\hat{s}_- + e^{-i\phi}\hat{s}_+) + \cos\theta\,\hat{s}_z, \qquad (20)$$

where $\mathbf{a} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a measurement vector and $\hat{\mathbf{s}} = (\hat{s}_x,\hat{s}_y,\hat{s}_z)$ is the pseudospin operator vector with $\hat{s}_\pm = \hat{s}_x \pm i\hat{s}_y$. Due to the bidimensional character of such

local operations, both a Bell-CHSH test and a Leggett one are possible.

The correlation function for (say) $|ECS_-\rangle$ with measurement vectors $\mathbf{a} = (\theta_A,\phi_A)$ and $\mathbf{b} = (\theta_B,\phi_B)$ is

$$
\begin{aligned}
E_-(\mathbf{a},\mathbf{b}) &= \langle ECS_-|(\mathbf{a}\cdot\mathbf{s})(\mathbf{b}\cdot\mathbf{s})|ECS_-\rangle \\
&= -\cos\theta_A\cos\theta_B - K(\alpha)\sin\theta_A\sin\theta_B\cos(\phi_A-\phi_B),
\end{aligned}
$$
$$(21)$$

with

$$K(\alpha) = \frac{2\alpha^2}{\sinh 2\alpha^2}\left[\sum_{n=0}^{\infty}\frac{\alpha^{4n}}{(2n)!\sqrt{2n+1}}\right]^2. \qquad (22)$$

Quantitatively, we have that $0.907 \leqslant K(\alpha) \leqslant 1$ and $K(\alpha) \to 1$ as $\alpha \to 0$ or ∞. In this limit we have $E_-(\mathbf{a},\mathbf{b}) = -\mathbf{a}\cdot\mathbf{b}$, which is exactly the correlation function of a PES [see Eq. (7)]. For the case of $|ECS_+\rangle$, if the direction of one of the vectors identifying a measurement is inverted, the correlation function becomes identical to Eq. (21) with $K(\alpha)$ replaced by $\tanh(2\alpha^2)K(\alpha)$. The local average for pseudospin measurements can be calculated, in analogy with Eq. (17), as

$$\overline{A}(\mathbf{u};\mathbf{a}) = \langle\alpha|(\mathbf{u}\cdot\mathbf{s})^\dagger(\mathbf{a}\cdot\mathbf{s})(\mathbf{u}\cdot\mathbf{s})|\alpha\rangle. \qquad (23)$$

Here, the additional pseudospin operator $\mathbf{u}\cdot\mathbf{s}$ with another unit vector $\mathbf{u}$ plays the role of the rotation operator as in Eq. (17). Note that $\mathbf{u}\cdot\mathbf{s}$ is unitary and for an arbitrary unit vector $\mathbf{a}'$, we can get $\mathbf{u} = (\mathbf{a}+\mathbf{a}')/\sqrt{2(1+\mathbf{a}\cdot\mathbf{a}')}$ satisfying

$$(\mathbf{u}\cdot\mathbf{s})^\dagger(\mathbf{a}\cdot\mathbf{s})(\mathbf{u}\cdot\mathbf{s}) = \mathbf{a}'\cdot\mathbf{s}. \qquad (24)$$

Clearly, the role of $\mathbf{u}\cdot\mathbf{s}$ is to rotate the axis of the pseudospin measurement from $\mathbf{a}$ to $\mathbf{a}'$. The local operator $\mathbf{u}\cdot\mathbf{s}$ transforms a coherent state $|\pm\alpha\rangle$ under the same assumption as in Eq. (13),

$$
\begin{aligned}
|\alpha\rangle &\to \sin\theta\cos\phi|\alpha\rangle - (\cos\theta + i\sin\theta\sin\phi)|-\alpha\rangle, \\
|-\alpha\rangle &\to -(\cos\theta - i\sin\theta\sin\phi)|\alpha\rangle - \sin\theta\cos\phi|-\alpha\rangle.
\end{aligned}
$$
$$(25)$$

With Eqs. (21) and (23) we test the Leggett inequality corresponding to the "3 + 6" and "3 + 7" settings.

