Large deviation function of the partially asymmetric exclusion process

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The large deviation function obtained recently by Derrida and Lebowitz [Phys. Rev. Lett. **80**, 209 (1998)] for the totally asymmetric exclusion process is generalized to the partially asymmetric case in the scaling limit. The asymmetry parameter rescales the scaling variable in a simple way. The finite-size corrections to the universal scaling function and the universal cumulant ratio are also obtained to the leading order. [S1063-651X(99)08006-X]

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I. INTRODUCTION

The asymmetric simple exclusion process (ASEP) is the simplest driven diffusive system where particles on a one-dimensional lattice hop with asymmetric rates under excluded volume constraints. Due to its simple but nontrivial out-of-equilibrium properties, it has attracted much attention recently. We refer to [1] for a review of recent developments.

For the prototype case of single-species, sequential updating dynamics, the time evolution operator of the probability distribution of particle configurations turns out to be the asymmetric XXZ chain [2,3]. The latter admits the Bethe ansatz solution for its eigenfunctions and eigenvalues when it is on a periodic ring. Due to its integrability, one can obtain many exact results of physical interest. In particular, the large deviation function (LDF) which describes the distribution of the total current has been obtained recently for a ring of N sites with P particles under a periodic boundary condition [4,5]. The LDF also describes the height distribution of the Kardar-Parisi-Zhang (KPZ)-type growth models and is believed to be universal. To confirm the universality of LDF, Derrida and Appert [5] compared a cumulant ratio obtained from the analytic LDF with numerical simulations of several stochastic models believed to belong to the KPZ universality class.

Since the LDF has been obtained in [4,5] for the totally asymmetric exclusion process (TASEP), where particle hopping occurs only to the right, it would be desirable to calculate it for the partially asymmetric exclusion process (PASEP), where the particles can hop both to the right and to the left but with different rates. In this paper, we report on this generalization using the crossover scaling functions of the XXZ chain obtained previously in [3]. Our method assumes from the outset that N is sufficiently large, but allows systematic evaluation of the finite-size corrections. We reproduce the universal scaling function of [4,5] for the PASEP and find that the asymmetry parameter rescales the scaling variable in a simple way. We also evaluate the leading order finite-size corrections to the universal scaling function and the cumulant ratio.

This paper is organized as follows. In Sec. II, we introduce the model and notation. In Sec. III, we make the connection between the present problem and the results of [3] and derive the LDF for the PASEP. The finite-size corrections are evaluated in Sec. IV. Sec. V contains the summary and discussions, while Appendix shows the equivalence of two representations of the crossover scaling functions.

II. MODEL AND THE LARGE DEVIATION FUNCTION

We consider the dynamics of the one-dimensional model in a periodic lattice (ring) of N sites with P particles [2]. Each site j $(1\leqslant j\leqslant N)$ is either occupied by a particle $(\sigma_j=-1)$ or vacant $(\sigma_j=+1)$. The PASEP considered in this work is defined by the following random sequential updating rule: During each time interval dt, each particle can hop to its right or left with probability $\frac{1}{2}(1+\epsilon)dt$ and $\frac{1}{2}(1-\epsilon)dt$, respectively, provided the target site is empty. $\epsilon=1$ corresponds to the TASEP considered in [4,5] and we work in the region $0<\epsilon\leqslant 1$. Interpreting $\sigma_j=\pm 1$ as the local slope of an interface in $(1+1)$ dimensions, one can map the model to the single step model [5,6], an archetype of the KPZ-class models. The quantity of main interest in this work is the total displacement Y_t which is the total number of hops of all particles to the right minus that to the left between time 0 and time t. In the single step model language, Y_t is the total number of particles deposited between time 0 and time t.

Let σ denote a system configuration $\{\sigma_1,\ldots,\sigma_N\}$ and $P_t(\sigma)$ the probability of finding the system in a configuration σ at time t. The master equation for the time evolution of $P_t(\sigma)$ can then be written as

$$\frac{dP_t(\sigma)}{dt}=-\sum_{\sigma'}\langle\sigma|H|\sigma'\rangle P_t(\sigma'), \tag{1}$$

where $\langle\sigma|H|\sigma'\rangle$ is the representation, on the basis where σ_j^z are diagonal, of the time evolution operator H given by

$$H=-\sum_{j=1}^{N}\left\{\left(\frac{1+\epsilon}{2}\right)\sigma_j^+\sigma_{j+1}^- +\left(\frac{1-\epsilon}{2}\right)\sigma_j^-\sigma_{j+1}^+ +\frac{1}{4}(\sigma_j^z\sigma_{j+1}^z-1)\right\}. \tag{2}$$

Here, $\sigma_j^\pm$ and σ_j^z are the Pauli spin operators and $\sigma_j=\pm 1$ are the eigenvalues of σ_j^z.

