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Quantum Fisher information on its own is not a valid measure of the coherence

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A B S T R A C T

We show that contrary to the claim in Feng and Wei (2017), the quantum Fisher information itself is not a valid measure of the coherence based on the resource theory because it can increase via an incoherent operation.

In Ref. [1], the authors claim that the quantum Fisher information (QFI) is a coherence measure that satisfies the conditions suggested by Baumgratz et al. [2]. Here we demonstrate the opposite with a clear counterexample for which the QFI increases via an incoherent operation.

The resource theory of coherence [2] with respect to a fixed basis $\{|i\rangle\}$ can be constructed by a set of incoherent states $\delta \in \Pi$ that contain only diagonal components, i.e., $\delta = \sum_i p_i\, |i\rangle\langle i|$ and a set of incoherent operations Φ which map every incoherent state into another incoherent state, i.e., $\Phi(\Pi) \subseteq \Pi$. A coherence measure $C(\rho)$ for state ρ should then satisfy the following conditions [2]:

- (C1) $C(\rho) \geqslant 0$ and $C(\rho) = 0$ iff $\rho \in \Pi$.
- (C2a) Non-increasing under an incoherent completely positive and trace preserving operation $\mathscr{O}_I$, i.e., $C(\rho) \geqslant C(\mathscr{O}_I[\rho])$.
- (C2b) Non-increasing on average by selective incoherent operations, i.e., $C(\rho) \geqslant \sum_n p_n C(A_n \rho A_n^\dagger / p_n)$, where Kraus operators A_n is an incoherent map, $\Phi(\rho) = \sum_n A_n \rho A_n^\dagger$ and $p_n = \mathrm{Tr}\rho A_n^\dagger A_n$.
- (C3) Convexity $\sum_n p_n C(\rho_n) \geqslant C(\sum_n p_n \rho_n)$.

The authors of Ref. [1] claim that the QFI with respect to a given Hamiltonain H

$$F(\rho, H) = 2\sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |\langle \lambda_i | H | \lambda_j \rangle|^2$$

satisfies all the conditions for a coherence measure (C1)–(C3) with respect to the eigenbasis of the Hamiltonian H, where λ_i and $|\lambda_i\rangle$ are eigenstates and eigenvalues of the quantum state ρ, respectively.

However, the proof of (C2) for the QFI is incorrect. There exists a counterexample in which an incoherent operation can increase the QFI. We consider a Hamiltonian in an N-level system with equal energy spacing,

$$H = \sum_{n=1}^{N} n\, |n\rangle\langle n|.$$

Assume that a quantum state $|\psi\rangle$ is initially given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

We consider an incoherent unitary operation

$$U = |N\rangle\langle 1| + |1\rangle\langle N| + |0\rangle\langle 0| + \sum_{n=2}^{N-1} |n\rangle\langle n| \tag{1}$$

which simply exchanges $|1\rangle$ and $|N\rangle$, while leaving the other eigenstates unchanged. It is important to note that U as an incoherent operation maps any incoherent state into another incoherent state.

Under this incoherent unitary U,$|\psi\rangle$ evolves to

$$U\,|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |N\rangle).$$

Using the fact that the QFI equals to four times of the variance of H for a pure quantum state, we can show that QFI before and after the incoherent unitary U is given by $F(|\psi\rangle, H) = 1$ and $F(U\,|\psi\rangle, H) = N^2$, respectively. It is thus immediately clear that the QFI can increase through an incoherent operation, and $F(U\,|\psi\rangle, H) > F(|\psi\rangle, H)$ for every $N > 2$. We conclude that the quantum Fisher information in general cannot be a valid coherence measure in the context of the resource theory of coherence formulated in Ref. [2].

