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Cascading toppling dynamics on scale-free networks

K.-I. Goh, D.-S. Lee, B. Kahng*, D. Kim

School of Physics and Center for Theoretical Physics, Seoul National University, Seoul 151-747, Republic of Korea

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Abstract

We study avalanche dynamics on scale-free networks, following a power-law degree distribution, $p_d(k) \sim k^{-\gamma}$, through the Bak–Tang–Wiesenfeld sandpile model. The threshold height of a node i is set to be $k_i^{1-\eta}$ with $0 \leqslant \eta < 1$. We obtain the exponents for the avalanche size and the duration distributions analytically as a function of γ and η by using the branching process approach. The analytic solution is checked with numerical simulations on both artificial uncorrelated networks such as the static model and real-world networks. While numerical results of the avalanche size distribution for artificial uncorrelated scale-free networks are in reasonable agreement with the analytic prediction, those for real-world networks are not, which may be attributed to non-trivial degree–degree correlations in real-world networks.

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*Corresponding author.
E-mail address: kahng@phya.snu.ac.kr (B. Kahng).

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1. Introduction

Avalanche dynamics, triggered by small initial perturbation, but spreading to other constituents successively, is one of the intriguing problems in physics [1–10]. Such avalanche dynamics manifests itself in diverse forms such as cultural fads [1], virus spreading [2], disease contagion [3], blackout in power transmission grids [4], data packet congestion in the Internet [5,6], and so on. In particular, avalanche dynamics on complex networks is more intriguing, because it can occur more easily due to close inter-connectivity among constituents. Here, we concern ourselves with the avalanche dynamics on complex networks with a power-law degree distribution, that is, $p_d(k) \sim k^{-\gamma}$, k being the degree, for avalanche dynamics, which is called scale-free (SF) networks.

Branching process approach is a powerful tool to study the avalanche dynamics on complex networks, which is valid when avalanche trails of successive toppling nodes do not form any loop structure. The validity of the assumption has been checked numerically for artificial SF networks such as the static model. Indeed, the number of loops is almost negligible in the thermodynamic limit. The static model was introduced to generate a SF network in a simple way. Each vertex i has a prescribed weight P_i summed to 1 and an edge can connect vertices i and j with rate P_iP_j. So, the degrees at each end of a given edge are uncorrelated. Due to the weights $\{P_i\}$, degrees of each vertex are heterogeneous. In this paper, we investigate how such SF network topology affects the avalanche dynamics through the Bak–Tang–Wiesenfeld (BTW) sandpile model [11].

The model was studied extensively as a prototypical system showing self-organized criticality (SOC), mostly on regular lattices in the Euclidean space. In the stationary state, without tuning a parameter, the system shows scale-invariant features in its avalanche size distribution $p_a(s)$ and duration or lifetime one $\ell(t)$ as

$$p_a(s) \sim s^{-\tau} \quad \text{and} \quad \ell(t) \sim t^{-\delta} . \tag{1}$$

Bonabeau has studied the sandpile dynamics on the Erdős–Rényi (ER) random networks [12] and found that the avalanche size distribution follows a power law with the exponent $\tau \simeq 1.5$. Interestingly, that value of τ is equal to the mean-field solution [13] that does not hold for the sandpile dynamics on low-dimensional regular lattices. Recently, Lise and Paczuski [14] have studied the Olami–Feder–Christensen model [15] on regular ER networks, where degree of each node is uniform but connections are random. They found the exponent to be $\tau \approx 1.65$. However, when degree of each node is not uniform, they found no criticality in the avalanche size distribution. Note that they assumed that the threshold of each node is uniform, whereas degree is not. Here we study the BTW sandpile model on SF networks, where the threshold at the node i is set to be $k_i^{1-\eta}$ with k_i the degree and $0 \leqslant \eta < 1$. Note that when $\eta = 1$, $z_i = 1$ for all i. Then the toppling dynamics is too trivial, so that this case is excluded. We find that the avalanche size distribution as well as the duration one depends on the degree exponent γ as $\tau = (\gamma - 2\eta)/(\gamma - 1 - \eta)$ and $\delta = (\gamma - 1 - \eta)/(\gamma - 2)$ for $\gamma < 3 - \eta$ while, for $\gamma > 3 - \eta$, they show the same behaviors as the conventional mean-field solutions as observed for the ER random networks. We also study the avalanche size distribution numerically for various modifications of the sandpile model on the static model. Finally, the numerical results for the sandpile model on uncorrelated SF networks are compared with those for real-world correlated networks such as the Internet and the coauthorship network.

