Phase Diagram of a Disordered Boson Hubbard Model in Two Dimensions

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We study the zero-temperature phase transition of a two-dimensional disordered boson Hubbard model. The phase diagram is constructed in terms of the disorder strength and the chemical potential. Via Monte Carlo simulations, we find a multicritical line separating the weak-disorder regime, where the Mott-insulator-to-superfluid transition occurs, from the strong-disorder regime, where the Bose-glass-to-superfluid transition occurs. On the multicritical line, the insulator-to-superfluid transition has the dynamical critical exponent $z = 1.35 \pm 0.05$ and the correlation length critical exponent $\nu = 0.67 \pm 0.03$. We suggest that the proliferation of the particle-hole pairs screens out the weak-disorder effects.

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The boson localization due to a random potential has continuously attracted significant attention as a paradigm of the zero-temperature quantum phase transition [1]. The superconductor-insulator transition has been believed to be a manifestation of the boson localization transition. Two-dimensional realization of this transition may be found in disordered thin-film superconductors [2], Josephson-junction arrays [3], and $^4$He films adsorbed in porous media [4]. An interacting boson model, called the boson Hubbard model, has been proposed [5] to describe the transition between an insulator and a superfluid (SF). In the disorder-free case, the insulating ground state is a Mott insulator (MI), which has commensurate boson density and a finite Mott gap which suppresses the excitations of freely moving particles or holes. As the gap vanishes the system becomes a superfluid. In the presence of disorder, even with a vanishing energy gap, particles or holes excited can be localized by a random potential, resulting in a Bose glass (BG) insulator.

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FIG. 1. Schematic phase diagram of a disordered boson Hubbard model. Here $\mu$ is the chemical potential, $\Delta$ is the strength of disorder, and $K$ is the parameter characterizing the hopping of bosons. The particle-hole symmetry is defined by the condition $\mu = 0$. The multicritical line separates the direct MI-SF transition region and the BG-SF transition region.
tip of the MI lobe for weak disorder. This means that the weak disorder is irrelevant for the localization transition in two-dimensional interacting boson systems that have near integer fillings. Strong disorder changes the nature of the transition to that of the BG-SF transition.

We consider a boson Hubbard model on the two-dimensional square lattice, given by the Hamiltonian

$$H = \frac{1}{2} \sum_i U n_i^2 - \sum_i \mu_i n_i - \frac{t}{2} \sum_{\langle i,j \rangle} (b_i^\dagger b_j + b_j^\dagger b_i^\dagger),$$  

(1)

where $b_i^\dagger (b_i)$ denotes the boson creation (destruction) operator at site $i$ $(n_i = b_i^\dagger b_i)$, $\mu_i$ denotes the local chemical potentials, $U$ denotes the on-site repulsion energy, $t$ denotes the hopping strength to the nearest neighbors, and finally the last sum is over nearest neighbor pairs. The disorder effect is embedded in local chemical potential as $\mu_i = \mu + \nu_i$, where $\nu_i$ are random variables independently and uniformly distributed in the range $[-\Delta, \Delta]$. Thus $\Delta$ characterizes the strength of disorder.

To perform Monte Carlo simulations, we follow the standard procedure [16] to transform the two-dimensional Hamiltonian in Eq. (1) to the $(2 + 1)$-dimensional classical action

$$S = \frac{1}{2K} \sum_{(x,y,\tau)} [\tilde{J}_x^2(x,y,\tau) - 2(\mu + \nu_\tau)J^x(x,y,\tau)],$$  

(2)

where $x$ and $y$ are spatial coordinates and $\tau$ is the imaginary-temporal coordinate. Here $K = \sqrt{1/U}$, analogous to the temperature in classical systems, and $\tilde{J}(x,y,\tau)$ is the integer current vectors which measure the fluctuations along the corresponding direction of the components. In the transformation from Eq. (1) to Eq. (2) one assumes that superfluidity is destroyed only by phase fluctuations.

