

Effect of anisotropy on the localization in a bifractal system

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A bifractal is a highly anisotropic structure where planar fractals are stacked to form a three-dimensional lattice. The localization lengths along a fractal structure for the Anderson model defined on a bifractal are calculated. The critical disorder and the critical exponent of the localization lengths are obtained from the finite-size scaling behavior. The numerical results are in a good agreement with previous results which have been obtained from the localization lengths along the perpendicular direction. This suggests that the anisotropy of the embedding lattice structure is irrelevant to the critical properties of the localization.

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Recently, there has been much attention focused¹⁻⁶ on the critical properties of the localization in anisotropic system. For a three-dimensional cubic system with anisotropic hopping matrix, it seems now that there is a general agreement⁶ that both the critical disorder (W_c) and the critical exponent (ν) of the localization length are independent of direction of measurement and that such a model belongs to the universality class of the isotropic Anderson model.

On the other hand, bifractals^{7,8} are constructed by stacking planar fractal lattices along the z direction so that they are of Euclidean structure only in the z direction. Anisotropy in these systems arises from the lattice structure itself. Therefore, it is by no means obvious whether the critical properties obtained from the localization lengths along the xy plane (λ_{xy}) are the same as those from the localization lengths along the z axis (λ_z). There is the possibility that even a mobility edge, one of the useful concepts in localization theory, may not exist in this intrinsically anisotropic system.

Therefore, in this paper, we study the critical disorder and the critical exponent along the fractal structure, i.e., along the xy plane, for a bifractal system. Our results are in excellent agreement with those obtained from λ_z 's, which have been reported in previous studies,^{8,9} suggesting that the anisotropy of the embedding lattice structure is irrelevant to the critical properties of the localization. In our study, one of the bifractals introduced in Ref. 8 is used as the model. We consider the Anderson Hamiltonian given as

$$\mathcal{H} = \sum_i \epsilon_i |i\rangle\langle i| + \sum_{\langle i,j \rangle} V(|i\rangle\langle j| + |j\rangle\langle i|), \quad (1)$$

where the random site energies ϵ_i are chosen from a box distribution of width W . The hopping energies V are set to 1 throughout this work and the second sum is over nearest-neighbor pairs of sites on the bifractal lattice. In Fig. 1(a), the lattice is schematically depicted. The cross section perpendicular to the z axis is a variant of the Sierpinski gasket and L is the number of fractal lattices that have been stacked. The number of iteration processes for the fractal lattice is denoted by n , e.g., $n = 2$ for Fig. 1(a). This is exactly the model called bifractal I in Ref. 8. The Green's function coupling two corner sites of the largest triangle, \mathbf{r} and \mathbf{r}' , is

denoted as $G_{\mathbf{r},\mathbf{r}'}(n,L)$. Then the localization length along the xy plane, $\lambda_{xy}(L)$, can be defined as follows:

$$\frac{1}{\lambda_{xy}(L)} = - \lim_{n \rightarrow \infty} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \ln |G_{\mathbf{r},\mathbf{r}'}(n,L)|, \quad (2)$$

where \mathbf{r} and \mathbf{r}' have the same z coordinates.

The main point of the calculation is to find $G_{\mathbf{r},\mathbf{r}'}(n,L)$'s for a sufficiently large value of n , which means that one should calculate elements of the inverse of a very large ran-

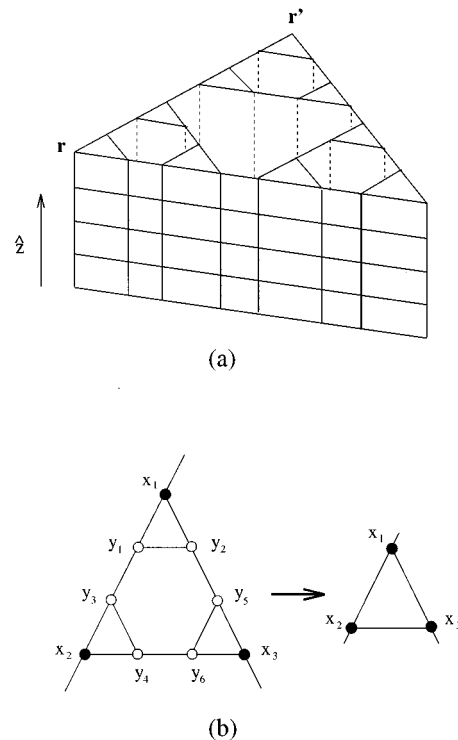


FIG. 1. (a) Schematic diagram of the lattice with the iteration number $n=2$ and $L=5$. Each vertex is a lattice site of the Hamiltonian given by Eq. (1). (b) Schematic diagram of the decimation process. The amplitudes of the sites represented by the empty circles are eliminated to modify the forms of the equations for the sites represented by the solid circles.

dom matrix, $(E - \mathcal{H})$. This is essentially the same problem as encountered in the transfer matrix method for quasi-one-dimensional systems.¹⁰ However a different recursive algorithm should be devised since we are considering a “quasi-two-dimensional system.”