2. Sandpile model

We present the dynamic rule of the BTW sandpile model on general networks.

(i) At each time step, a grain is added at a randomly chosen node i.
(ii) If the height at the node i reaches or exceeds a prescribed threshold z_i, then it becomes unstable and the z_i grains at the node topple to its randomly chosen z_i adjacent nodes among k_i ones; $h_i \to h_i - z_i$, and $h_j = h_j + 1$, where j is the randomly chosen neighbor node.
(iii) If this toppling causes any of the adjacent nodes receiving grains to be unstable, subsequent topplings follow on those nodes in parallel until there is no unstable node left. This process defines an avalanche.
(iv) Repeat (i)–(iii).

Here the threshold z_i of i node is given by

$$z_i = k_i^{1-\eta} \,. \tag{2}$$

To prevent overflow of the system, a small number of grains needs to be removed from the system in actual numerical simulations. This will be mentioned in Section 3. The BTW model with the threshold in Eq. (2) is the generalized one of that in Ref. [16]. We study this model analytically as well as numerically. We concentrate on the distribution of (a) the avalanche area A, i.e., the number of distinct nodes participating in a given avalanche, (b) the avalanche size S, i.e., the number of toppling events in a given avalanche, and (c) the duration T of a given avalanche.

2.1. Branching process approach

The mapping of each avalanche to a tree provides a useful way of understanding the statistics of avalanche dynamics analytically. Whenever an avalanche occurs, one can draw a corresponding tree: the node where an avalanche is triggered corresponds to the originator of the tree and the other nodes participating in the avalanche subsequently to the descendants. In the tree structure, a descendant born at time t is located away from the originator by distance t along the shortest pathway. The tree stops to grow when no further avalanche proceeds. Then the ensemble of avalanches can be identified with that of trees grown by the branching process. In this mapping, the avalanche duration T is equal to the lifetime of the tree minus one, and the avalanche size S differs from the tree size only by the number of boundary nodes of the tree, which is relatively small when the overall tree size is very large. If one assumes that branching events at different nodes occur independently and that there is no loop in the tree, the tree size and lifetime distribution can be obtained analytically [17,18]. Those distributions are expected to share the same asymptotic behaviors with the avalanche size and duration distribution, respectively, due to the near-equivalence between an avalanche and its corresponding tree in their scales as mentioned above.

The branching probability $q(k)$ that a node generates k branches is the only parameter of a given branching process. For the BTW model in the Euclidean space with $z_i = k_i$, $q(k)$ has a finite cut-off such that $q(k) = 0$ for $k > k_c$ since the degree of each node is uniform and finite. Consequently, the avalanche size and duration distribution in Eq. (1) are described in terms of the mean-field exponents $\tau = 3/2$ and $\delta = 2$ [17,18]. These results are known to hold for the BTW model on regular lattices with dimensions larger than 4 [13].

In SF networks, the avalanches usually do not form a loop but have tree-structures: according to the numerical simulations of the BTW model with $z_i = k_i$ on SF networks, the probability distributions of the two quantities A and S are nearly equal when they are large; for example, the maximum area and size ($A_{\max}$, $S_{\max}$) among avalanches are (5127, 5128), (12 058, 12 059), and (19 692, 19 692) for the networks having the power-law degree distributions with the degree exponents $\gamma =$ 2.01, 3.0, and ∞, respectively. A and S being almost the same implies that the avalanche structure can be treated as a tree. From now on, we shall not distinguish A and S, and use s to represent either A or S. Thus it is justified to use the branching process approach to understand the avalanche dynamics on SF networks.