A. 3 + 7 setting Leggett inequality test

In order to test Leggett inequality with an ECS, we first obtain numerically the nonlocal realistic constraint $f_{min}(\varphi)$ in Eq. (8). We then compare the (nonoptimized) Leggett function with such bound, as shown in Fig. 4 for $|ECS_-\rangle$ and $\alpha = 5,50$. As expected, the bound depends on the measurement settings. Moreover, there is a dependence on α as well, although this becomes very weak as soon as $\alpha \gtrsim 5$. Evidently, there is a range of values of φ where the inequality is violated. The degree of violation is maximized at $\varphi \simeq 0.25$, regardless of α and for both $|ECS_-\rangle$ and $|ECS_+\rangle$. Such value agrees exactly with that maximizing the degree of violation when the PES in Eq. (7) is used. This should be expected as, for $\alpha \to 0$ or ∞, $E_-(\mathbf{a},\mathbf{b}) \to -\mathbf{a}\cdot\mathbf{b}$.

Retaining $\varphi = 0.25$ in our calculations, we optimize the Leggett function by rigid-body rotations of the measurement vectors, as illustrated before. As can be seen in Figs. 5(a) and 5(b), the optimization really helps Leggett function to grow larger within the range of α considered in our study, although
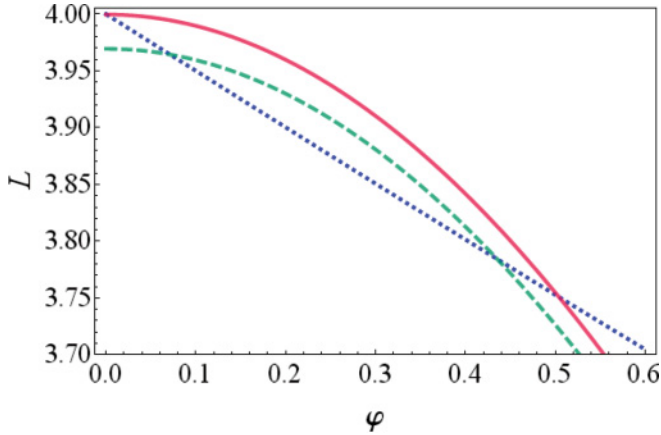
FIG. 4. (Color online) Leggett function L for $|ECS_-\rangle$ against φ at $\alpha = 5$ (green dashed) and 50 (red solid). The nonlocal realistic bound (blue dotted) is also plotted. Note that the bounds corresponding to $\alpha = 5$ and 50 are practically coincident. This can also be seen in Figs. 5(c) and 5(d). The violations are maximized at $\varphi \simeq 0.25$ regardless of the actual value of α.

the enhancement progressively decreases as $\alpha \to 0$ or ∞. In the spirit of the investigations performed in Refs. [22,23], in Figs. 5(c) and 5(d) we also compare the Leggett and optimized CHSH functions for $|ECS_\pm\rangle$ so as to elucidate the relation between the two inequalities. As proven earlier in this paper, the parameter region where Leggett inequality is violated stays within the region where the CHSH inequality is violated too. As also mentioned in Refs. [22,23], there is an interesting region of values of α where, while the CHSH inequality is violated, Leggett's is not. It is worth stressing the effectiveness of the optimization procedure adopted in our analysis. For instance, the minimum amplitude α of $|ECS_+\rangle$ for the violation of Leggett inequality was 7.5 in Ref. [23], while it is lowered down to 2.9 here, as a result of the optimization over rigid-body rotations. In the next subsection, it is shown that for the "3 + 6 settings" inequality, such threshold value is even smaller and equal to 1.8.

