

Fluctuation-Driven Dynamics of the Internet Topology

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We study the dynamics of the Internet topology based on empirical data on the level of the autonomous systems. It is found that the fluctuations occurring in the stochastic process of connecting and disconnecting edges are important features of the Internet dynamics. The network's overall growth can be described approximately by a single characteristic degree growth rate $g_{\text{eff}} \approx 0.016$ and the fluctuation strength $\sigma_{\text{eff}} \approx 0.14$, together with the vertex growth rate $\alpha \approx 0.029$. A stochastic model which incorporates these values and an adaptation rule newly introduced reproduces several features of the real Internet topology such as the correlations between the degrees of different vertices.

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Recently many studies on complex systems [1,2] paid attention to complex networks [3,4]. An interesting feature emerging in such complex systems is a power-law behavior in the degree distribution, $P_D(k) \sim k^{-\gamma}$ [5,6], where the degree k is the number of edges incident upon a given vertex. Recently, Barabási and Albert (BA) [6] introduced an evolving network model to illustrate such networks, called the scale-free (SF) networks, in which the number of vertices N increases linearly with time, and a newly introduced vertex is connected to already existing vertices following the so-called preferential attachment (PA) rule.

Huberman and Adamic (HA) [7] proposed another scenario for SF networks. They argued that the fluctuation effect arising in the process of connecting and disconnecting edges between vertices, is the essential feature to describe the dynamics of the Internet topology correctly. In this model, the total number of vertices $N(t)$ increases exponentially with time as

$$N(t) = N(0) \exp(\alpha t). \quad (1)$$

Next, it is assumed that the degree k_i at a vertex i evolves through the multiplicative process [8],

$$k_i(t+1) = k_i(t)[1 + \zeta_i(t+1)], \quad (2)$$

where $\zeta_i(t)$ is the growth rate of the degree k_i at time t , which fluctuates from time to time. Thus, one may divide the growth rate $\zeta_i(t)$ into two parts,

$$\zeta_i(t) = g_{0,i} + \xi_i(t), \quad (3)$$

where $g_{0,i}$ is the mean value over time, and $\xi_i(t)$ the rest part, representing fluctuations [9]. $\xi_i(t)$ is assumed to be a white noise satisfying $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t) \xi_j(t') \rangle = \sigma_{0,i}^2 \delta_{i,i'} \delta_{t,t'}$, where $\sigma_{0,i}^2$ is the variance. Here $\langle \dots \rangle$ is the sample average. For later convenience, we denote the logarithm of the growth factor as $G_i(t) \equiv \ln[1 + \zeta_i(t)]$. Then a simple application of the central limit theorem ensures that $k_i(t)/k_i(t_0)$, t_0 being a reference time, follows the log-normal distribution for sufficiently large t . To get the degree distribution, one needs to collect all contributions from different ages τ_i , growth rates $g_{0,i}$, standard

deviations $\sigma_{0,i}$, and initial degree $k_i(t_0)$. HA first assumed that ζ_i are identically distributed so that $g_{0,i} = g_0$ and $\sigma_{0,i} = \sigma_0$ for all i . Then the conditional probability for degree, $P_D(k, \tau | k_0)$, that $k_i = k$ at time $t = t_0 + \tau$, given $k_i = k_0$ at $t = t_0$ is given by

$$P_D(k, \tau | k_0) = \frac{1}{k \sqrt{2\pi \sigma_{\text{eff}}^2 \tau}} \times \exp\left\{-\frac{[\ln(k/k_0) - g_{\text{eff}}\tau]^2}{2\sigma_{\text{eff}}^2 \tau}\right\}, \quad (4)$$

where $g_{\text{eff}} \equiv \langle G_i(t) \rangle$ and $\sigma_{\text{eff}}^2 \equiv \langle [G_i(t) - \langle G_i(t) \rangle]^2 \rangle$. g_{eff} and σ_{eff}^2 are related to g_0 and σ_0^2 as $g_{\text{eff}} \approx g_0 - \sigma_0^2/2$ and $\sigma_{\text{eff}}^2 \approx \sigma_0^2$, respectively [11]. Since the density of vertices with age τ is proportional to $\rho(\tau) \sim \exp(-\alpha\tau)$, the degree distribution collected over all ages becomes

$$P_D(k) \sim \int d\tau \rho(\tau) P_D(k, \tau | k_0) \sim k^{-\gamma}, \quad (5)$$

where

$$\gamma = 1 - \frac{g_{\text{eff}}}{\sigma_{\text{eff}}^2} + \frac{\sqrt{g_{\text{eff}}^2 + 2\alpha\sigma_{\text{eff}}^2}}{\sigma_{\text{eff}}^2}. \quad (6)$$

Therefore, it is instructive to know the effective values of g_{eff} and σ_{eff} to determine the degree exponent.