Next, following [4,5], we introduce $P_t(\sigma,Y)$, the joint probability that the system is in a configuration σ and Y_t, the total displacement, takes the value Y at time t, and let

$$F_t(\sigma;\alpha)=\sum_{Y=-\infty}^{\infty} e^{\alpha Y}P_t(\sigma,Y). \quad (3)$$

Then, $F_t(\sigma;\alpha)$ evolves according to

$$\frac{dF_t(\sigma;\alpha)}{dt}=\sum_{\sigma'} \langle\sigma|M|\sigma'\rangle F_t(\sigma';\alpha), \quad (4)$$

where

$$M=\sum_{j=1}^{N} \left\{ e^{\alpha}\left(\frac{1+\epsilon}{2}\right)\sigma_j^{+}\sigma_{j+1}^{-} + e^{-\alpha}\left(\frac{1-\epsilon}{2}\right)\sigma_j^{-}\sigma_{j+1}^{+}+\frac{1}{4}(\sigma_j^z\sigma_{j+1}^z-1)\right\}. \quad (5)$$

The ''Hamiltonian'' $-M$ is the asymmetric XXZ chain Hamiltonian studied, e.g., in [3]. Let $\lambda(\alpha)$ denote the largest eigenvalue of M, regarded as a function of α. Then, one can show that

$$\langle e^{\alpha Y_t}\rangle=\sum_{\sigma} F_t(\sigma;\alpha)\sim e^{\lambda(\alpha)t} \quad (6)$$

as $t\to\infty$ and the long time behaviors of all cumulants of Y_t are derived from $\lambda(\alpha)$.

The LDF f describes the long time behavior of the distribution of Y_t/t and is defined by

$$f(y)=\lim_{t\to\infty}\frac{1}{t}\ln\left\{\text{Prob}\left[\frac{Y_t}{t}=\overline{v}+y\right]\right\}, \quad (7)$$

where $\overline{v}=\lim_{t\to\infty}\langle Y_t\rangle/t$ is the mean current for a ring of finite size N. Note that $\overline{v}=d\lambda(\alpha)/d\alpha|_{\alpha=0}$. This can be easily obtained from a first order perturbation calculation as

$$\overline{v}=\epsilon\rho(1-\rho)N\frac{N}{N-1}. \quad (8)$$

Our definition of $f(y)$ is slightly different from that of [4,5] in that we use the exact value of $\overline{v}$, Eq. (8), in Eq. (7) while [4,5] use its bulk value $\epsilon\rho(1-\rho)N$. Since $\langle e^{\alpha Y_t}\rangle\sim e^{\lambda(\alpha)t}$ on the one hand, and $\langle e^{\alpha Y_t}\rangle=\Sigma_{Y=-\infty}^{\infty}\text{Prob}[Y_t=Y]e^{\alpha Y}\sim\max_y e^{t(f(y)+\alpha\overline{v}+\alpha y)}$ on the other, the LDF is related to $\lambda(\alpha)-\alpha\overline{v}$ by the Legendre transformation

$$f(y)=[\lambda(\alpha)-\alpha\overline{v}]-\alpha y, \quad (9)$$

$$y=\frac{d}{d\alpha}[\lambda(\alpha)-\alpha\overline{v}]. \quad (10)$$

Therefore, the largest eigenvalue $\lambda(\alpha)$ of the asymmetric XXZ chain M determines the LDF.

III. $\lambda(\alpha)$ IN THE SCALING LIMIT

In [4,5], $\lambda(\alpha)$ for the case of $\epsilon=1$ is obtained for arbitrary N and P. Then, one takes the scaling limit, $N\to\infty$, $\alpha\to 0$, with the scaling variable $\alpha N^{3/2}$ and the density $\rho\equiv P/N$ fixed. In this scaling limit, $\lambda(\alpha)$ takes the parametric form

$$\lambda(\alpha)=\alpha N\rho(1-\rho)+\sqrt{\frac{\rho(1-\rho)}{2\pi N^3}}f_{5/2}(C)(\epsilon=1), \quad (11)$$

$$\alpha\sqrt{2\pi\rho(1-\rho)N^3}=f_{3/2}(C), \quad (12)$$

where $f_k(C)$ are defined as

$$f_{3/2}(C)=-\sum_{n=1}^{\infty}\frac{(-C)^n}{n^{3/2}}, \quad (13)$$

$$f_{5/2}(C)=-\sum_{n=1}^{\infty}\frac{(-C)^n}{n^{5/2}}, \quad (14)$$

for $|C|\leqslant 1$. To probe the region $\alpha\sqrt{2\pi\rho(1-\rho)N^3}<f_{3/2}(-1)$, $f_k(C)$ are analytically continued as

$$f_{3/2}(C)=-4\sqrt{\pi}[-\ln(-C)]^{1/2}-\sum_{n=1}^{\infty}\frac{(-C)^n}{n^{3/2}}, \quad (15)$$

