Scale-free random branching trees in supercritical phase

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Abstract

We study the size and the lifetime distributions of scale-free random branching trees in which k branches are generated from a node at each time step with probability $q_k \sim k^{-\gamma}$. In particular, we focus on finite-size trees in a supercritical phase, where the mean branching number $C = \sum_k k q_k$ is larger than 1. The tree-size distribution $p(s)$ exhibits a crossover behaviour when $2 < \gamma < 3$. A characteristic tree size s_c exists such that for $s \ll s_c$, $p(s) \sim s^{-\gamma/(\gamma-1)}$ and for $s \gg s_c$, $p(s) \sim s^{-3/2} \exp(-s/s_c)$, where s_c scales as $\sim (C-1)^{-(\gamma-1)/(\gamma-2)}$. For $\gamma > 3$, it follows the conventional mean-field solution, $p(s) \sim s^{-3/2} \exp(-s/s_c)$ with $s_c \sim (C-1)^{-2}$. The lifetime distribution is also derived. It behaves as $\ell(t) \sim t^{-(\gamma-1)/(\gamma-2)}$ for $2 < \gamma < 3$, and $\sim t^{-2}$ for $\gamma > 3$ when branching step $t \ll t_c \sim (C-1)^{-1}$, and $\ell(t) \sim \exp(-t/t_c)$ for all $\gamma > 2$ when $t \gg t_c$. The analytic solutions are corroborated by numerical results.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

A tree is a graph with no loop within it. Owing to the simplicity of its structure and amenability of analytic studies, tree graphs have drawn considerable attention in many disciplines of scientific research, e.g., the Galton–Watson trees in the probability theory [1]. A scale-free (SF) random branching tree, in which the number of branches k generated from a node is stochastic following a power-law distribution, $q_k \sim k^{-\gamma}$, is particularly interesting here. Such trees can be found in various phenomena such as the trajectories of cascading failure in the sandpile model on SF networks [2], epidemic spreading on SF networks [3, 4], aftershock propagation in earthquakes [5, 6], random spanning tree or skeleton of SF networks [7],

phylogenetic tree [8], etc. Here, the SF network is the network with the degree distribution following a power law $P_d(k) \sim k^{-\lambda}$ [9–11]. So far, several analytic studies have been performed to understand structural properties of SF branching trees [12]. The size distribution of the trees is the key to understanding those branching trees, which corresponds to the first-passage time distribution in a suitably defined random walk [13]. However, most works are focused on the critical case, where the mean branching number $C \equiv \sum_k k q_k$ is equal to 1, motivated by universal feature of scale invariance observed in nature and society.

Recent studies have, however, shown that the structure of real-world networks may have been designed upon supercritical trees [7]. Supercritical trees, where the mean branching number $C > 1$, turn out to act as a skeleton of some fractal networks such as the world-wide web. Here skeleton [14] is defined as a spanning tree formed by edges with highest betweenness centrality or loads [15, 16]. A supercritical branching tree can grow indefinitely with a nonzero probability, which is the most marked difference from critical ($C = 1$) or subcritical ($C < 1$) tree that cannot grow infinitely. Moreover, the total number of offsprings $s(t)$ generated from a single root (ancestor) up to a given generation t can increase exponentially in supercritical trees and this is reminiscent of the small-world behaviour: the mean distance between nodes scales logarithmically as a function of the total number of nodes [12].

Due to the mean branching number being larger than 1, some supercritical trees may be alive in a very long time limit. The tree-size distribution of those surviving trees in the supercritical phase has been derived in the mean-field framework [17], which follows a power law, $p(s) \sim s^{-2}$. Here, we consider finite-size trees in the supercritical phase. In spite of the large mean branching number, some trees do not grow infinitely even in the supercritical phase. For such finite-size trees in the supercritical phase, we derive the tree size and the lifetime distributions using the generating function technique [18]. Distinguished from the critical case, the generating function of the tree-size distribution exhibits two singular behaviours in the supercritical phase and thereby a crossover behaviour of the tree-size distribution can arise when $2 < \gamma < 3$. We present in detail the derivation of all these analytic solutions in the following sections. The tree-size and lifetime distributions predicted by analytic solutions are confirmed by numerical simulations. This is important in itself for understanding the branching trees whose structure changes drastically depending on the phase. Since the branching tree approach can be applied to numerous systems, our results should be useful for future diverse applications as well.