The Internet topology on the level of the autonomous systems (AS) has been recorded by the National Laboratory for Applied Network Research (NLANR) [12] since November 1997, which enables one to study its evolution. Here a node represents an AS, which is a unit of router policy in the Internet, consisting of either a single domain or a group of domains. Analysis of these data sets has been performed by several research groups. Some of the findings are as follows: First, the numbers of vertices $N(t)$ and edges $L(t)$ increase exponentially with time [13], and $L(t) \sim N(t)^{1+\theta}$ with $\theta > 0$, showing the so-called accelerated growth [14]. Second, the degree distribution follows a power law with exponent, $\gamma \approx 2.2 \pm 0.1$ [13,15]. Third, there occurs the PA behavior in the evolution process [16]. Last, there exist certain nontrivial correlations

between the degrees of different vertices [15]. Meanwhile, Capocci *et al.* [17] considered the Internet on the levels of both the Internet Service Providers (ISPs) and hosts, in which the relative frequency of the creation of the ISPs and hosts determines the degree exponent.

In this Letter, we analyze the empirical data of the Internet topology from the viewpoint of fluctuation-driven adaptive dynamics arising in the process of connecting and disconnecting edges. We find that the fluctuation effect is indeed essential to the dynamics of the Internet topology. We measure the growth rate of vertices α and the effective values of g_{eff} and σ_{eff} , and determine the degree exponent γ using Eq. (6), which is well compared with the directly measured value. Moreover, using the measured values, we construct a stochastic model following the HA idea. In addition, we include in our model an adaptive process that favors rewiring towards vertices with higher degree. The network structure constructed in this way reproduces the topological features of the real Internet, such as the correlations between the degrees of different vertices.

Numerical analysis.—In our analysis, the data are selected monthly, so that time is discretized with one month as a unit time step. We have analyzed the data from November 1997 through January 2000, corresponding to $t = 0$ and 26, respectively. First we examined the total number of vertices $N(t)$ existing at time t , which grows exponentially with time as $N(t) = N(0)\exp(\alpha t)$ with $\alpha \approx 0.029(1)$ (see Fig. 1). The number of edges $L(t)$ also increases exponentially with time as $L(t) = L(0)\exp(\beta t)$ with $\beta \approx 0.034(2)$ (see Fig. 1), leading to the relation $L(t) = N(t)^{1+\theta}$ with $\theta \approx 0.16(4)$. While the total number of vertices increases with time, some vertices disappear from the data as time goes on due to permanent or temporary shutdown of the corresponding AS. Thus the number of vertices $N_0(t)$ introduced earlier than $t = 0$

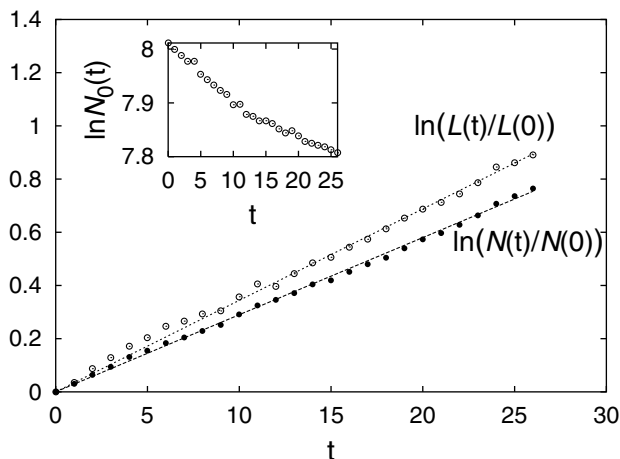


FIG. 1. Semilogarithmic plot of the total number of vertices $N(t)$ (\bullet) and edges $L(t)$ (\circ) as a function of time. The dashed and dotted lines obtained via the least-squares fit have slopes 0.029 and 0.034, respectively. Inset: Semilogarithmic plot of $N_0(t)$ versus t .

but still remaining at time t , decreases with time. Note that the decreasing rate of $N_0(t)$ is considerably reduced across $t \approx 14$ (see the inset of Fig. 1). The hub, the vertex with the largest degree, remains identical to the initial one throughout the period we studied.