B. 3 + 6 setting Leggett inequality test

We complete our study by addressing now the form of Leggett inequality given in Eq. (9), which needs only six measurement settings at the B site and needs no optimization. We follow the same procedure described above for the "3 + 7 settings" case, except that we skip the unnecessary
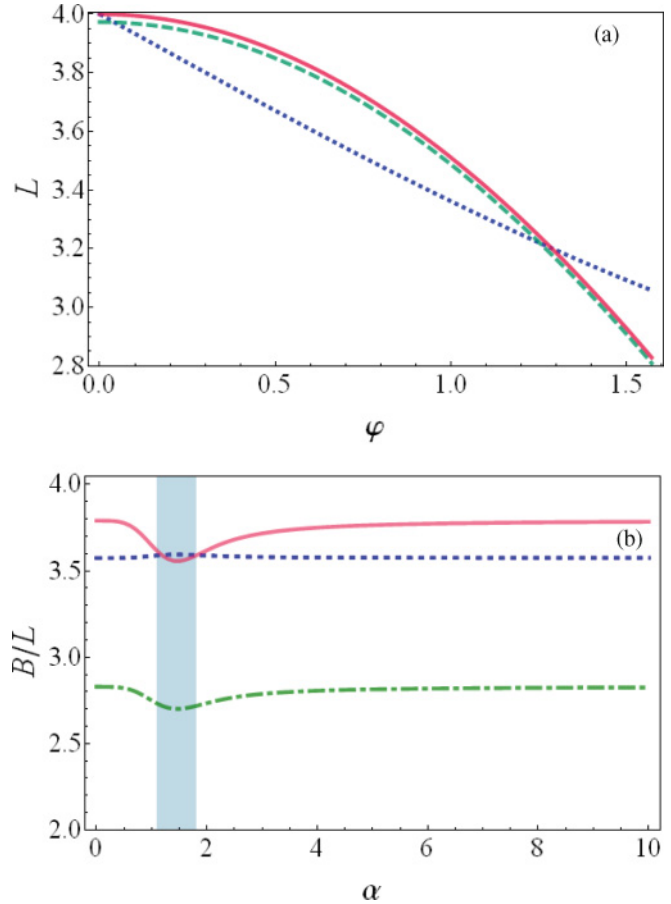

FIG. 6. (Color online) (a) Leggett function L at $\alpha = 5$ (green dashed) and 50 (red solid) and the nonlocal realistic bound (blue dotted) as functions of φ. The parameters used for this plot are the same as in Fig. 5(a), although the degree of violation of Leggett inequality is maximized at $\varphi \simeq 0.65$. (b) Optimized CHSH function B (green dot-dashed), maximized Leggett function L (red solid), and the nonlocal realistic bound (blue dotted) for $|ECS_-\rangle$.

optimization. As can be appreciated by examining Fig. 6(a), the value of φ maximizing the degree of violation of the Leggett inequality is $\varphi \simeq 0.65$, which agrees with the one found for a PES [Eq. (7)] [6]. Here, too, we retain this value and study both the CHSH and Leggett functions for $|ECS_-\rangle$ [we omit the case of $|ECS_+\rangle$ for the sake of brevity]. Figure 6(b) shows the optimal nature of the "3 + 6 settings" inequality: The region where Leggett inequality is satisfied is halved with
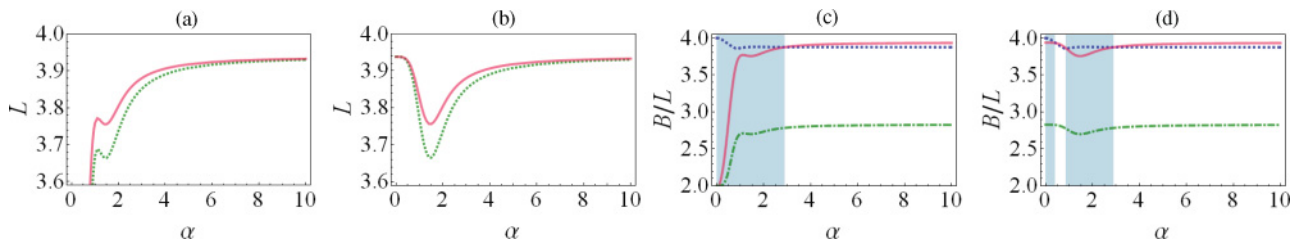