One can handle the problem by decimating recursively the amplitudes of the sites characterized by the largest iteration number, as shown in Fig. 1(b). The following scheme is for the case of $L = 1$ but the extension of the method for $L \geq 2$ is straightforward. Let \mathbf{x} (\mathbf{y}) be a vector, the elements of which are the amplitudes of the sites represented by the solid (empty) circles in Fig. 1(b). Then a matrix equation for \mathbf{x} and \mathbf{y} can be constructed in the form,

$$\begin{pmatrix} H_x(E) & V_{xy}(E) \\ V_{xy}^t(E) & H_y(E) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{z} \\ \mathbf{0} \end{pmatrix}, \quad (3)$$

where $H_x(E)$, $H_y(E)$, and $V_{xy}(E)$ are matrices and contributions from remaining sites other than shown in Fig. 1(b) are contained in a vector \mathbf{z} . The vector $\mathbf{0}$ in the right-hand side of Eq. (3) represents the fact that \mathbf{y} is directly coupled only with \mathbf{x} . Initially, when the sites have been indexed as in Fig. 1(b), the explicit forms of the three matrices are as follows: (i) $H_{x,ij} = \delta_{ij}(E - \epsilon_{x_i})$, (ii) $H_{y,ii} = E - \epsilon_{y_i}$, $H_{y,12} = H_{y,21} = H_{y,13} = H_{y,31} = H_{y,25} = H_{y,52} = H_{y,34} = H_{y,43} = H_{y,46} = H_{y,64} = H_{y,56} = H_{y,65} = -V$, and $H_{y,ij} = 0$, otherwise, (iii) $V_{xy,11} = V_{xy,12} = V_{xy,23} = V_{xy,24} = V_{xy,35} = V_{xy,36} = -V$, and $V_{xy,ij} = 0$, otherwise. Eliminating the amplitudes of the internal sites, i.e., \mathbf{y} , we get

$$(H_x - V_{xy}H_y^{-1}V_{xy}^t)\mathbf{x} = \mathbf{z}. \quad (4)$$

By performing the decimation process of Eq. (4) for every triangle consisting of nine sites, the number of the whole eigenvalue equations reduces by a factor of 3. The 3×3 matrix in the left-hand side of Eq. (4) defines the renormalized hopping energies within the smallest triangles of the new lattice and the renormalized on-site terms. Since the hopping energies are modified only within the smallest triangle one can cast the matrix equation for the remaining sites again in the form of Eq. (3). It should be also noted that at the first step of the iteration, the elements of V_{xy} are independent of E as can be seen from (iii), while after the decimation process, i.e., Eq. (4), they become functions of E , in general. Therefore, the problem has been reduced to another eigenvalue equation problem on the Sierpinski gasket with the iteration number smaller than the original by 1. Now we can iterate the above procedure until three linear equations for the amplitudes of the three outermost sites are left in the case of $L = 1$. For general L , we have $3 \times L$ linear equations instead of 3. Then the inverse of the matrix constructed by the coefficients of the amplitudes is the Green's function for the outermost sites.

In principle, this algorithm can be iterated infinitely. However, one of the technical problems in the real calculation is that some off-diagonal elements of the matrix become very small compared with, say, $\mathcal{O}(1)$ diagonal elements, as the iteration proceeds. In general, these small elements contain the essential information for our problem since they are directly related to the Green's function connecting two different sites far away from each other. For example, let us

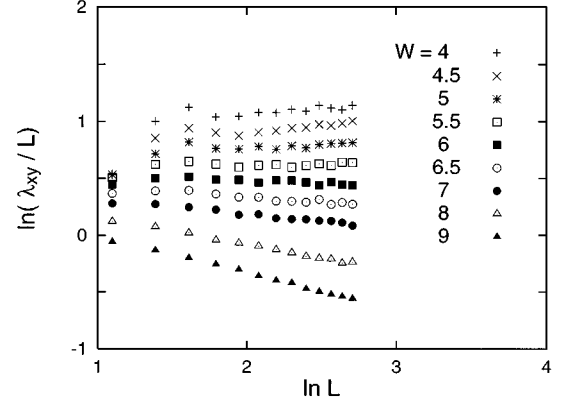


FIG. 2. Log-log plot of the renormalized localization length along the Sierpinski gasket as a function of L for various values of W . The uncertainty of each data point is less than the symbol size.

