Percolation Transitions in Scale-Free Networks under the Achlioptas Process

Y. S. Cho, J. S. Kim, J. Park, B. Kahng, and D. Kim

1Center for Theoretical Physics and Department of Physics and Astronomy, Seoul National University, Seoul 151-747, Korea
2School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea

(Received 1 July 2009; revised manuscript received 31 August 2009; published 23 September 2009)

It has been recently shown that the percolation transition is discontinuous in Erdős-Rényi networks and square lattices in two dimensions under the Achlioptas process (AP). Here, we show that when the structure is highly heterogeneous as in scale-free networks, a discontinuous transition does not always occur: a continuous transition is also possible depending on the degree distribution of the scale-free network. This originates from the competition between the AP that discourages the formation of a giant component and the existence of hubs that encourages it. We also estimate the value of the characteristic degree exponent that separates the two transition types.

DOI: 10.1103/PhysRevLett.103.135702 PACS numbers: 64.60.ah, 64.60.aq, 68.35.Rh

The Achlioptas process (AP) is a network evolution process in which the number of vertices is fixed as $N$ and edges are added one by one at each time step following a given rule that prevents the formation of a target pattern. Recently, Achlioptas et al. [1] studied the percolation transition (PT) for the Erdős-Rényi (ER) model [2] following an AP rule, called the product rule (PR) in which the number of vertices is fixed as $N$ and the transition is first order. More recently, Ziff [3] found the same first-order transition in the two-dimensional bond percolation clusters under AP. Similar explosive transition pattern has also been observed in a jamming transition model of Internet packets [4].

Here, we study the PT in a model scale-free (SF) network under the AP rule. SF networks contain heterogeneous degrees, and their distribution follows a power law, $P_d(k) \sim k^{-\lambda}$. To construct artificial SF networks, a stochastic model called the Chung and Lu (CL) model [5] is used. Similar to the ER model and the static model [6], the CL model starts with a fixed number of $N$ vertices indexed $i = 1, \ldots, N$. Then, a vertex $i$ is assigned a weight of $w_i = (i + i_0 - 1)^{-\mu}$, where $\mu \in [0, 1)$ is a control parameter, and $i_0 \approx N^{1-1/\mu}$ [7] for $1/2 < \mu < 1$ and $i_0 = 1$ for $\mu < 1/2$. Then, two different vertices $(i, j)$ are selected with their probabilities equal to the normalized weights, $w_i/\sum_k w_k$ and $w_j/\sum_k w_k$, respectively, and an edge is added between them unless one already exists. This process is repeated until $pN$ edges are created in the system. The obtained network is SF in degree distribution with the exponent $\lambda = 1 + 1/\mu$. Henceforth, we will use the CL model to study the percolation transition of scale-free networks in PR (SFPR).

The mechanism by which a giant component in PT forms in conventional SF networks with $2 < \lambda < 3$ is different from that in ER networks. In an ER network, as the number of edges $L = pN$ increases in the system, multiple isolated small components are created and merged together. This process continues up to the finite percolation threshold $p_c$ where a single giant component emerges through an abrupt coalescence of those small components. On the contrary, in SF networks with $2 < \lambda < 3$, the percolation threshold is zero in the thermodynamic limit. Thus, the giant component initially develops with the largest degree vertex as the seed, and grows continuously by aggregating small-size components. The development and growth of the giant component is the result of relatively high probability of a vertex being chosen in the giant component [8]. In the SFPR, on the other hand, two vertex pairs are selected according to the aforementioned weights. During the network growth, if two vertices get selected from the same component, an edge is created between them with no change in component size. Thus, the existence of a giant component implies that even under AP, the probability of growing the giant component is very high. This leads us to ask the following question: what is the impact of introducing the AP rule on the nature of the percolation transition in SF networks?

We obtain the following results by performing extensive numerical simulations for the SFPR model: There exists a tricritical point $\lambda_c$, estimated to be between $2.3 < \lambda_c < 2.4$, such that when $2 < \lambda \leq \lambda_c$, the transition point $p_c$ is zero in the thermodynamic limit, and the PT is second order as in conventional SF networks. When $\lambda > \lambda_c$, however, $p_c$ is finite, and the transition is first order. The jump in the giant component size at the first-order transition point decreases as $p_c$ decreases. The phase diagram is depicted in Fig. 1. In finite-size systems, however, $p_c(N)$ is finite even when $\lambda < \lambda_c$ and the transition is first order.
In addition to this new feature, many other unexpected behaviors emerge.