In particular, we point out some misleading points given in Ref. [1]. In the proof of (C2a) in Ref. [1], the following unitary transformation was introduced:

$$\rho(\theta) = U_\theta \rho U_\theta^\dagger,$$

where $U_\theta = \exp(-i\theta H)$. Then an operator monotone function

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$f: \mathbb{R}^+ \to \mathbb{R}^+$ defines a positive linear mapping $\mathbb{J}_\rho^f = f(\mathbb{L}_\rho \mathbb{R}_\rho^{-1})\mathbb{R}_\rho$ with $\mathbb{L}_\rho(A) = \rho A$ and $\mathbb{R}_\rho(A) = A\rho$ that leads to a generalized QFI function [3]

$$F_\rho^f(\theta) = \mathrm{Tr}[\partial_\theta \rho(\theta)(\mathbb{J}_\rho^f)^{-1}(\partial_\theta \rho(\theta))].$$

Based on the monotonicity of f, the following inequality holds for the linear mapping $\mathbb{J}_\rho^f$

$$\mathrm{Tr}[\Phi(\partial_\theta \rho(\theta))(\mathbb{J}_{\Phi(\rho)}^f)^{-1}\Phi(\partial_\theta \rho(\theta))] \leqslant \mathrm{Tr}[\partial_\theta \rho(\theta)(\mathbb{J}_\rho^f)^{-1}\partial_\theta \rho(\theta)], \quad (2)$$

where the proof is given in Ref. [3]. However, it is important to notice that the QFI after acting incoherent operation Φ is actually given by

$$F(\Phi(\rho),H) = \mathrm{Tr}[\partial_\theta \Phi(\rho)(\theta)(\mathbb{J}_{\Phi(\rho)}^f)^{-1}\partial_\theta \Phi(\rho)(\theta)]$$

by choosing an appropriate operator monotone function f. Thus the following condition

$$\Phi(\rho(\theta)) = \Phi(\rho)(\theta) = U_\theta \Phi(\rho) U_\theta^\dagger \quad (3)$$

is additionally required for an incoherent operation to prove (C2a) completely by following the proof given in Ref. [1]. We point out that this is not the case in general, since $\Phi(U_\theta \rho U_\theta^\dagger) = U_\theta \Phi(\rho) U_\theta^\dagger$ is a stronger condition than $\Phi(\Pi) \subseteq \Pi$ when H has nondegenerate eigenvalues. In fact, the unitary operation given by Eq. (1) is an example of an incoherent operation that satisfies $\Phi(\Pi) \subseteq \Pi$ but does not satisfy Eq. (3).

A similar issue can be raised concerning the proof of (C2b). The authors in Ref. [1] assumed that the dynamic process of a system can be expressed by a unitary operation in addition to an ancillary state $|\psi\rangle_B$ and Hamiltonian H_B,

$$\rho_{AB}(t) = V(\rho_A(0) \otimes |\psi\rangle_B\langle\psi|)V^\dagger,$$

where the unitary operator V satisfies

$$[V, H_A \otimes I_B + I_A \otimes H_B] = 0. \quad (4)$$

The selective operation is then defined by the projection $\{|\beta_l\rangle_B\}$ onto the eigenstates of H_B. Again, U given by Eq. (1) for $H_A = \sum_{n=1}^N n\, |n\rangle\langle n|$ does not satisfy Eq. (4) although it is a valid incoherent operation, i.e. $[U, H_A \otimes I_B + I_A \otimes H_B] \neq 0$ regardless of the choice of H_B and $|\psi\rangle_B$. This implies that a unitary evolution $V(\rho_A \otimes |\psi\rangle_B\langle\psi|)V^\dagger$ in addition to the projection onto the support of H_B is not sufficient to describe every selective incoherent operation suggested in Ref. [2].

Nevertheless, the QFI and f-dependent QFI functions studied in Ref. [1] may be useful quantities to characterize and quantify coherence when we restrict incoherent operations to a set of translationally-covariant operations satisfying Eq. (3). In this case, the quantifiable amount of coherence resource can be interpreted as the degree of broken symmetry under a group transformation [4]. In this point of view, beginning from its foremost application in quantum metrology [5], the QFI and related asymmetry measures have been studied in various contexts including reference frame alignment [6], quantum speed limit [7], and quantum macroscopicity [8,9].

As we described in this paper, however, it should be carefully addressed in which regime quantum coherence is characterized among different sets of incoherent operations, especially between a incoherent map related to a fixed set of incoherent basis $\{|i\rangle\}$ and a set of translationally-covariant operations with respect to some generator H. These difference notions of coherence have been well described in Ref. [10].

We finally point out that although the QFI itself is not a valid measure of the coherence, a proper coherence measure can be defined by optimizing the QFI as $\max_\Phi F(\Phi(\rho),H)$ over all possible incoherent operations Φ as detailed in Ref. [11].

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