assume that the localization length is of the order of unity and the distance between two sites is, say, $1000 \times$ (lattice constant). Then the Green's function connecting the two sites is of order $e^{-1000} \sim 10^{-430}$, while the diagonal elements of the Green's function are of the order of unity. Therefore one is investigating an asymptotic behavior of vanishingly small matrix elements, while the order of magnitudes of some other elements of the matrix is by far much larger than them. Direct manipulation of such matrix leads to loss of information on the smaller elements. One of the possible techniques to overcome this difficulty is to decompose the matrix into two parts as

$$A = A_0 + e^{-\alpha}A', \quad (5)$$

where the matrix elements of both A_0 and A' are within the range safely handled by computers. α is a number and should be modified whenever the matrix is manipulated. When a matrix is decomposed as above, the inverse of the matrix can be calculated by the formula

$$(A_0 + e^{-\alpha}A')^{-1} = A_0^{-1} + A_0^{-1} \sum_{n=1}^{\infty} (-e^{-\alpha}A'A_0^{-1})^n. \quad (6)$$

For our calculation, it turns out to be sufficient to retain terms up to second order in $e^{-\alpha}$.

We calculate $\lambda_{xy}(L)$'s for $E = 0$ and several values of W in the range $4.0 \leq W \leq 9.0$. In our calculation, the Sierpinski gasket with $n = 11$ (3^{12} sites/cross section) is used and L is varied within the range $3 \leq L \leq 15$. For a given set of parameters, configurational averages are performed over 4–70 different realizations to control the uncertainty of $\lambda_{xy}(L)$ within 1%. A periodic boundary condition is imposed in the z direction. The results are shown in Fig. 2. As L increases, the renormalized localization length, $\lambda_{xy}(L)/L$, increases for smaller values of W , while for larger values it keeps decreasing. This implies that in the macroscopic limit, i.e., $L \rightarrow \infty$, there exists a transition from an extended state to a localized state as W varies, e.g., from 5.0 to 6.0. Estimates of W_c and ν are obtained by fitting the data to the scaling form

$$\ln(\lambda_{xy}(L)/L) = a + b(W - W_c)L^{1/\nu}, \quad (7)$$

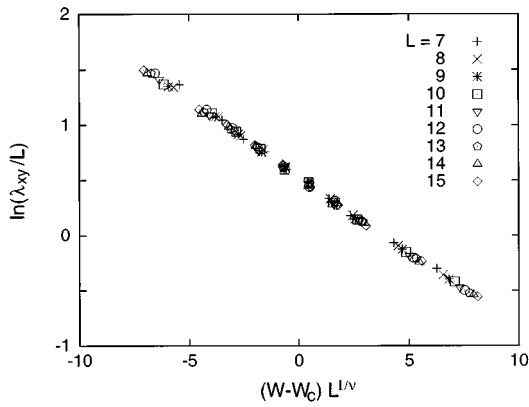


FIG. 3. Scaling behavior of $\ln(\lambda_{xy}/L)$ versus $(W-W_c)L^{1/\nu}$ with $W_c=5.79$ and $\nu=2.92$.

where a and b are constants. Several sets of data for $4.5 \leq W \leq 7$ and $7 \leq L \leq 15$, have been fitted to Eq. (7) and finally we get $W_c=5.79 \pm 0.04$ and $\nu=2.92 \pm 0.14$. The errors are the dispersions between different sets of data. The

scaling plot with these values is shown in Fig. 3. Our numerical results are in good agreement with those obtained from the analysis of λ_z 's, i.e., $W_c=5.8$ and $\nu=3.0 \pm 0.2$ (Ref. 9) for the same model. This supports the idea that W_c and ν are independent of the direction of measurement for this bifractal system.

That W_c 's are the same along the two directions indicates that there exists a well-defined mobility edge in spite of the intrinsic anisotropy in this model. In addition we can expect that the anisotropy is irrelevant to the localization transition in a somewhat strong sense, that is, *the critical properties of the localization are insensitive not only to the anisotropy in the energy parameters but also to that of the embedding lattice structure.*

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