FIG. 5. (Color online) (a) and (b) Leggett function (green dotted) for (a) $|ECS_+\rangle$ and (b) $|ECS_-\rangle$, its optimized one (red solid) as functions of α. (c) and (d) Optimized CHSH function B (green dot-dashed), optimized Leggett function L (red solid) for (c) $|ECS_+\rangle$, and (d) $|ECS_-\rangle$ plotted, together with the nonlocal realistic bound (blue dotted), against α. As the local realistic bound is 2, the CHSH inequality is violated for any value of α within this range except at $\alpha = 0$ for $|ECS_+\rangle$. Leggett inequality is violated for $\alpha \gtrsim 2.9$.

the degree of violation being doubled as compared with the "3 + 7 settings" case.

As for the qualitative similarity between the curves of CHSH and Leggett functions, which is observed in common in Figs. 5(c) and 5(d) and Fig. 6(b), some considerations can be drawn. First, the small dip is found around $\alpha \simeq 1.5$ and common to all the graphs is solely due to the similar behavior that $K(\alpha)$ has in the two cases. Second, one finds a similar sudden drop of the curves associated with $|ECS_+\rangle$ as $\alpha \rightarrow 0$ [we only show it for the "3 + 7 settings" case in Fig. 5(c)]. This is due to the fact that $|ECS_+\rangle \rightarrow |0,0\rangle$ as $\alpha \rightarrow 0$. The similarity between the Leggett corresponding to the "3 + 7 settings" and "3 + 6 settings" is somehow to be expected, given that the two functions have been constructed using the same assumptions on nonlocal realism. On the contrary, we believe the analogies between the behavior of the CHSH and Leggett functions are striking in consideration of the different arguments at the basis of the two inequalities.

V. CONCLUSION

We have performed the first step toward the construction of a formal apparatus for the rigorous extension of Leggett's test for nonlocal realism to the CV scenario, therefore going significantly beyond the efforts performed in Ref. [23]. Technically, our tools have been borrowed from the considerable body of studies performed so far on the violation of local realism by this class of states and include on or off, parity, and pseudospin measurements. The requirements for local state definiteness at the basis of Leggett's arguments impose some fundamental constraints resulting in the necessity for the research for a new local-realistic bound, when CV states are used. While generalizing Leggett inequality to a form suitable for ECS resources, we have analytically clarified the relation between the violation of the CHSH and Leggett inequality and such a relation has been exemplified by studying the behavior of the CHSH and Leggett functions against the sizes of ECSs. We believe that our study contributes significantly to the understanding of the facets of nonlocal realism, in particular, with respect to the generalization of Leggett's original formulation to entangled CV states. Our results highlight the necessity for a more general way to construct nonlocal realistic tests applicable to any quantum-mechanical entangled states, an intriguing task that would be interesting to pursue both theoretically and experimentally.

[1] J. S. Bell, Physics **1**, 195 (1964); J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).

[2] Y. Hasegawa, R. Loidl, G. Badurek, M. Baron, and H. Rauch, Nature (London) **425**, 45 (2003); D. N. Matsukevich, P. Maunz, D. L. Moehring, S. Olmschenk, and C. Monroe, Phys. Rev. Lett. **100**, 150404 (2008); D. Salart, A. Baas, C. Branciard, N. Gisin, and H. Zbinden, Nature (London) **454**, 861 (2008); N. Gisin, Science **326**, 1357 (2009); M. Ansmann *et al.*, Nature (London) **461**, 504 (2009); P. Sekatski, N. Brunner, C. Branciard, N. Gisin, and C. Simon, Phys. Rev. Lett. **103**, 113601 (2009).