The average number of degree $k_{\text{new}}(t)$ of the vertices newly introduced at time t fluctuates in time about a positive mean $\langle k_{\text{new}} \rangle_t \approx 1.34$ with a standard deviation $\sigma_{\text{new}} \approx 0.05$, where $\langle \dots \rangle_t$ denotes the average over a time interval T . This result suggests that each newly introduced vertex connects to only one or two existing vertices, and internal links between existing vertices are created actively as time goes on. So, the Internet becomes much more interwoven, and $L(t)$ grows faster than $N(t)$.

We consider the dynamics of the degree $k_i(t)$ at each vertex i as a function of time. For convenience, we deal with only the vertices existing all the time from $t = 0$ to $t = 26$, and the set composed of such vertices is denoted by S . Thus the degree growth rate of a vertex i , $G_i(t) = \ln[k_i(t)/k_i(t-1)]$ is well defined, because $k_i(t) \neq 0$ for any vertex $i \in S$ for all t . The measured value of $G_i(t)$ fluctuates in time about a finite value. Let $g_i \equiv \langle G_i \rangle_t$ and $\sigma_i^2 \equiv \langle (G_i - \langle G_i \rangle_t)^2 \rangle_t$. If the dynamics follows that of the HA model, a histogram of g_i for many vertices would show the Gaussian distribution with mean g_{eff} and variance σ_{eff}^2/T . We find that $\{g_i\}$ show some correlations with the degree of the vertex. In particular, the behavior of $G_h(t)$ at the hub is interesting (see the inset of Fig. 2). It is found that the fluctuation of $G_h(t)$ is drastically reduced across $t \approx 15$, February 1999. We obtain $g_h \approx 0.037(20)$ by averaging over the earlier period from $t = 1$ to 15, while $g_h \approx 0.031(6)$ over the later period from $t = 16$ to 26. Thus, the degree of the hub depends on $N(t)$ as $k_h(t) \sim N^\eta(t)$, where the exponent η is related to g_h as $\eta = g_h/\alpha$, exhibiting a crossover behavior from $\eta \approx 1.3(1)$ to $1.0(1)$ as directly measured (see Fig. 2). Note that $\eta = 0.5$

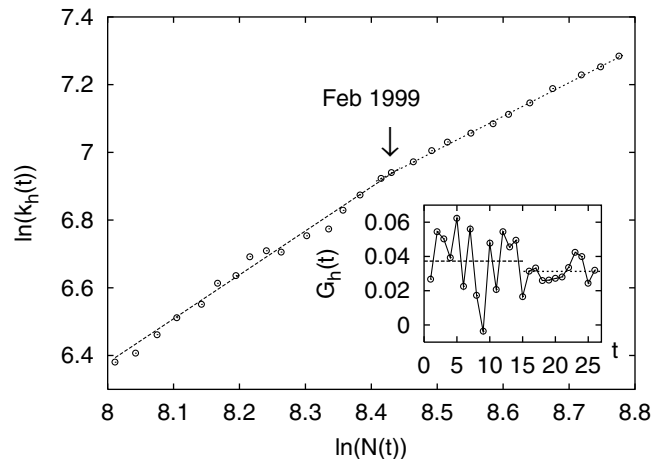


FIG. 2. Plot of the degree of the hub $k_h(t)$ versus the number of vertices $N(t)$. Inset: Plot of the growth rate $G_h(t)$ of the degree at the hub versus t . The dotted lines are the average values for given intervals.

in the BA model. Despite the apparent correlation of the growth rate with degree, we do not attempt further analysis but rather focus on the distribution of $\{g_i\}$ below in accordance with the HA idea. Data shown in the insets of Figs. 1 and 2 suggest that the Internet topology has become much stabilized around $t \approx 15$. Therefore, we will use only the data of the later period for further discussions.

We measured the mean growth rate g_i and the corresponding standard deviation σ_i defined above for each vertex i ($i \in S$) by taking the average over the period from $t = 16$ to 26 ($T = 10$). The measured values $\{g_i\}$ ($i \in S$) are distributed as shown in the inset of Fig. 3. In this inset, an abnormal peak is located at $g = 0$, which is mostly contributed by the vertices whose degree is a few and never changes at all during the period we studied. Thus those vertices may be regarded as the ones located at dangling ends in the graph, and be ignored for the dynamics of the Internet topology. In the inset of Fig. 3, we fit the data other than the $g = 0$ peak to the Gaussian form with mean $\bar{g} \approx 0.016(2)$ and standard deviation $\sigma_g \approx 0.04$. On the other hand, the measured values $\{\sigma_i\}$ are also distributed but with small dispersion around the mean $\bar{\sigma} \approx 0.12(6)$. In the HA model, \bar{g} and σ_g would correspond to g_{eff} and $\sigma_{\text{eff}}/\sqrt{T}$, respectively.