2. Tree-size distribution

Let us consider the branching process that each node generates k offsprings with probability q_k,

$$q_k = \begin{cases} 1 - \dfrac{C\zeta(\gamma)}{\zeta(\gamma - 1)} & \text{for} \quad k = 0, \\ \dfrac{C}{\zeta(\gamma - 1)} k^{-\gamma} & \text{for} \quad k \geqslant 1, \end{cases} \tag{1}$$

where C is constant in the range of $0 < C < \zeta(\gamma - 1)/\zeta(\gamma)$ with the Riemann-zeta function $\zeta(x)$, and γ is larger than 2, ensuring that $\zeta(\gamma - 1)$ is finite. Then, C is automatically identical to the mean branching number, i.e. the average number of offsprings $C = \sum_{k=0}^{\infty} k q_k$ generated from a node. When $C < 1$, the number of offsprings decreases on average as branching proceeds and it vanishes eventually. Thus, the branching tree has a finite lifetime with probability 1. When $C > 1$, as branching proceeds, the number of offsprings can increase exponentially with non-zero probability. The case of $C = 1$ is marginal: Offsprings

persist, neither disappear nor flourish on average. A branching tree generated through the stochastic process (1) is a SF branching tree, because its degree distribution follows a power law, $P_d(k_d) \sim k_d^{-\gamma}$ asymptotically. Degree k_d of each node in the tree is related to the branching number k of that node as $k_d = k + 1$ but for the root, $k_d = k$.

2.1. Generating function method

A tree grows as each of the youngest nodes generates their offsprings following the probability q_k in equation (1). This evolution is regarded as a process in a unit time step. When a node generates no offspring with probability q_0, it remains inactive in further time steps. We define $p_t(s)$ as the fraction of trees with total number of nodes s at time t. By definition, $p_0(s) = \delta_{s,1}$. Then, $p_{t+1}(s)$ can be written in terms of $p_t(s)$ as

$$p_{t+1}(s) = \sum_{k=0}^{\infty} q_k \sum_{s_1,s_2,\ldots,s_k} p_t(s_1) p_t(s_2) \cdots p_t(s_k) \delta_{\sum_{i=1}^{k} s_i, s-1}. \tag{2}$$

Defining the generating functions, $\mathcal{Q}(\omega) = \sum_{k=0}^{\infty} q_k \omega^k$ and $\mathcal{P}_t(y) = \sum_{s=1}^{\infty} p_t(s) y^s$, and applying them to (2), one can obtain that

$$\mathcal{P}_{t+1}(y) = y \mathcal{Q}(\mathcal{P}_t(y)). \tag{3}$$

Let us consider the tree-size distribution in the $t \to \infty$ limit, i.e., $p(s) = \lim_{t \to \infty} p_t(s)$ and its generating function $\mathcal{P}(y) = \lim_{t \to \infty} \mathcal{P}_t(y)$. However, some trees may grow infinitely in the supercritical phase, which makes $\mathcal{P}(y) = \sum_s p(s) y^s$ ill-defined at $y = 1$. So we limit the summation in $\mathcal{P}(y)$ over finite trees only, i.e., $\mathcal{P}(y) = \sum_{\text{finite } s} p(s) y^s$. This is equivalent to defining $\mathcal{P}(1) = \lim_{y \to 1} \mathcal{P}(y)$. Then, equation (3) gives the relation in the $t \to \infty$ limit,

$$\mathcal{P}(y) = y \mathcal{Q}(\mathcal{P}(y)). \tag{4}$$

The next step is to extract a singular part of $\mathcal{P}(y)$ from equation (4), and then to derive the behaviour of $p(s)$ for $s \gg 1$.