Specifically, numerical simulations are performed for the CL model with the PR. At each time step, two candidate edges, \( e_1 \) and \( e_2 \), are drawn from the system with respective probabilities as described previously, and added is the edge that minimizes the product of the component sizes on each side of respective edge (PR) is chosen. In (ii), the edge \( e_2 \) is chosen, leading to no change in component size. For (iii), an edge is chosen randomly between the two. A schematic picture of the selection rule in the AP is depicted in Fig. 2. Henceforth, \( \lambda = 1 + 1/\mu \) is a control parameter of simulation: We find that if SFPR, \( \lambda \) is not the resulting degree exponent, unlike in the conventional CL model (see below).

We measure the fraction of vertices in the giant component, denoted as \( G \), averaged over \( 10^2 \sim 10^4 \) different network configurations, as a function of \( p \). We define the PT point, denoted by \( p_c(N) \), in a system of finite size \( N \) as the point at which the local slope of \( G \) is maximal. This position is consistent with the peak position of the susceptibility defined below. We also define the discontinuity of \( G \) [9], denoted as \( \partial G \), as the height of the intersection point of two tangent lines, one from the rapidly increasing transition region and the other from the smoothly increasing curve after the jump. Indeed, \( G \) shows the first-order phase transition at \( p_c(N) \) in finite-size systems as shown in Fig. 3. As the parameter \( \lambda = 2 \) (equivalently \( \mu = 1 \)), the transition point \( p_c(N) \) and the jump \( \partial G \) decrease. To understand the behavior of \( G(p) \) in the \( \to \infty \) limit, numerical simulations are performed for various system sizes in Fig. 4. We find that there exists a critical value \( \lambda_c \), estimated to be between \( 2.3 < \lambda < 2.4 \), such that for \( \lambda < \lambda_c \), \( p_c(N) \) decreases to zero as \( N \) increases [Fig. 4(a)] in a power-law manner \( p_c(N) \sim N^{-1/\xi} \) with \( \xi > 0 \) [inset of Fig. 4(a)], and thus \( p_c(N \to \infty) \to 0 \). The exponent \( \xi \) depends on \( \lambda \). For example, \( 1/\xi = 0.15(1) \) for \( \lambda = 2.2 \). The jump \( \partial G \) at \( p_c(N) \) decreases to zero as \( \partial G \sim N^{-\beta/\xi} \), where the exponent \( \beta \) also depends on \( \lambda \). For example, \( \beta/\xi = 0.23(1) \) for \( \lambda = 2.2 \) [Fig. 4(b)]. Thus, we conclude that the PT is continuous in the thermodynamic limit, and Achlioptas suppression is not effective in this case. When \( \lambda > \lambda_c \), however, \( p_c(N \to \infty) \) converges to a finite value [inset of Fig. 4(d)]. The estimated values of \( p_c(N) \) for different \( N_s \) and \( p_c(\infty) \) are listed in Table I. In finite-size systems, \( p_c(N) - p_c(\infty) \sim N^{-1/\xi} \). For example, estimated value of the exponent \( 1/\xi = 0.29(1) \) for \( \lambda = 2.8 \). \( \xi \neq 1 \) indicates that the first-order transition for \( \lambda = 2.8 \) is not criti-

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**FIG. 1** (color online). Phase diagram of the percolation transition in the SFPR network. Here, \( p = L/N \) is the edge density, and \( \lambda \) is the control parameter corresponding to the degree exponent of non-PR SF networks. A second-order (first-order) PT is represented by a solid line (dashed line). The tricritical exponent of non-PR SF networks. A second-order (first-order) transition region and the other from the smoothly increasing curve after the jump. Indeed, \( G \) shows the first-order phase transition at \( p_c(N) \) in finite-size systems as shown in Fig. 3. As the parameter \( \lambda = 2 \) (equivalently \( \mu = 1 \)), the transition point \( p_c(N) \) and the jump \( \partial G \) decrease. To understand the behavior of \( G(p) \) in the \( \to \infty \) limit, numerical simulations are performed for various system sizes in Fig. 4. We find that there exists a critical value \( \lambda_c \), estimated to be between \( 2.3 < \lambda < 2.4 \), such that for \( \lambda < \lambda_c \), \( p_c(N) \) decreases to zero as \( N \) increases [Fig. 4(a)] in a power-law manner \( p_c(N) \sim N^{-1/\xi} \) with \( \xi > 0 \) [inset of Fig. 4(a)], and thus \( p_c(N \to \infty) \to 0 \). The exponent \( \xi \) depends on \( \lambda \). For example, \( 1/\xi = 0.15(1) \) for \( \lambda = 2.2 \). The jump \( \partial G \) at \( p_c(N) \) decreases to zero as \( \partial G \sim N^{-\beta/\xi} \), where the exponent \( \beta \) also depends on \( \lambda \). For example, \( \beta/\xi = 0.23(1) \) for \( \lambda = 2.2 \) [Fig. 4(b)]. Thus, we conclude that the PT is continuous in the thermodynamic limit, and Achlioptas suppression is not effective in this case. When \( \lambda > \lambda_c \), however, \( p_c(N \to \infty) \) converges to a finite value [inset of Fig. 4(d)]. The estimated values of \( p_c(N) \) for different \( N_s \) and \( p_c(\infty) \) are listed in Table I. In finite-size systems, \( p_c(N) - p_c(\infty) \sim N^{-1/\xi} \). For example, estimated value of the exponent \( 1/\xi = 0.29(1) \) for \( \lambda = 2.8 \). \( \xi \neq 1 \) indicates that the first-order transition for \( \lambda = 2.8 \) is not criti-