[3] A. J. Leggett, Found. Phys. **33**, 1469 (2003).

[4] S. Gröblacher, T. Paterek, R. Kaltenbaek, C. Brukner, M. Żukowski, M. Aspelmeyer, and A. Zeilinger, Nature (London) **446**, 871 (2007).

[5] T. Paterek, A. Fedrizzi, S. Gröblacher, Th. Jennewein, M. Żukowski, M. Aspelmeyer, and A. Zeilinger, Phys. Rev. Lett. **99**, 210406 (2007); C. Branciard, A. Ling, N. Gisin, C. Kurtsiefer, A. Lamas-Linares, and V. Scarani, *ibid.* **99**, 210407 (2007).

[6] C. Branciard, N. Brunner, N. Gisin, C. Kurtsiefer, A. Lamas-Linares, A. Ling, and V. Scarani, Nat. Phys. **4**, 681 (2008).

[7] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).

[8] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).

[9] Z. Y. Ou and L. Mandel, Phys. Rev. Lett. **61**, 50 (1988).

[10] K. Banaszek and K. Wódkiewicz, Phys. Rev. A **58**, 4345 (1998); Phys. Rev. Lett. **82**, 2009 (1999).

[11] B. C. Sanders, Phys. Rev. A **45**, 6811 (1992).

[12] S. J. van Enk and O. Hirota, Phys. Rev. A **64**, 022313 (2001).

[13] H. Jeong, M. S. Kim, and J. Lee, Phys. Rev. A **64**, 052308 (2001).

[14] X. Wang, Phys. Rev. A **64**, 022302 (2001).

[15] H. Jeong and M. S. Kim, Phys. Rev. A **65**, 042305 (2002).

[16] H. Jeong and M. S. Kim, Quantum Inf. Comput. **2**, 208 (2002).

[17] T. C. Ralph, A. Gilchrist, G. J. Milburn, W. J. Munro, and S. Glancy, Phys. Rev. A **68**, 042319 (2003).

[18] N. B. An, Phys. Rev. A **68**, 022321 (2003).

[19] A. P. Lund, T. C. Ralph, and H. L. Haselgrove, Phys. Rev. Lett. **100**, 030503 (2008).

[20] N. B. An, Phys. Lett. A **373**, 1701 (2009).

[21] P. Marek and J. Fiurášek, Phys. Rev. A **82**, 014304 (2010).

[22] H. Jeong, M. Paternostro, and T. C. Ralph, Phys. Rev. Lett. **102**, 060403 (2009).

[23] M. Paternostro and H. Jeong, Phys. Rev. A **81**, 032115 (2010).

[24] A second relation also introduced by Leggett, namely,

$$-1 + |\overline{A} + \overline{B}| \leqslant \overline{AB}, \qquad (26)$$

is redundant. It is easily obtained by taking $B \rightarrow -B$ in Eq. (1) and gives rise to the same consequences as Eq. (1).

[25] The derivation of Leggett inequality given here skips a number of technical and nontrivial steps whose presentation is immaterial for the tasks and purposes of our work. The reader interested in such details could consult the original paper by Leggett [3] as well as Refs. [4–6].

[26] H. Jeong, W. Son, M. S. Kim, D. Ahn, and C. Brukner, Phys. Rev. A **67**, 012106 (2003).

[27] M. Stobinska, H. Jeong, and T. C. Ralph, Phys. Rev. A **75**, 052105 (2007).

[28] H. Jeong, Phys. Rev. A **78**, 042101 (2008).

[29] C.-W. Lee and H. Jeong, Phys. Rev. A **80**, 052105 (2009).

[30] Z.-B. Chen, J.-W. Pan, G. Hou, and Y.-D. Zhang, Phys. Rev. Lett. **88**, 040406 (2002).