$$f_{5/2}(C)=\frac{8}{3}\sqrt{\pi}[-\ln(-C)]^{3/2}-\sum_{n=1}^{\infty}\frac{(-C)^n}{n^{5/2}}, \quad (16)$$

for $-1\leqslant C<0$, while for $\alpha\sqrt{2\pi\rho(1-\rho)N^3}>f_{3/2}(1)$, one may use the integral forms

$$f_{3/2}(C)=2\pi\int_0^{\infty}ds\frac{\sqrt{s}}{C^{-1}e^{\pi s}+1}, \quad (17)$$

$$f_{5/2}(C)=2\pi\int_0^{\infty}ds\sqrt{s}\ln(1+Ce^{-\pi s}). \quad (18)$$

To generalize Eqs. (11) and (12) to the case of PASEP $(0<\epsilon\leqslant 1)$, we limit our attention only to the scaling limit and use the results of Kim [3]. In [3], the low-lying eigenvalues of the asymmetric XXZ chain near the stochastic line [$\alpha=0$ in Eq. (5)] have been expressed as perturbative expansions in $N^{-1/2}$ with a scaling variable, which is essentially the same as $\alpha N^{-3/2}$, held constant. Therefore the results of [3] applied to the ground state energy (denoted as E_N^0 in [3]) can be used immediately to obtain the LDF.

The notations q, $\tilde{\Delta}$, s, H, and ν used in [3] translate into the present ones as ρ, $(\cosh\alpha+\epsilon\sinh\alpha)^{-1}$, $(\sinh\alpha+\epsilon\cosh\alpha)/(\cosh\alpha+\epsilon\sinh\alpha)$, $(\alpha+\tanh^{-1}\epsilon)/2$, and $\tanh^{-1}\epsilon$, respectively. Using these and taking care of different normalization $[\lambda(\alpha)=-E_N^0/\tilde{\Delta}]$, one can rewrite Eq. (58a) of [3] as

$$\lambda(\alpha)=\epsilon\sum_{m=1}^{\infty}\sum_{k=1}^{m}\frac{b_{m,k}}{(1-x_c)^{k+1}}Y_m^0(z)\left(\frac{\pi}{N}\right)^{m/2}, \quad (19)$$

and Eq. (54a) of [3] as

$$\alpha=\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{m}b_{m,k}\frac{(-1)^k}{kx_c^k}Y_m^0(z)\left(\frac{\pi}{N}\right)^{(m+2)/2}. \quad (20)$$

In the above sums, Y_m^0 for even m vanishes for the ground state and only odd-m terms are needed. For m odd, $Y_m^0(z)$ with real z are defined as

$$Y_m^0(z)=\frac{m+2iz}{2(m+2)}(-i\sqrt{z+i})^m+\frac{m-2iz}{2(m+2)}(i\sqrt{z-i})^m+\frac{1}{2i}\int_0^\infty dt\frac{(-i\sqrt{z+i+t})^m-(-i\sqrt{z+i-t})^m}{e^{\pi t}-1}-\frac{1}{2i}\int_0^\infty dt\frac{(i\sqrt{z-i+t})^m-(i\sqrt{z-i-t})^m}{e^{\pi t}-1}. \quad (21)$$

The coefficients x_c and $b_{m,k}$ are recursively determined order by order in $N^{-1/2}$ from a set of equations, as explained in [3] and $x_c=-\rho/(1-\rho)+O(N^{-5/2})$, $b_{m,k}=b_{m,k}^0+O(N^{-3/2})$. $b_{m,k}$ is the coefficient of x^m in the series expansion of $(\Sigma_{m=1}^\infty a_m x^m)^k$, where $a_m=a_m^0+O(N^{-3/2})$ and the first few values of a_m^0 needed in this work are given by

$$a_1^0=\sqrt{\frac{2\rho}{(1-\rho)^3}},$$

$$a_2^0=-\frac{2}{3}\frac{(1+\rho)}{(1-\rho)^2},$$

$$a_3^0=\sqrt{\frac{2\rho}{(1-\rho)^3}}\frac{1+11\rho+\rho^2}{18\rho(1-\rho)}. \quad (22)$$

The eigenvalue expression, Eq. (19), is a power series expansion in $N^{-1/2}$ with the scaling variable $\alpha N^{3/2}>0$ and $\epsilon>0$ fixed. (If $\alpha>0$ and finite, the asymmetric XXZ chain is in the critical phase and hence the ground state energy and the low lying excitations possess finite-size corrections analytic in N^{-1}.) When $\epsilon\to 0$ with another crossover scaling variable $\epsilon\sqrt{N}$ fixed, the infinite series Eq. (19) reduces to a series in $1/(\epsilon\sqrt{N})$.