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**FIG. 2 (color online).** A schematic diagram of the selection rules in AP for cases (i)–(iii) defined in the text. In case (i), two intercomponent edges are drawn at random, and one of them is chosen to be connected according to the product rule (PR). In case (ii), one edge is intercomponent and the other edge is intracomponent. The latter is chosen. In cases (iii-a) and (iii-b), two intracomponent edges are drawn, and one is randomly chosen to be connected.

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**FIG. 3 (color online).** The fraction \( G \) of the giant component versus the edge density \( p \) for (a) the CL model under AP, and (b) the conventional CL model. Data were obtained for networks with various control parameters \( \lambda = [2.2, 2.4, 2.6, 2.8, 3.0, 4.0] \) from left to right. System size was fixed at \( N = 10^7 \).
We find that there exists a scaled quantity \(\Delta \equiv L_1 - L_0\), which converges to a finite value as \(N \to \infty\) for \(\lambda = 2.8\) [Fig. 4(e)]. The scaling factor \(N^{0.8} < N\) indicates that the transition is of first order \([1]\). It is interesting to note that the susceptibility, defined as \(\chi \equiv \sum s^2 n_s\), with \(n_s\), the number of \(s\)-size components per node, and the sum excluding the largest component, diverges as \(N \to \infty\) even when the transition is first order. We find that \(\chi_{\text{max}} \equiv \chi[p_c(N)] \sim N^{\gamma/\xi}\) with \(\gamma/\xi = 0.4\) and 0.7 for \(\lambda = 2.2\) and 2.8, respectively, shown in Figs. 4(c) and 4(f). Interestingly, \(\gamma/\xi = 0.7\) remains unchanged for \(\lambda = 4.0\) and \(\infty\).

Since the second-order and the first-order transitions meet at \(\lambda_c\), \(\lambda_c\) is a tricritical point. To estimate the position of \(\lambda_c\), we measure successive slopes of the function \(p_c(N)\) with respect to \(N\) for several values of \(\lambda\) and plot them as a function of \(1/N\) in the inset of Fig. 5. We find that the successive slopes decrease to zero for \(\lambda = 2.4\) and 2.5, while they converge to a finite negative value for \(\lambda = 2.3\). Thus, we conclude that the tricritical point is between \(2.3 < \lambda_c < 2.4\), shown in Fig. 5.

The relative frequencies of occurrence of the three cases of (i)–(iii) of Fig. 2 during the evolution is related to the degree effectiveness of AP. We find that the case (i) occurs dominantly with a probability nearly one during the period \(p < p_c(N)\), in which an attached edge connects two isolated components, merging them into a larger component. Above \(p_c(N)\), it decays rapidly since a giant component is already there. The cases (ii) and (iii) begin to occur when \(p\) is close to \(p_c\). Next, we examine the component-size distribution during the evolution. In early time regime \(p \ll p_c(N)\), the component-size distribution exhibits an exponential decaying behavior. As \(p\) is increased, the distribution develops a hump in a large-size region, which is made through the coalescence of small-size components, result-

![FIG. 4 (color online). (a) Same plot as Fig. 3 but for various system sizes \(N = 10^5, 10^6, 10^7\), from right to left. Control parameter \(\lambda = 2.2\). Inset: Plot of \(p_c(N)\) versus \(1/N\). The solid line is a guideline with slope 0.15, indicating that \(p_c(\infty) \to 0\). (b) Plot of the jump \(\Delta G\) around \(p_c(N)\) versus \(1/N\). Solid line is a guideline with slope 0.23. (c) Susceptibility versus \(1/N\). Same as the inset of (a) for \(\lambda = 2.8\). Inset: The peak value versus \(1/N\). (d) Same as (a) for \(\lambda = 2.8\). Inset: Same as the as the inset of (a) for \(\lambda = 2.8\). Solid line is a guideline with slope 0.0, indicating that \(p_c(\infty)\) is finite. (e) Scaling plot of \(\Delta/N^{0.8}\) versus \(N\) for \(\lambda = 2.8\), where \(\Delta/N = p_1 - p_0\) with \(p_1\) and \(p_0\) being the edge densities when the fractions of the giant component reach \(G = 0.3\) and \(G = N^{-1/2}\) for the first time, respectively. (f) Same as (c) for \(\lambda = 2.8\). Error bars in each data point are within symbol sizes.