We are interested in the phase diagram of Eq. (2) in the space of $(K, \mu, \Delta)$. The transition described by the classical action is studied by Monte Carlo simulations. We perform the simulations on the lattice of various sizes, denoted by $L \times L \times L_\tau$, where $L$ and $L_\tau$ are sizes of the systems along a spatial and the imaginary-temporal axis, respectively. The periodic boundary conditions are adopted. In order to extract the critical properties of the transition, we analyze the data using the finite-size scaling theory. An important quantity which indicates the onset of the superfluidity is the superfluid stiffness, which is measured by the formula

$$\rho = \frac{1}{L_\tau} \langle n_x^2 \rangle_{av},$$  

(3)

where $n_x = (1/L) \sum_{(x,y,\tau)} J^x(x,y,\tau)$ is the winding number along the $x$ direction. Here $[\ldots]_{av}$ denotes the average over different realizations of disorder. The finite-size scaling behavior of the superfluid stiffness is given by [17]

$$\rho = L^{-(d+z-2)} \tilde{\rho}(L^{1/r} D, L_\tau/L_\tau^z),$$  

(4)

where $\delta = (K - K_c)$ is the distance from the critical point $K_c$ for each $\Delta$ and $\mu$, $\tilde{\rho}$ is a scaling function, and $d$ is the spatial dimension. Throughout this work, we set $d = 2$. Another useful quantity is the compressibility, which directly shows whether the Mott gap vanishes at the transition. It is given by the formula

$$\kappa = \frac{L_\tau^z}{L^z} \frac{\langle n_x^2 \rangle - \langle n_x^2 \rangle_{av}}{\langle n_x^2 \rangle_{av}}$$  

(5)

with $n_x = (1/L_\tau) \sum_{(x,y,\tau)} J^x(x,y,\tau)$, and is assumed to take the scaling form

$$\kappa = L^{z-\delta} \tilde{\kappa}(L^{1/r} D, L_\tau/L_\tau^z)$$  

(6)

with another scaling function $\tilde{\kappa}$.

In order to investigate the scaling behavior of the superfluid stiffness and the compressibility, one must specify the dynamical critical exponent, $z$, in advance so as to fix the aspect ratio $L_\tau/L_\tau^z$ throughout the simulations. We have tried various values of $z$ and have chosen the one which gives the best scaling behavior satisfying Eq. (4). For lattices with noninteger $L_\tau$, the superfluid stiffness is obtained by a simple interpolation of the two values measured in the lattices of nearby integer sizes.

The effect of disorder at various strengths is investigated on the $\mu = 0$ plane. For strong disorder, say for $\Delta > 0.45$, the onset of superfluidity follows the BG-SF transition behavior with $z = 2$ and $\nu = 0.9$, and the compressibility is finite at the transition as suggested by the scaling argument [5] and confirmed by subsequent numerical simulations [16]. For $\Delta < 0.35$, however, the transitions take the signature of the direct MI-SF transition with $z = 1$ and $\nu = 0.67$, as reported in the recent simulations [6], while the compressibility vanishes at the transition. At the intermediate strength of disorder, naive scaling analysis using either $z = 1$ or $z = 2$ fails. In this work, we take a different approach and allow the possibility of intermediate values of $z$ by varying the aspect ratio $L_\tau/L_\tau^z$ until the scaling plots collapse onto a single curve.

A new scaling behavior emerges, at the intermediate strength of disorder, which has $z = 1.35 \pm 0.05$ and $\nu = 0.67 \pm 0.03$. This scaling behavior is shown in Fig. 2. We observe the best scaling when $z = 1.35$. In addition, with this value of $z$, $K_c$’s obtained from the scaling behavior of the superfluid stiffness and the compressibility data curves are consistent with each other. The compressibility vanishes at the transition as expected for $z < d$ [see Fig. 2(b)]. Kisker and Rieger reported [6] $z = 1.4$ for the intermediate strength of disorder, $\Delta = 0.4$. However, they interpreted the result as simply measuring an effective exponent. As we can see in Fig. 2, the robust finite-size scaling behavior for various sizes strongly suggests that the transition truly has new values of the critical exponents. In order to rule out the possibility that we are disguised by finite-size effects, we have performed simulations adopting...
transition by measuring $\nu$ as well as by observing the compressibility. Previously Park et al. have reported [7] that even in the presence of weak disorder, the generic transition survives near the tip. As $\Delta$ increases, the generic MI-SF transition ends at the critical value, $\Delta_c(\mu)$, above which the BG-SF transition occurs.

The critical strength of disorder, $\Delta_c(\mu)$, therefore defines a multicritical line, separating the critical surface $K_c(\mu, \Delta)$ into the two regions: the strong-disorder region and the weak-disorder one. Figure 3 is the measured $\Delta_c(\mu)$. Numerically we find good scaling behaviors using the above exponents for a range of parameters (see Fig. 3 inset). The error bars in Fig. 3 show such ranges. We believe that, in larger systems, the finite range should shrink to a point. Otherwise, the existence of an incompressible localized phase other than the BG phase is required, which is unreasonable.