We consider the BTW model on SF networks with the threshold in Eq. (2) and the degree distribution $p_d(k) \sim k^{-\gamma}$. The branching probability consists of two factors: one is the probability $q_1(k)$ that a node has the threshold $k-1 < z_i \leqslant k$ and the other is $q_2(k)$ that the total number of grains at the node reaches or exceeds the threshold. Since $z_i = k_i^{1-\eta}$, $q_1(k)$ is the probability that the node i at the one end of a randomly chosen edge has its degree k_i in the region $\left((k-1)^{1/(1-\eta)}, k^{1/(1-\eta)}\right]$, and thus $q_1(k) = \sum_{k'=\lceil (k-1)^{1/(1-\eta)}\rceil}^{\lfloor k^{1/(1-\eta)}\rfloor} k p_d(k)/\langle k\rangle \sim k^{(1-\gamma+\eta)/(1-\eta)}$ for large k, where $\lceil x\rceil(\lfloor x\rfloor)$ is the smallest (largest) integer not smaller (larger) than x. $q_2(k)$ is the probability that the node i has height $k-1$ at the moment of gaining the grain from one of its neighbors. In the inactive state, there is no typical height of the node but every number of grain from 0 to $k-1$ is equally probable. Thus $q_2(k) = 1/k$. As a result, the branching probability $q(k)$ for large k is given asymptotically as

$$q(k) \sim k^{-\gamma'}, \quad \left(\gamma' = \frac{\gamma - 2\eta}{1-\eta}\right). \tag{3}$$

Notice that when $z_i = k_i$ or $\eta = 0$, γ' is equal to γ.

Using the independence of the branchings from different parent-nodes, one can derive the following self-consistent relation for the tree size distribution $p(s)$ as [17,18]

$$p(s) = \sum_{k=0}^{\infty} q(k) \sum_{s_1=1}^{\infty} \sum_{s_2=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} p(s_1)p(s_2)\ldots p(s_k)\delta_{\sum_{i=1}^{k} s_i, s-1}. \tag{4}$$

This relation can be written in a more compact form by introducing the generating functions, $\mathscr{P}(y) = \sum_{s=1}^{\infty} p(s)y^s$ and $\mathscr{Q}(\omega) = \sum_{k=0}^{\infty} q(k)\omega^k$ as

$$\mathscr{P}(y) = y\mathscr{Q}(\mathscr{P}(y)) . \tag{5}$$

Then $\omega = \mathscr{P}(y)$ is obtained by inverting $y = \mathscr{P}^{-1}(\omega) = \omega/\mathscr{Q}(\omega)$.

The distribution of duration, i.e., the lifetime of a tree growth, can be evaluated similarly [17,18]. Let $r(t)$ be the probability that a branching process stops at or prior to time t. Then following the similar steps leading to Eq. (4), one has

$$r(t) = \mathscr{Q}(r(t-1)) . \tag{6}$$

For large t, $r(t)$ comes close to 1. One can obtain $\omega = r(t-1)$ by solving $\mathrm{d}\omega/\mathrm{d}t \simeq r(t) - r(t-1) = Q(\omega) - \omega$. Then the lifetime distribution $\ell(t)$ is obtained through $\ell(t) = r(t) - r(t-1) \simeq \mathrm{d}\omega/\mathrm{d}t$.

2.2. *Avalanche size and duration distribution*

The growth of a tree depends on the average number of branches defined as

$$C = \sum_{k=1}^{\infty} kq(k) . \tag{7}$$

When $C > 1$ ($C < 1$), a tree can (cannot) grow infinitely in the probabilistic sense. The value $C = 1$ is thus the critical point distinguishing such distinct growth of a tree. One can see that with the branching probability $q(k)$ of the BTW sandpile model with $z_i = k_i^{1-\eta}$, the average number of branches C is 1 on arbitrary networks, which means that the sandpile model on them shows the SOC.