Inserting the zeroth order values of x_c and $b_{m,k}$, and keeping only the leading order terms in Eqs. (19) and (20), one then obtains

$$\sqrt{\frac{2\pi N^3}{\rho(1-\rho)}}[\lambda(\alpha)-\alpha\overline{v}]=\epsilon\left\{\left(-\frac{4\pi^2}{3}Y_3^0(z)\right)-[2\pi Y_1^0(z)]\right\}\quad (0<\epsilon\leqslant 1), \quad (23)$$

$$\alpha\sqrt{2\pi\rho(1-\rho)N^3}=[2\pi Y_1^0(z)]. \quad (24)$$

Here $\overline{v}$ is the exact average current, $\epsilon\rho(1-\rho)N^2/(N-1)$. The second term on the right-hand side of Eq. (23) appears due to our choice of the exact $\overline{v}$ on the left-hand side of Eq. (23). Except for that, the similarity of Eq. (23) to Eq. (11) is obvious. One simply needs to relate $Y_m^0(z)$ to $f_k(C)$. In the Appendix, we show, by changing the integration contours of Eq. (21), that $2\pi Y_1^0(z)$ is indeed nothing but a different form of $f_{3/2}(C)$, provided the variables z and C are related by $C=e^{\pi z}$. So is $-4\pi^2 Y_3^0(z)/3$ of $f_{5/2}(C)$. Moreover, we show in the Appendix that the analytic continuation of Eq. (21) to the region Im $z>1$ naturally reproduces the analytically continued forms of Eqs. (15) and (16). Therefore, the generalization of Eq. (11) to $\epsilon\neq 1$ is achieved by a factor ϵ multiplying the right-hand side of Eq. (11). Consequently, by Eqs. (9), (10), (23), and (24), one obtains the LDF in the form

$$f(y)\simeq\epsilon\sqrt{\frac{\rho(1-\rho)}{\pi N^3}}H\left(\frac{y}{\epsilon\rho(1-\rho)}\right), \quad (25)$$

where the universal scaling function $H(x)$ is given in the parametric form satisfying the relation

$$H(x)=\frac{f_{5/2}(C)f_{3/2}{}'(C)-f_{5/2}{}'(C)f_{3/2}(C)}{\sqrt{2}f_{3/2}'(C)}, \quad (26)$$

$$x=\frac{f_{5/2}{}'(C)-f_{3/2}{}'(C)}{f_{3/2}{}'(C)}, \quad (27)$$

with $'$ denoting the derivative with respect to C. $H(x)$, as defined here, is $H(x+1)$ of [4,5], the difference originating from using exact $\overline{v}$ in Eqs. (7) and (23). Thus it has the following asymptotic behaviors:

$$H(x)\simeq\begin{cases}-x^2 & \text{for } |x|\ll 1,\\ -\frac{2}{5}\sqrt{\frac{3}{\pi}}x^{5/2} & \text{for } x\to\infty,\\ -\frac{4}{3}\sqrt{\pi}|x|^{3/2} & \text{for } x\to-\infty.\end{cases} \quad (28)$$

Equation (25) is the generalization of the result of [4,5].

IV. FINITE-SIZE CORRECTIONS IN DISCRETE DYNAMICS

Finite-size correction is useful in comparing theoretical predictions with simulation data. In simulations, particle configurations are updated in discrete time steps, and to describe such situations, Eqs. (1) and (4) should be replaced by their discrete time versions. For example, Eq. (4) is replaced by

$$F_{\tau+1}(\sigma;\alpha)-F_\tau(\sigma;\alpha)=\frac{1}{N}\sum_{\sigma'}\langle\sigma|M|\sigma'\rangle F_\tau(\sigma';\alpha), \quad (29)$$

where one update interval is set as $dt=1/N$ and $t=\tau/N$. These difference equations reduce to the continuous time versions, Eq. (1) and Eq. (4), in the limit $N\to\infty$. Thus the leading terms in N of all quantities are the same in both versions. However, there appear differences in the finite-size corrections and we work in the discrete version. Using Eq. (29), Eq. (6) is then modified to $\langle e^{\alpha Y_t}\rangle=\Sigma_\sigma F_t(\sigma;\alpha)\sim e^{\mu(\alpha)t}$, where

$$\mu(\alpha)=N\ln\left(1+\frac{\lambda(\alpha)}{N}\right). \quad (30)$$

Therefore the LDF is the Legendre transformation of $\mu(\alpha)-\alpha\overline{v}$.