It is most likely that \bar{g} and $\bar{\sigma}$ have a distribution among vertices. To obtain more accurate degree distribution, Eq. (4) has to be averaged over those distributions. Not knowing such details, however, we try to approximate the growth process by a single process whose effective mean growth rate and standard deviation are g_{eff} and σ_{eff} , respectively. For this purpose, we plot in Fig. 3 the distribution $P[k_i(t)/k_i(t_0)]$ in terms of the scaled variables x and

y defined as

$$x \equiv \frac{\ln[k_i(t)/k_i(t_0)] - g_d(t - t_0)}{\sqrt{2\sigma_d^2(t - t_0)}}, \quad (7)$$

and

$$y \equiv P[k_i(t)/k_i(t_0)][k_i(t)/k_i(t_0)]\sqrt{2\pi\sigma_d^2(t - t_0)}, \quad (8)$$

using a semilogarithmic scale, where g_d and σ_d are fitting parameters. We choose $t_0 = 0$, and the data shown in Fig. 3 are for times $t > 15$. It appears that the data for different times collapse onto the curve $\ln y = -x^2$ reasonably well for small x with our best choice of $g_d = 0.016$ and $\sigma_d = 0.14$ as shown in Fig. 3. Larger deviations for large x are due to t being finite and are caused by the rare statistics of a few nodes whose degree increases by an anomalously large factor. The values g_d and σ_d are close to \bar{g} and $\bar{\sigma}$, respectively. Also, $\sigma_d/\sqrt{T} \approx 0.044$ is consistent with σ_g . We also checked σ_d by measuring the variance of $P[k_i(t)/k_i(t_0)]$ for each time, and plotting them as a function of time. The slope of the asymptotic line in the plot corresponds to σ_d^2 . Using this method, we also obtain $\sigma_d \approx 0.14(1)$, which is in agreement with the one obtained through the data-collapse method. Thus, the values $g_{\text{eff}} \approx 0.016$ and $\sigma_{\text{eff}} \approx 0.14$ may be regarded as the effective values representing the growth process as a single stochastic process. Applying those values to the formula Eq. (6), we obtain the degree exponent $\gamma \approx 2.1$, which is in agreement with the directly measured one $\gamma_{\text{AS}} \approx 2.2(1)$ [13,15].

Stochastic model.—Using the measured values, α , g_{eff} , and σ_{eff} , and following the HA idea, we construct a stochastic model evolving through the following three rules: (i) Geometrical growth: At time t , a geometrically increased number of new vertices, $\alpha N(t - 1)$, are introduced in the system, and following the fact $\langle k_{\text{new}} \rangle_t \approx 1.34$, each of them connects to one or two existing vertices according to the PA rule. (ii) Accelerated growth: Each existing vertex increases its degree by the factor $g_0 \approx g_{\text{eff}} + \sigma_{\text{eff}}^2/2$. These internal edges are also connected following the PA rule. (iii) Fluctuation and adaptation: Each vertex disconnects existing edges randomly (respectively, connects new edges following the PA rule) when the noise $[\xi_i(t)$ in Eq. (3)] is chosen to be negative (respectively, positive). This fluctuation has the variance $\sigma_0^2 \approx \sigma_{\text{eff}}^2$. When connecting, the PA rule is applied only within the subset of the existing vertices consisting of those having more degree than the one previously disconnected. This last constraint accounts for the adaptation process in our model. Through this adaptation process, the Internet becomes more efficient.

With this stochastic model, we first measure the degree exponent to be $\gamma_{\text{model}} \approx 2.2$, close to the empirical result $\gamma_{\text{AS}} \approx 2.2(1)$. Second, the clustering coefficient is measured to be $C_{\text{model}} \approx 0.15(7)$, comparable to the empirical value $C_{\text{AS}} \approx 0.25$ (see also Ref. [15]). Note that without