The inverse function $\mathscr{P}^{-1}(\omega)$ satisfies $\mathscr{P}^{-1}(1) = 1$. When $C = 1$, the first-order derivative $\partial\mathscr{P}^{-1}(\omega)/\partial\omega$ at $\omega = 1$ is zero and thus $\mathscr{P}(y)$ becomes singular at $y = 1$; it is expanded around $y = 1$ as $\mathscr{P}(y) \simeq 1 - a(1-y)^{\phi}$ with $0 < \phi < 1$. The asymptotic behavior of the avalanche or tree size distribution $p(s)$ for large s is then given by $p(s) \sim s^{-\phi-1}$ since

$$(1-y)^{\phi} = \sum_{s=0}^{\infty} a_s y^s, \quad \text{where} \quad a_s = \frac{\Gamma[s-\phi]}{\Gamma[s+1]\Gamma[-\phi]} \sim s^{-\phi-1}(s \to \infty) . \tag{8}$$

The functional form of the branching probability $q(k)$ determines the singularity of $\mathscr{P}(y)$ represented by the exponent ϕ. To understand it, first consider $q(k)$ decaying exponentially or with a finite cut-off. Then its generating function $\mathscr{Q}(\omega)$ is analytic for $\omega < \rho$ with $\rho > 1$ and $1 - \mathscr{P}^{-1}(\omega)$ around $\omega = 1$ has a second-order term in $(1-\omega)$ as the leading term . Thus

$$1 - \mathscr{P}(y) \sim (1-y)^{1/2} \quad \text{and} \quad p(s) \sim s^{-3/2} . \tag{9}$$

This is the conventional mean-field solution for the avalanche size distribution [13] and has been shown to hold for the BTW model on the ER random networks [12]. On the other hand, with a slowly-decaying branching probability such as in Eq. (3), its generating function $\mathscr{Q}(\omega)$ is singular at $\omega = 1$. For $q(k)$ in Eq. (3), the expansion of

$\mathcal{Q}(\omega)$ around $\omega = 1$ is given as [19]

$$\mathcal{Q}(\omega) \simeq 1 - (1-\omega) + \begin{cases} (1-\omega)^{\gamma'-1} & (2<\gamma<\gamma_c)\,, \\ -(1-\omega)^2 \ln(1-\omega) & (\gamma = \gamma_c)\,, \\ (1-\omega)^2 & (\gamma>\gamma_c)\,, \end{cases} \tag{10}$$

where γ' given in Eq. (3) and $\gamma_c = 3 - \eta$. Note that the singular term $(1-\omega)^{\gamma'-1}$ is the second leading term of $1 - \mathcal{Q}(\omega)$ for $\gamma<\gamma_c$. Using the relation $\mathcal{P}^{-1}(\omega) = \omega/\mathcal{Q}(\omega)$ in Eq. (5), the behavior of $\mathcal{P}(y)$ around $y = 1$ is obtained for each region of γ from Eq. (10), and in turn, using Eq. (8), we have

$$p(s) \sim \begin{cases} s^{-(\gamma-2\eta)/(\gamma-1-\eta)} & (2<\gamma<\gamma_c)\,, \\ s^{-3/2}(\ln s)^{-1/2} & (\gamma = \gamma_c)\,, \\ s^{-3/2} & (\gamma>\gamma_c)\,. \end{cases} \tag{11}$$

In other words, the exponent τ is given as $\tau = (\gamma - 2\eta)/(\gamma - 1 - \eta)$ for $2<\gamma<\gamma_c$ and $\tau = 3/2$ for $\gamma \geqslant \gamma_c$.

Also obtained is $r(t)$ from Eq. (10) using Eq. (6). The duration distribution $\ell(t)$ is the derivative of $r(t)$ and found to be

$$\ell(t) \sim \begin{cases} t^{-(\gamma-1-\eta)/(\gamma-2)} & (2<\gamma<\gamma_c)\,, \\ t^{-2}(\ln t)^{-1} & (\gamma = \gamma_c)\,, \\ t^{-2} & (\gamma>\gamma_c)\,. \end{cases} \tag{12}$$

That is, the exponent δ is given as $\delta = (\gamma - 1 - \eta)/(\gamma - 2)$ for $2<\gamma<\gamma_c$ and $\delta = 2$ for $\gamma \geqslant \gamma_c$.