From Eqs. (19), (20), and (30), $\mu(\alpha)-\alpha\overline{v}$ is written as, including its next leading term,

$$\sqrt{\frac{2\pi N^3}{\rho(1-\rho)}}[\mu(\alpha)-\alpha\overline{v}]$$
$$\simeq \epsilon\{f_{5/2}(C)-f_{3/2}(C)\}-\frac{\epsilon^2}{2\sqrt{2}}\sqrt{\frac{\rho(1-\rho)}{\pi N}}f_{3/2}(C)^2. \tag{31}$$

The last term on the right-hand side of Eq. (31) arises from the first nonlinear term in the expansion $\mu(\alpha)=\lambda(\alpha)-\lambda(\alpha)^2/2N+\cdots$. The leading correction term in $\mu(\alpha)-\alpha\overline{v}$ is of order $N^{-1/2}$, while that in $\lambda(\alpha)-\alpha\overline{v}$ is of order N^{-1}. Since the leading correction to $\alpha\sqrt{2\pi\rho(1-\rho)N^3}$ is also of order N^{-1}, using Eq. (24) and Eq. (31), one finds that

$$f(y)=\epsilon\sqrt{\frac{\rho(1-\rho)}{\pi N^3}}\Bigg[H\left(\frac{y}{\epsilon\rho(1-\rho)}\right)$$
$$+\epsilon\sqrt{\frac{\rho(1-\rho)}{\pi N}}H_1\left(\frac{y}{\epsilon\rho(1-\rho)}\right)+O(N^{-1})\Bigg], \tag{32}$$

where $H_1(x)$ is determined from Eq. (27) and

$$H_1(x)=-\frac{f_{3/2}(C)^2}{4}. \tag{33}$$

The correction term shows dependence on the particle density and the asymmetry parameter, and hence is not universal. The asymptotic behaviors of $H_1(x)$ are

$$H_1(x)\simeq\begin{cases}-2x^2 & \text{for } |x|\ll 1\\ -3x^3/(2\pi) & \text{for } x\to\infty\\ -2\pi|x| & \text{for } x\to-\infty.\end{cases} \tag{34}$$

Another quantity of interest concerning the finite-size correction is the cumulant ratio considered in [5]. It is defined as

$$\lim_{t\to\infty}R_t=\lim_{t\to\infty}\frac{\langle Y_t^3\rangle_c^2}{\langle Y_t^2\rangle_c\langle Y_t^4\rangle_c}, \tag{35}$$

where $\langle Y_t^n\rangle_c$ are the cumulants of Y_t and are evaluated from

$$\lim_{t\to\infty}\frac{\langle Y_t^n\rangle_c}{t}=\left.\frac{d^n\mu(\alpha)}{d\alpha^n}\right|_{\alpha=0}. \tag{36}$$

Using Eqs. (24) and (31), we find

$$\lim_{t\to\infty}\frac{\langle Y_t^2\rangle_c}{t}=\epsilon N^{3/2}[\rho(1-\rho)]^{3/2}\frac{\sqrt{\pi}}{2}$$
$$\times\left[1-2\epsilon\sqrt{\frac{\rho(1-\rho)}{\pi N}}+O(N^{-1})\right],$$

$$\lim_{t\to\infty}\frac{\langle Y_t^3\rangle_c}{t}=\epsilon N^3[\rho(1-\rho)]^2\pi\left(\frac{3}{2}-\frac{8\sqrt{3}}{9}\right)[1+O(N^{-1})],$$

$$\lim_{t\to\infty}\frac{\langle Y_t^4\rangle_c}{t}=\epsilon N^{9/2}[\rho(1-\rho)]^{5/2}\pi^{3/2}\left(\frac{15}{2}+\frac{9\sqrt{2}}{2}-8\sqrt{3}\right)$$
$$\times[1+O(N^{-1})]. \tag{37}$$

Therefore the cumulant ratio has an $O(N^{-1/2})$ correction term,

$$\lim_{t\to\infty}R_t=2\frac{\left(\frac{3}{2}-\frac{8\sqrt{3}}{9}\right)^2}{\left(\frac{15}{2}+\frac{9\sqrt{2}}{2}-8\sqrt{3}\right)}$$
$$\times\left(1+2\epsilon\sqrt{\frac{\rho(1-\rho)}{\pi N}}+O(N^{-1})\right). \tag{38}$$

We note in passing that in the continuous time version, our method shows

$$\lim_{t\to\infty}\frac{\langle Y_t^2\rangle_c}{t}=\epsilon N^{3/2}[\rho(1-\rho)]^{3/2}\frac{\sqrt{\pi}}{2}$$
$$\times\left[1+\frac{1+11\rho-11\rho^2}{8\rho(1-\rho)N}+O(N^{-3/2})\right]. \tag{39}$$

This is in exact agreement with the expansion derived, with the help of Stirling's formula, from Eq. (6) of Derrida and Mallick [7].

V. DISCUSSION

The main results of this paper are Eqs. (25), (32), and (38). The universal scaling function of the LDF, $H(x)$, first defined in [4,5] for the TASEP, is reproduced for the PASEP in Eq. (25). The only change in this generalization is the modification of the scaling variable by a simple factor ϵ, the asymmetry parameter. Physically, this is equivalent to a rescaling of time by ϵ. Nontrivial ϵ-dependence of $\lambda(\alpha)$ appears only in higher orders of $N^{-1/2}$ in Eq. (19). To compare analytic results with simulation data, the finite-size correction terms in the discrete time dynamics are important. They are derived for the LDF and the cumulant ratio in Eqs. (32) and (38), respectively. One sees that the finite-size corrections in the discrete time dynamics are of $O(N^{-1/2})$. Also they depend on ρ and ϵ explicitly in both versions implying that they are non-universal.