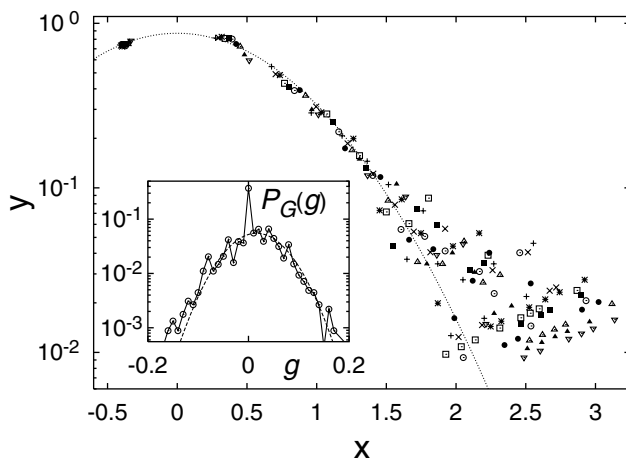


FIG. 3. Plot of $P[k_i(t)/k_i(0)]$ versus $k_i(t)/k_i(0)$ for different times in terms of y and x defined in the text. The dotted line is our best fit of which the peak is located at $g_d = 0.016$ and the standard deviation is $\sigma_d = 0.14$. Inset: Plot of the distribution of $g_i = \langle G_i \rangle_t$, $P_G(g)$, versus $g = g_i$ for the vertices in the set S . The dashed line is our best fit of the central part following the Gaussian distribution of which the peak is located at $\bar{g} = 0.016$ and the standard deviation is $\sigma_g = 0.04$.

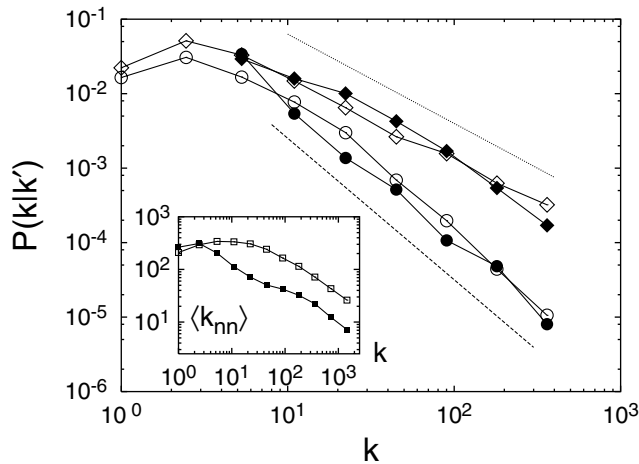


FIG. 4. Plot of the conditional probability $P(k|k')$ for the dangling vertices with $k' = 1$ (diamonds) and the hub (circles). Data points with filled (open) symbols are from the real Internet data (the model simulations). The dashed (dotted) line has a slope -1.9 (-1.1), drawn for a guide to the eye. Inset: Plot of the average degree of nearest neighbors of a vertex whose degree is k , $\langle k_{nn} \rangle = \sum_{k'} k' P(k'|k)$, as a function of k from the model simulation (open square) and the real data (filled square).

the adaptation rule, we get only $C \approx 0.01(1)$. Third, we measure the conditional probability, $P(k|k')$, that the degree of a vertex is k given that it is connected from a vertex with degree k' . For both our model and the real data, it is obtained that $P(k|k') \sim k^{-1.1(1)}$ for small k' and $\sim k^{-1.9(1)}$ for large k' (see Fig. 4). Note that, for linearly growing networks with the PA rule, it is known that the probability that vertices of degree k (ancestor) and k' (descendent) are connected scales as $\sim k^{-(\gamma-1)} k'^{-2}$ [4,18]. Finally, we measure the average degree of the nearest neighbors of a vertex whose degree is k , $\langle k_{nn} \rangle = \sum_{k'} k' P(k'|k)$, as a function of k . It exhibits a decaying behavior for large k , in agreement with the observation for the real Internet topology, as shown in the inset of Fig. 4 (see also Ref. [15]). This is in contrast with the k -independent behavior occurring in the BA model. The adaptive feature in rule (iii) is crucial to reproduce such detailed agreements between our model and the real data: While the degree exponent $\gamma_{\text{model}} \approx 2.2$ can be obtained without the adaptation rule, other results cannot be obtained without it. So adaptation and fluctuation are essential ingredients to describe the real Internet topology correctly. Meanwhile, due to the adaptation effect, the network is more centralized to the hub, so that it spreads diseases more quickly via the hub [19] and becomes more vulnerable to the attacks [20].

Summary.—The Internet topology evolves exponentially in the number of vertices and edges as time goes on. The degree of each vertex also increases exponentially with time, but its growth rate fluctuates strongly from time to time. The effect induced by such fluctuations is essential [21]. This has not been incorporated properly

so far in most scale-free network models. Based on the numerical measurement, we construct a stochastic model following the HA idea. In addition, the adaptation process arising in the evolution of edges has been newly taken into account in our model, through which we can reproduce the correlations between the degrees of different vertices in the real Internet.

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