The existence of the multicriticality strongly suggests that in the presence of disorder two different universality classes exist, in one of which disorder is irrelevant. The irrelevance of weak disorder may be understood from the abundance of the particle-hole pairs excited. At the tips, the proliferation of particle-hole pairs near the transition modifies the particle propagation to yield $z = 1$ (otherwise $z = 2$) in the pure case. These particle-hole pairs may screen the disorder effects. In order to directly check the possibility that the proliferation of particle-hole pairs may screen out the weak-disorder effects, we have repeated Monte Carlo simulations of the classical model

![Diagram](image1)

**FIG. 2.** The finite-size scaling behavior of (a) the superfluid stiffness and (b) the compressibility at the tip of the MI lobe ($\mu = 0$) with $\Delta = 0.40$. We set $z = 1.35$. The curves cross at $K_c = 0.292$. The insets show the scaled data along the $K$ direction with $\nu = 0.67$.

a different aspect ratio and find the same critical exponents. The value of the correlation length critical exponent happens to be the same as the pure transition at the tip.

This behavior persists even off the tip ($\mu \neq 0$). The off-tip transition of the pure case, usually called the generic transition, is of a mean-field type, so that $z = 2$ and $\nu = 1/2$ for the MI-SF transition. If disorder is relevant, we expect in general that the correlation length critical exponent has a different value [18]. Thus, we can numerically identify the BG-SF transition from the MI-SF

![Diagram](image2)

**FIG. 3.** The critical strength of disorder, $\Delta_c(\mu)$, as a function of the chemical potential $\mu$. At the tip ($\mu = 0$), we have tuned the value of $\Delta$ to find the critical one. The error bar indicates the range in which the scaling plots show the behavior expected on the multicritical point. Off the tip $\mu$ is tuned instead, while $\Delta$ is fixed. Inset: the superfluid stiffness scaling curves away from the tip ($\mu = 0.175$) at $\Delta = 0.2$ with $z = 1.35$ and $\nu = 0.67$. 

given by Eq. (2) under the condition that the background particle-hole fluctuations are completely suppressed. For simulations of the system governed by the action of Eq. (2), one should generate all possible current configurations \( \{ \bar{J}(x, y, \tau) \} \) [16]. The suppression of the particle-hole fluctuations is realized by disallowing any Monte Carlo updating which generates local particle-hole current loops. In the spatial planes, however, all possible current configurations are allowed as usual. Thus the allowed current configurations, if they are not restricted in a spatial plane, percolate in the \( \tau \) direction (the imaginary temporal direction).

Figure 4 shows the scaling behavior of the superfluid stiffness when the particle-hole fluctuations are suppressed at the tip (\( \mu = 0 \)) for relatively weak disorder (\( \Delta = 0.3 \)). The scaling shows that the critical exponents are \( z = 2 \) and \( \nu = 0.9 \), indicating the BG-SF transition occurs. Note that, with the particle-hole fluctuations, the transition has the behavior of the MI-SF transition [6,7] at this strength of disorder. This simulation result, therefore, is consistent with the argument that the particle-hole pair fluctuations make the weak disorder irrelevant to the transition.

Now one may question how the particle-hole fluctuations screen the disorder. In the mean-field picture, ignoring the particle-hole fluctuations, particles (holes) are localized by arbitrarily weak disorder due to the coherent impurity backscattering. However, the proliferation of the particle-hole pairs could hinder the coherent backscattering. A particle propagating could be captured and destroyed by holes, and a new particle starts to propagate. A phase shift which accompanies such a particle-exchange process may destroy the coherent backscattering of the particle. As a result, this effect could screen out a weak random potential, making it irrelevant. Slightly off the tip, there are still abundant particle-hole pairs fluctuating. Thus this screening effect will survive even off the tip.

In summary, via quantum Monte Carlo simulations of the phase-only model of a disordered boson Hubbard model, we have found that there is a multicritical line which divides the universality of this model into the weak-disorder regime and the strong-disorder regime. In the weak-disorder regime, disorder is irrelevant for the insulator-to-superfluid transition. On the multicritical line, the dynamical critical exponent \( z = 1.35 \pm 0.05 \) and the correlation exponent \( \nu = 0.67 \pm 0.03 \) are obtained. Our result strongly suggests the direct MI-SF transition survives for weak disorder. We argue that the particle-hole pairs fluctuating near the tip will screen out disorder to make a weak random potential irrelevant.

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