Instead of the statistics of Y_t, the total displacement, one could have asked for the statistics of the displacement across one bond. In this case, one has to deal with an asymmetric XXZ chain with a twisted boundary condition, $\sigma_{N+1}^{\pm}=e^{\mp\alpha}\sigma_1^{\pm}$ and $\sigma_{N+1}^z=\sigma_1^z$, and analysis similar to that presented here can be carried out [8]. In particular, if J_t is the displacement across the Nth bond, one can show that $\langle e^{\alpha J_t}\rangle\sim\langle e^{\alpha Y_t/N}\rangle\sim e^{\lambda(\alpha/N)t}$ (in the continuous time notation). Therefore, the LDF and the long time behaviors of the cumulants of J_t are the same as those of Y_t/N. This is why Eq.

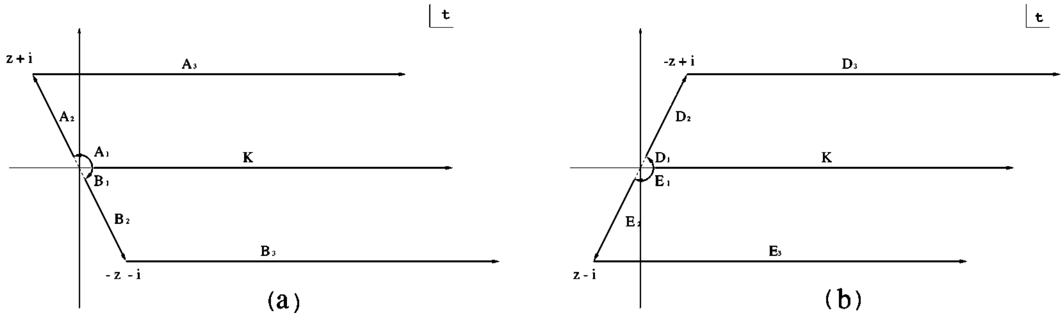

FIG. 1. Contours for I_1 (a) and I_2 (b). Re z is assumed to be negative and $|\mathrm{Im}\, z|<1$. The semicircles about the origin have the small radii δ, which are set to be zero in the last step.

(39) agrees with the result of [7] where $\lim_{t\to\infty}\langle J_t^2\rangle_c/t$ is obtained. However, $\langle J_t^2\rangle_c-\langle (Y_t/N)^2\rangle_c$, the surface width in the growth model language, saturates to a finite value of $O(N)$ as $t\to\infty$.

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APPENDIX: PROPERTIES OF $Y_m^0(z)$

In this Appendix, we show the equivalence of $f_{1+m/2}(C)$ and $Y_m^0(z)$ $(m=1,3,5,\dots)$. The former will be defined later extending the definitions of $f_{3/2}(C)$ and $f_{5/2}(C)$, and the latter is defined in Eq. (21). We take Eq. (21) as defining $Y_m^0(z)$ for any complex z.

1. Simple form of $Y_1^0(z)$

We first pay attention to $Y_1^0(z)$ since $Y_m^0(z)$ $(m=3,5,\dots)$ can be evaluated from $Y_1^0(z)$ through the recursion relation, $dY_{m+2}^0(z)/dz=-(m+2)Y_m^0(z)/2$ [3]. Equation (21) for $m=1$ is written as

$$2Y_1^0(z)=-i(z+i)^{1/2}+\frac{2}{3}(z+i)^{3/2}+\int_0^\infty dt\,\frac{(z+i-t)^{1/2}-(z+i+t)^{1/2}}{e^{\pi t}-1}+i(z-i)^{1/2}+\frac{2}{3}(z-i)^{3/2}+\int_0^\infty dt\,\frac{(z-i-t)^{1/2}-(z-i+t)^{1/2}}{e^{\pi t}-1}. \tag{A1}$$

Suppose $|\mathrm{Im}\, z|<1$ and let I_1 and I_2 be the first and the second integrals in Eq. (A1), respectively. The two integrations are over the positive real axis of the complex-t plane, denoted by K in Fig. 1. Our method is to deform the contours in the complex-t plane such that only simple integrals remain and the additive terms cancel out. Each integral has two terms. For the first term of I_1, K is deformed to $A_1+A_2+A_3$ as shown in Fig. 1(a), while for the second term of I_1, to $B_1+B_2+B_3$ in Fig. 1(a). Similarly, K is deformed to $E_1+E_2+E_3$ and $D_1+D_2+D_3$ for the first and second terms of I_2, respectively, as shown in Fig. 1(b). We then have