3. Numerical simulations

In this section, we perform numerical simulations of the sandpile model to check the analytical prediction. Unlike the Euclidean lattices, the boundary sites are not explicitly defined in complex networks, without which the system will be overloaded in the end. Here, we consider two different choices of boundary assignment, the annealed assignment and the quenched one. For the former, we allow with a certain probability f grains to leave the system in the process of grain transfers in avalanche. In the latter case, we consider a certain set of nodes, for example, peripheral nodes (those with $k_i = 1$), as the boundary nodes at which grains leave the system upon their arriving. We generate SF networks through the static model [20] and simulate the BTW model with $z_i = k_i$ ($\eta = 0$) on them. Then the avalanche area distribution is shown in Figs. 1 and 2. In Fig. 1, we show for an ER network that the two boundary assignments produce the same scaling results. For the annealed boundary assignment, we used $f = 10^{-4}$. The only possible difference would be in the values of the cut-off, which is determined by f in the annealed case and by $p_d(1)$ in the quenched case. In Fig. 2, we show the avalanche area distributions for the SF networks with various degree exponents using the annealed boundary assignment. The numerical

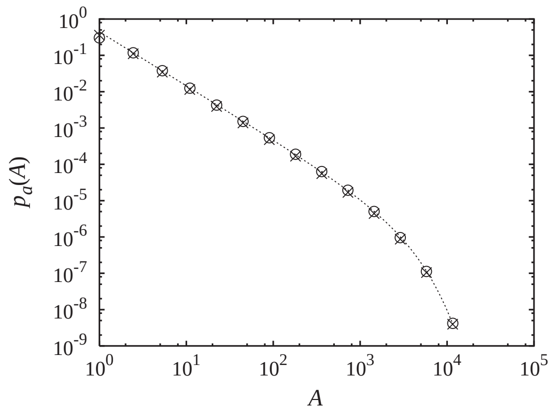

Fig. 1. The avalanche size distribution for the BTW model on the ER network with two different boundary assignments, the annealed boundary assignment (◯) and the quenched one (×).

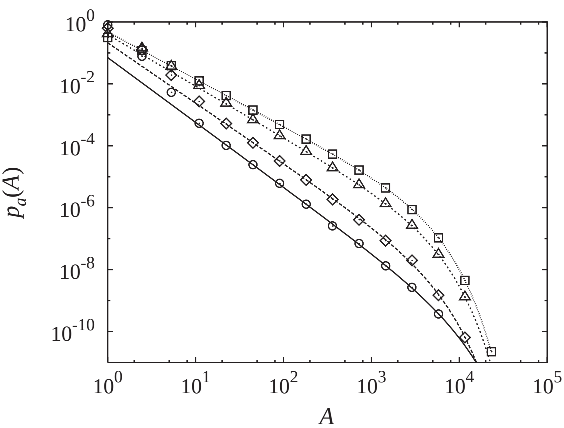

Fig. 2. Avalanche size distributions for the BTW models with the threshold $z_i = k_i$ ($\eta = 0$) on the SF networks with size $N = 10^6$ and $\gamma = \infty$ (□), 3.0 (△), 2.2 (◇), and 2.0 (◯) with the annealed boundary assignment.The data are logarithmically binned and fitted with the form $p_a(A) = aA^{-\tau}\exp(-A/A_c)$. The fitted values of τ are 1.52(1) (ER), 1.66(2) ($\gamma = 3.0$), 1.95(9) ($\gamma = 2.2$), and 2.09(8) ($\gamma = 2.0$), respectively, which are to be compared with the prediction $\tau = 3/2$ for $\gamma > 3$ and $\tau = \gamma/(\gamma - 1)$ for $2 < \gamma < 3$.

simulation results are in reasonable agreement with the analytical prediction, Eq. (11), although the agreement gets poorer for $\gamma \leqslant 3$. The discrepancy can be attributed to the finite-size effect and the presence of short circuits in the SF networks with $\gamma < 3$. In Fig. 3, we show the avalanche area distribution for a couple of values of η, $\eta = 0.4$ and 0.8 on the SF networks with $\gamma = 2.4$. We can clearly see the avalanche area distribution changes with η as theory predicts: τ should decrease as η increases and it finally reduces to the mean-field value for $\eta > 3 - \gamma$, but the precise agreement with the predicted τ is again imperfect.