The power-law form of q_k in equation (1) results in the expansion of $\mathcal{Q}(\omega)$ around $\omega = 1$. (i) For $\gamma > 3$ in equation (1), $\mathcal{Q}(\omega)$ is expanded as

$$\mathcal{Q}(\omega) = 1 - C(1-\omega) + \frac{B(\gamma)}{2}(1-\omega)^2 + \cdots$$
$$+ \begin{cases} A(\gamma)(1-\omega)^{\gamma-1} & (\gamma \neq \text{integer}) \\ \dfrac{(-1)^{\gamma}}{\Gamma(\gamma)}(1-\omega)^{\gamma-1} \ln(1-\omega) & (\gamma = \text{integer}) \end{cases} + \cdots, \tag{5}$$

where $B(\gamma) = C[\zeta(\gamma-2)/\zeta(\gamma-1) - 1]$, and $A(\gamma) = C\Gamma(1-\gamma)/\zeta(\gamma-1)$ with the Gamma function $\Gamma(x)$. (ii) For $\gamma = 3$,

$$\mathcal{Q}(\omega) = 1 - C(1-\omega) - \tfrac{1}{2}(1-\omega)^2 \ln(1-\omega) + \cdots, \tag{6}$$

(iii) For $2 < \gamma < 3$,

$$\mathcal{Q}(\omega) = 1 - C(1-\omega) + A(\gamma)(1-\omega)^{\gamma-1} + \cdots. \tag{7}$$

The inverse function $y = \mathcal{P}^{-1}(\omega)$ is then expanded (i) for $\gamma > 3$ as

$$y = \mathcal{P}^{-1}(\omega) = \frac{\omega}{\mathcal{Q}(\omega)} = 1 + \Delta(1-\omega) - \frac{B(\gamma)}{2}(1-\omega)^2 + \cdots$$
$$- \begin{cases} A(\gamma)(1-\omega)^{\gamma-1} & (\gamma \neq \text{integer}) \\ \dfrac{(-1)^{\gamma}}{\Gamma(\gamma)}(1-\omega)^{\gamma-1} \ln(1-\omega) & (\gamma = \text{integer}) \end{cases} + \cdots, \tag{8}$$
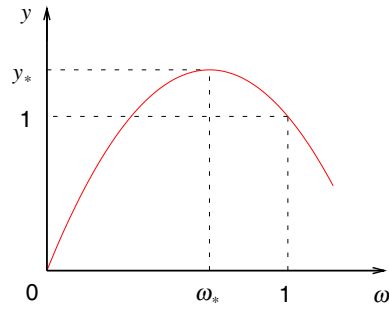
Figure 1. Schematic plot of the function $y = \omega/\mathcal{Q}(\omega)$ in the supercritical phase. The $\mathrm{d}y/\mathrm{d}\omega = 0$ occurs at $\omega = \omega_* < 1$.

where $\Delta \equiv C - 1$. (ii) For $\gamma = 3$,

$$y = 1 + \Delta(1 - \omega) - \tfrac{1}{2}(1 - \omega)^2 \ln(1 - \omega) + \cdots. \tag{9}$$

(iii) For $2 < \gamma < 3$,

$$y = 1 + \Delta(1 - \omega) - A(\gamma)(1 - \omega)^{\gamma-1} + \cdots. \tag{10}$$

We recall that Δ is positive (negative) in the supercritical (subcritical) regime and 0 in the critical case. Here we focus on the supercritical case of $\Delta > 0$ and being very small, but the obtained result can be naturally extended to large-Δ cases.

2.2. The singularity at $y = y_* > 1$

Let us investigate how y behaves as ω decreases from 1 to 0. For $\Delta > 0$, as ω decreases from 1 to ω_*, y increases from 1 to y_* and then decreases to zero as shown in figure 1, where ω_* satisfying $(\mathrm{d}/\mathrm{d}\omega)[\omega/\mathcal{Q}(\omega)]|_{\omega=\omega_*} = 0$ locates less than 1. This feature is distinguished from the solution $\omega_* = 1$ for the critical case. It is obtained that ω_* depends on Δ as

$$1 - \omega_* \equiv \varepsilon_* \sim \begin{cases} \Delta & \text{for} \quad \gamma > 3, \\ \Delta/\ln(1/\Delta) & \text{for} \quad \gamma = 3, \\ \Delta^{1/(\gamma-2)} & \text{for} \quad 2 < \gamma < 3. \end{cases} \tag{11}$$

The value y_*, determined by the relation $y_* = \omega_*/\mathcal{Q}(\omega_*)$, locates at

$$y_* - 1 \equiv \delta_* \sim \begin{cases} \Delta^2 & \text{for} \quad \gamma > 3, \\ \Delta^2/\ln(1/\Delta) & \text{for} \quad \gamma = 3, \\ \Delta^{(\gamma-1)/(\gamma-2)} & \text{for} \quad 2 < \gamma < 3. \end{cases} \tag{12}$$

The curve $y = \omega/\mathcal{Q}(\omega)$ in the region $\omega > \omega_*$ is just the analytic continuation of the inverse function $y = \mathcal{P}^{-1}(\omega)$ that is analytic for $\omega < \omega_*$ [19].