$$\begin{aligned}I_1&=\int_{A_1+A_2+A_3}dt\frac{(z+i-t)^{1/2}}{e^{\pi t}-1}-\int_{B_1+B_2+B_3}dt\frac{(z+i+t)^{1/2}}{e^{\pi t}-1}\\&=\int_0^\theta d\theta'\frac{i(z+i)^{1/2}}{\pi}-\int_0^{-(\pi-\theta)}d\theta'\frac{i(z+i)^{1/2}}{\pi}\\&\quad+\int_0^1 d\xi\frac{(z+i)^{3/2}(1-\xi)^{1/2}}{e^{\pi(z+i)\xi}-1}-\int_0^1 d\xi\frac{-(z+i)^{3/2}(1-\xi)^{1/2}}{e^{-\pi(z+i)\xi}-1}\\&\quad+\int_0^\infty ds\frac{(-s)^{1/2}}{e^{\pi(z+i)}e^{\pi s}-1}-\int_0^\infty ds\frac{s^{1/2}}{e^{-\pi(z+i)}e^{\pi s}-1}\\&=i(z+i)^{1/2}-\frac{2}{3}(z+i)^{3/2}+\int_0^\infty ds\frac{(-s)^{1/2}}{e^{\pi(z+i)}e^{\pi s}-1}\\&\quad-\int_0^\infty ds\frac{s^{1/2}}{e^{-\pi(z+i)}e^{\pi s}-1},\end{aligned}\tag{A2}$$

where $\theta=\mathrm{Arg}(z+i)$. Similarly,

$$\begin{aligned}I_2&=\int_{E_1+E_2+E_3}dt\frac{(z-i-t)^{1/2}}{e^{\pi t}-1}-\int_{D_1+D_2+D_3}dt\frac{(z-i+t)^{1/2}}{e^{\pi t}-1}\\&=\int_0^{-(\pi-\theta)}d\theta'\frac{i(z-i)^{1/2}}{\pi}-\int_0^\theta d\theta'\frac{i(z-i)^{1/2}}{\pi}\\&\quad+\int_0^1 d\xi\frac{(z-i)^{3/2}(1-\xi)^{1/2}}{e^{\pi(z-i)\xi}-1}-\int_0^1 d\xi\frac{-(z-i)^{3/2}(1-\xi)^{1/2}}{e^{-\pi(z-i)\xi}-1}\\&\quad+\int_0^\infty ds\frac{(-s)^{1/2}}{e^{\pi(z-i)}e^{\pi s}-1}-\int_0^\infty ds\frac{s^{1/2}}{e^{-\pi(z-i)}e^{\pi s}-1}\\&=-i(z-i)^{1/2}-\frac{2}{3}(z-i)^{3/2}+\int_0^\infty ds\frac{(-s)^{1/2}}{e^{\pi(z-i)}e^{\pi s}-1}\\&\quad-\int_0^\infty ds\frac{s^{1/2}}{e^{-\pi(z-i)}e^{\pi s}-1},\end{aligned}\tag{A3}$$

with $\theta=\mathrm{Arg}(-z+i)$.

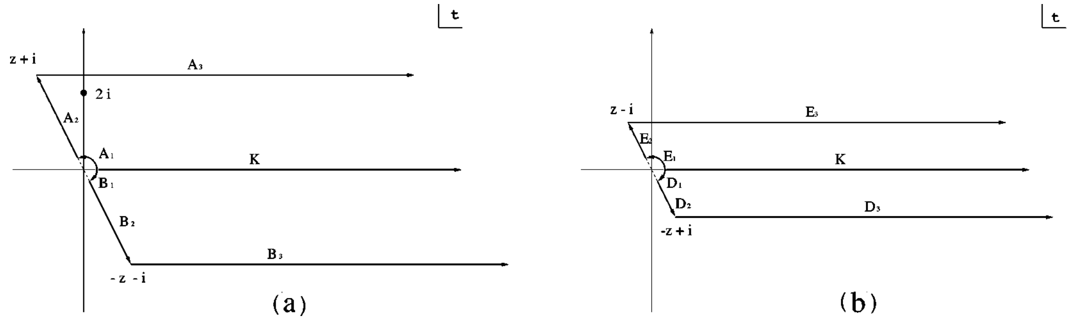

FIG. 2. Contours for I_1 (a) and I_2 (b). $\mathrm{Re}\,z$ is assumed to be negative and $1<\mathrm{Im}\,z<3$. Compared with Fig. 1, a pole is placed inside the contour for I_1 and the integration over the semicircle about the origin makes a different (sign-changed) value in I_2.