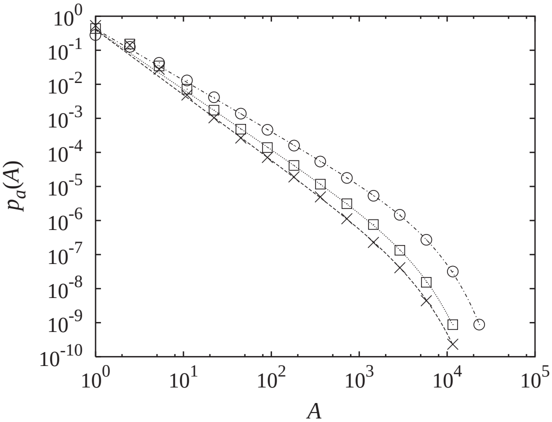

Fig. 3. The avalanche size distributions of the generalized BTW sandpile model with $\eta = 0$ ($\times$), 0.4 ($\square$), and 0.8 ($\bigcirc$) on the static model with $\gamma = 2.4$ and $N = 10^6$. The data are fitted to the formula $p_a(A) = aA^{-\tau}\mathrm{e}^{-A/A_c}$ and the fitted values of τ are 1.89(3) for $\eta = 0$, 1.76(4) for $\eta = 0.4$, and 1.51(4) for $\eta = 0.8$, respectively, to be compared with the predicted values 1.71, 1.60, and 1.5, respectively.

Next, we address the question of how robust these scaling results are against the variations in the dynamic rules of the model. First we modify rule (i) as

(i′) At each time step, a grain is added at a node i chosen preferentially, i.e., proportional to its degree k_i.

As is shown in Fig. 4(a), this modification does not affect the scaling behavior of the model. We also check the fluctuation effect in the threshold z_i. To maintain the heterogeneity of the threshold, we set $z_i = \xi_i k_i$, where ξ_i is a quenched noise uniformly distributed in $(0, 1)$. We find that such modification does not change the scaling behavior either (Fig. 4(b)).

4. Sandpile on real-world networks

It is very interesting how the results so far apply to the case of the real-world networks. There are two important ingredients to be taken into account in real-world networks to compare the analytic prediction with numerical results. One is finite-size effect and the other is degree–degree correlations. Due to the finiteness in system size, the branching probability is cut at a finite value of k, and the critical exponents for uncorrelated networks are expected to take the mean-field values, e.g., $\tau = 1.5$ in all cases. However, due to the degree correlations, the corresponding branching process becomes non-Markovian, which is beyond the applicability of the present framework of branching process based on the assumption of independence of consecutive toppling events.

In Fig. 5, we report the simulation results of the case $z_i = k_i$ for two real-world networks, the Internet at the autonomous systems level as of January 2000 and the

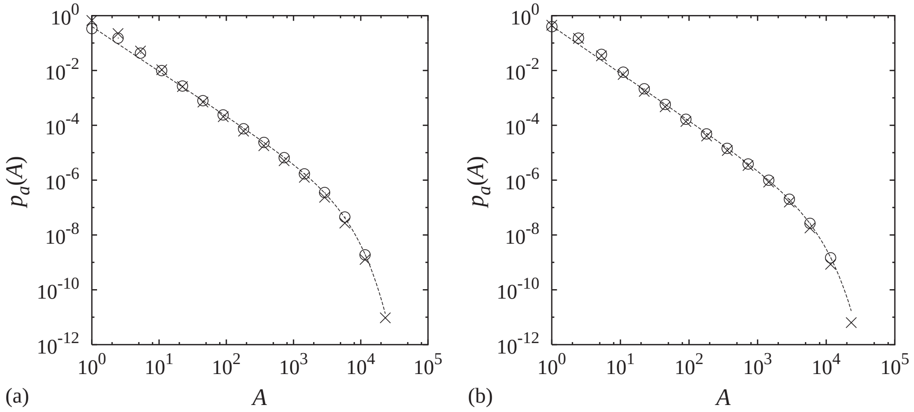

Fig. 4. The avalanche size distributions of the modified BTW sandpile model (○) for (a) preferential piling at $\eta = 0$ and (b) the noisy threshold case where $z_i = \xi_i k_i$, compared with that of the original model (×). Simulations are performed for the static model with $\gamma = 2.6$ and $N = 10^6$.