![FIG. 5 (color online). Plot of \(p_c(N)\) versus \(1/N\) for \(\lambda = 2.3\) (○), 2.4 (●), and 2.5 (□). Error bars in each data point are within symbol sizes. Inset: plot of successive slopes of \(p_c(N)\) versus \(1/N\). For \(\lambda = 2.4\) (●) and 2.5 (□), the successive slopes approach zero, indicating that \(p_c(\infty)\) is finite. For \(\lambda = 2.3\) (○), the successive slopes approach a finite negative value, indicating \(p_c(\infty) = 0\).

### Table I. Estimated percolation threshold \(p_c\) values for finite \((N_1 = 10^6\) and \(N_2 = 10^7\)) and infinite system sizes, and the obtained degree exponents \(\lambda'\) at \(p_c(\infty)\) for various \(\lambda\). Errors in the last decimal points are given in parentheses.

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(p_c(N_1))</th>
<th>(p_c(N_2))</th>
<th>(p_c(\infty))</th>
<th>(\lambda'(p_c))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2</td>
<td>0.33(1)</td>
<td>0.23(1)</td>
<td>0</td>
<td>2.8(1)</td>
</tr>
<tr>
<td>2.3</td>
<td>0.42(1)</td>
<td>0.33(1)</td>
<td>0</td>
<td>3.0(1)</td>
</tr>
<tr>
<td>2.4</td>
<td>0.49(1)</td>
<td>0.42(1)</td>
<td>0.18(1)</td>
<td>3.1(1)</td>
</tr>
<tr>
<td>2.6</td>
<td>0.60(1)</td>
<td>0.57(1)</td>
<td>0.52(1)</td>
<td>3.5(1)</td>
</tr>
<tr>
<td>2.8</td>
<td>0.68(1)</td>
<td>0.66(1)</td>
<td>0.65(1)</td>
<td>3.8(1)</td>
</tr>
<tr>
<td>3.0</td>
<td>0.73(1)</td>
<td>0.73(1)</td>
<td>0.72(1)</td>
<td>4.2(1)</td>
</tr>
<tr>
<td>4.0</td>
<td>0.83(1)</td>
<td>0.83(1)</td>
<td>0.83(7)</td>
<td>6.3(1)</td>
</tr>
</tbody>
</table>
In summary, we have studied the percolation transition in the evolution of SF networks governed by AP. The nature of the phase transition changes from continuous to discontinuous as the degree-exponent parameter $\lambda$ is tuned past a tricritical value $\lambda_c$ (Fig. 1). This phenomenon originates from a competition between AP that discourages the formation of a giant component and the existence of hubs in SF networks that encourages it.

This work is supported by KOSEF grant Acceleration Research (CNRC) (Grant No. R17-2007-073-01001-0) and NAP of KRCF. Thank Dr. Fortunato for sending us their paper and an anonymous referee for introducing Ref. [11].

Note added.—Shortly after the submission of this Letter, we became aware of a similar work [13] under preparation. It uses a different model from ours, the configuration model, exhibiting similar properties with some differences.

Note.—There is an obvious typo: $\deg^G = dG = dp_1 \equiv r(dG/dp_1)$ should read $\deg^G = dG = dp_1 \equiv r(dG/dp_1)_{p_1}$.

In this Letter, we consider a variant of the SF network where the network is grown by preferential attachment (AP) on the network at each step, as follows: for $N > N_c$ and $p < p_c$, there are two possible events: 1) A new node is added to the network with an average degree of $\lambda_0 = \lambda_0(N_c)$; 2) A random node is selected, and a new edge is added plus an edge from the new node to the selected node. This alternative method does not change the $N$ dependence of $\deg^G$, regardless of $r$.

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[7] We use the proportionality constant $10^\sqrt{(1-\mu)/1}$ to eliminate the degree-degree correlation.
[9] $\delta G$ may be measured as $\delta G_2 = G(p_2) - G(p_1)$, where $p_1$ and $p_2$ are defined through $dG/dp_1 = rdG/dp_1$. Here, $r(\leq 1)$ is a tuning parameter. We find that this alternative method does not change the $N$ dependence of $\delta G$, regardless of $r$.
[10] The exponent $\zeta$ corresponds to $d\nu$ in the Euclidean space, where $d$ is spatial dimension and $\nu$ is the correlation length exponent. When the first-order transition is critical, it is known that $\nu$ becomes $1/d$ and thus, $\zeta = 1$ [11,12]; however, we obtain numerically that $1/\zeta \approx 0.29$, 0.69, and 0.72 for $\lambda = 2.8$, 4.0, and $\infty$, respectively. Thus, the feature of the first-order transition occurring in the current disordered system is different from the one in thermal systems.