We note that the branch cuts for the square-root functions in I_1 and I_2 are in the opposite directions [3], so the two integrals having the factor $(-s)^{1/2}$ in their integrands cancel out when I_1 and I_2 are added. Therefore we arrive at the conclusion that

$$Y_1^0(z)=\int_0^\infty ds\,\frac{s^{1/2}}{e^{-\pi z}e^{\pi s}+1}\;(|\mathrm{Im}\,z|<1). \tag{A4}$$

Next, consider the region $1<|\mathrm{Im}\,z|<3$. If $1<\mathrm{Im}\,z<3$, the contours shown in Fig. 1 change to those shown in Fig. 2. Compared with Fig. 1, a pole at $t=2i$ is placed inside the contour for I_1 and the direction of the integration over the semicircle about the origin is reversed for I_2. These changes produce extra contributions $2i(z-i)^{1/2}$ for both I_1 and I_2. Therefore, we obtain

$$\begin{aligned}Y_1^0(z)&=2i(z-i)^{1/2}+\int_0^\infty ds\,\frac{s^{1/2}}{e^{-\pi z}e^{\pi s}+1}\\&=-2(-z+i)^{1/2}+\int_0^\infty ds\,\frac{s^{1/2}}{e^{-\pi z}e^{\pi s}+1}\;(1<\mathrm{Im}\,z<3).\end{aligned} \tag{A5}$$

Similarly, for $-3<\mathrm{Im}\,z<-1$, we find

$$\begin{aligned}Y_1^0(z)&=-2i(z+i)^{1/2}+\int_0^\infty ds\,\frac{s^{1/2}}{e^{-\pi z}e^{\pi s}+1}\\&=-2(-z-i)^{1/2}+\int_0^\infty ds\,\frac{s^{1/2}}{e^{-\pi z}e^{\pi s}+1}\\&\qquad(-3<\mathrm{Im}\,z<-1).\end{aligned} \tag{A6}$$

When $|\mathrm{Im}\,z|$ increases further, more and more poles are placed inside the contours for I_1 and I_2, and corresponding residues should be added. But recalling that $Y_m^0(z)$ is used to express real α and $\lambda(\alpha)$, Eqs. (A4) and (A5) are sufficient for our purpose.

2. Relation between $f_{1+m/2}(C)$ and $Y_m^0(z)$

We now make the identification $C=e^{\pi z}$ with C real. $C>0$ if z is real, and $-1<C<0$ if $z=-x+i^-$ with $x>0$. In both cases, we have, from Eq. (A4),

$$Y_1^0(z)=\int_0^\infty ds\,\frac{s^{1/2}}{C^{-1}e^{\pi s}+1}. \tag{A7}$$

Equation (A7) admits a series expansion

$$Y_1^0(z)=\frac{1}{2\pi}\sum_{n=1}^\infty\frac{(-1)^{n+1}C^n}{n^{3/2}}, \tag{A8}$$

when $|C|<1$. The second branch in the region $-1<C<0$ is obtained if $z=-x+i^+$ with $x>0$. In this case, using Eq. (A5),

$$\begin{aligned}Y_1^0(z)&=-\frac{2}{\sqrt{\pi}}[-\ln(-C)]^{1/2}+\int_0^\infty ds\,\frac{s^{1/2}}{C^{-1}e^{\pi s}+1}\\&=-\frac{2}{\sqrt{\pi}}[-\ln(-C)]^{1/2}+\frac{1}{2\pi}\sum_{n=1}^\infty\frac{(-1)^{n+1}C^n}{n^{3/2}}.\end{aligned} \tag{A9}$$

Comparing Eqs. (A7) and (A9) with Eqs. (13) and (15), we have

$$f_{3/2}(C)=2\pi Y_1^0(z), \tag{A10}$$

provided $C=e^{\pi z}$, the two branches of $-1<C<0$ corresponding to $|\mathrm{Im}\,z|=1^-$ and $|\mathrm{Im}\,z|=1^+$, respectively.

Next, we define $f_{1+m/2}(C)(m=1,3,5,\ldots)$ by Eq. (A10) and the recursion relation

$$\frac{d}{d(\ln C)}f_{1+(m+2)/2}(C)=f_{1+m/2}(C), \tag{A11}$$

with the initial condition $f_{1+m/2}(0)=0$. Then $f_{1+m/2}(C)$ takes the form

$$f_{1+m/2}(C)=\sum_{n=1}^\infty\frac{(-1)^{n+1}C^n}{n^{1+m/2}}, \tag{A12}$$

in the first branch and

$$f_{1+m/2}(C)=(-1)^{(1+m)/2}\frac{(1/2)!}{(m/2)!}4\sqrt{\pi}[-\ln(-C)]^{m/2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}C^n}{n^{1+m/2}}, \quad (A13)$$

in the second branch. Comparison of the recursion relations of $Y_m^0(z)$ and $f_{1+m/2}(C)$ then leads to the identification

$$f_{1+m/2}(C)=(-\pi)^{(m-1)/2}2\pi\frac{(1/2)!}{(m/2)!}Y_m^0(z). \quad (A14)$$

For example, $f_{3/2}(C)=2\pi Y_1^0(z)$, $f_{5/2}(C)=-4\pi^2 Y_3^0(z)/3$, $f_{7/2}(C)=8\pi^3 Y_5^0(z)/15$, etc.

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