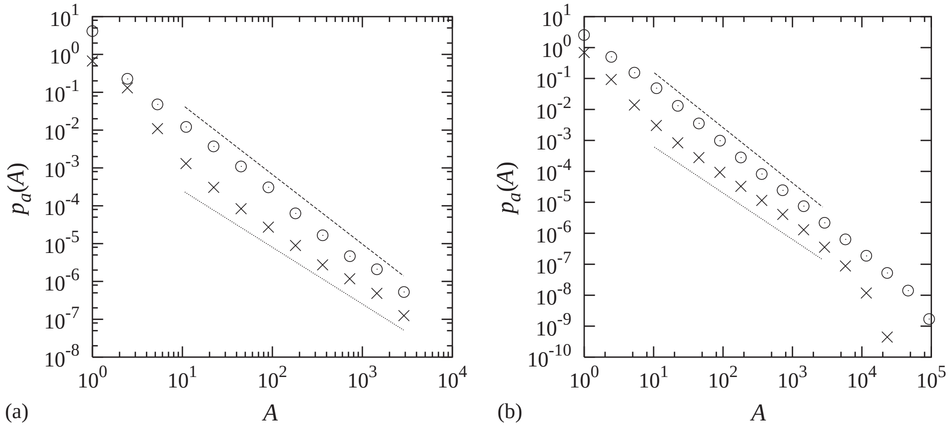

Fig. 5. The avalanche size distributions of the sandpile model on real-world networks (○), (a) the Internet and (b) the coauthorship network, and on their randomized configurations (×). Dashed lines have a slope −1.85 (a) and −1.8 (b) and both the dotted lines have slope −1.5, drawn for the eye. The data for real-world networks (○) are shifted vertically by the factor of 5 for visual clarity.

coauthorship network in neuroscience [22]. The Internet has the degree exponent about $\gamma \approx 2.1$ and shows dissortative mixing with the assortativity index [23] $r = -0.18$. The neuroscience coauthorship network has $\gamma \approx 2.2$ with additional cut-off decaying faster than a power law and shows strong assortative mixing with $r = 0.60$ (only the largest component is considered). The exponent of the avalanche size distribution is obtained to be $\tau \approx 1.85$ and 1.8 for the Internet and the coauthorship network, respectively. While the numerical values seem to be consistent with the theoretical prediction, $\tau = \gamma/(\gamma - 1) \approx 1.83$ for $\gamma = 2.2$, this conclusion is not correct.

We perform numerical simulations on the randomized network [24] of the real-world networks. Here, two edges are selected randomly and their connections are interchanged so as to remove the degree–degree correlation. We find that the avalanche size exponent $\tau \approx 1.5$ is observed for both randomized real-world networks. This result implies that the exponents τ for the randomized Internet is determined by the finite-size effect, and that for the randomized coauthorship network is determined by the non-SF nature of the network. So the deviations of the exponents for the real-world networks from $\tau = 3/2$ would originate from other effects. The degree–degree correlation should be one of them. Further investigations for this issue will be published elsewhere.

5. Conclusion

We have studied the BTW sandpile model on SF networks to understand the influence of the heterogeneity in degree on the avalanche dynamics.The main results are the avalanche size and duration distribution represented in Eqs. (11) and (12). The exponents τ and δ appearing in those distributions increase as γ decreases, which means that the hubs play a role of reservoir, that is, sustain large amount of grains to make the SF network resilient under avalanche phenomena. This is reminiscent of the structural resilience of the SF network under random removal of nodes for $\gamma \leqslant 3$ [21,25,26].

We also studied avalanche dynamics with modifications of avalanche dynamic rule, finding that the result of the sandpile avalanche rule is robust in many cases when embedded network structure exhibits no degree–degree correlation. However, numerical results of the avalanche size distribution for real-world networks are inconsistent with the theoretical prediction, which would come from the degree–degree correlations. To elucidate this inconsistence, one has to modify the branching process approach in an appropriate way, which remains as further studies.

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