The right-hand sides of equations (8)–(10) for $\omega < \omega_*$ are expanded around ω_* as

$$y \simeq y_* + \sum_{n=2}^{\infty} \frac{D_n(\gamma)}{n!}(\omega_* - \omega)^n, \tag{13}$$

when ω is close to ω_* such that

$$\max_{n \geqslant 2} \frac{D_{n+1}(\gamma)}{D_n(\gamma)(n+1)}(\omega_* - \omega) \ll 1. \tag{14}$$

Here $D_n(\gamma)$ is the nth derivative of $\omega/\mathcal{Q}(\omega)$ at ω_*. For $n = 2$,

$$D_2(\gamma) \sim \begin{cases} -B(\gamma) & \text{for} \quad \gamma > 3, \\ \ln \Delta & \text{for} \quad \gamma = 3, \\ -\Delta^{(\gamma-3)/(\gamma-2)} & \text{for} \quad 2 < \gamma < 3. \end{cases} \tag{15}$$

This result is used for future discussions. Keeping only the quadratic term $(\omega_* - \omega)^2$ in equation (13), one obtains the leading singular behaviour of $\mathcal{P}(y)$ at y_*,

$$\omega = \mathcal{P}(y) \sim \omega_* - \sqrt{\frac{2(y_* - y)}{|D_2(\gamma)|}}. \tag{16}$$

In fact such a square-root singularity at $y = y_*$ is generic regardless of the form of the branching probability when $q_0 + q_1 < 1$ [19], yielding the asymptotic behaviour of $p(s)$ given by

$$p(s) \sim b(\Delta)s^{-3/2}\exp(-s/s_*), \tag{17}$$

where the coefficient $b(\Delta) \sim \Delta^{-(\gamma-3)/[2(\gamma-2)]}$ for $2 < \gamma < 3$, $1/\sqrt{\ln(1/\Delta)}$ for $\gamma = 3$ and constant for $\gamma > 3$, and $s_* = (\ln y_*)^{-1}$.

2.3. The singularity at $y = 1$

When ω is far from ω_* such that the linear term with the coefficient Δ is not comparable to the next-order term, another singularity becomes dominant. The next-order term is the quadratic term for $\gamma > 3$ and the non-analytic term for $2 < \gamma \leqslant 3$. To be precise, if the condition, $1 - \omega \gg \Delta$ for $\gamma > 3$, $-(1 - \omega)\ln(1 - \omega) \gg \Delta$ for $\gamma = 3$, and $1 - \omega \gg \Delta^{1/(\gamma-2)}$ for $2 < \gamma < 3$ holds, then the linear term is negligible compared with the next order terms, and then equations (8)–(10) are reduced to

$$y \sim 1 - \begin{cases} \dfrac{B(\gamma)}{2}(1 - \omega)^2 & \text{for} \quad \gamma > 3, \\[2mm] -\dfrac{1}{2}(1 - \omega)^2 \ln(1 - \omega) & \text{for} \quad \gamma = 3, \\[2mm] A(\gamma)(1 - \omega)^{\gamma-1} & \text{for} \quad 2 < \gamma < 3. \end{cases} \tag{18}$$

The generating function $\mathcal{P}(y)$ then behaves as

$$\omega = \mathcal{P}(y) \sim 1 - \begin{cases} \sqrt{\dfrac{2(1 - y)}{B(\gamma)}} & \text{for} \quad \gamma > 3, \\[3mm] \sqrt{\dfrac{4(1 - y)}{|\ln(1 - y)|}} & \text{for} \quad \gamma = 3, \\[3mm] \left(\dfrac{1 - y}{A(\gamma)}\right)^{1/(\gamma-1)} & \text{for} \quad 2 < \gamma < 3. \end{cases} \tag{19}$$

From this result, one can obtain the tree-size distribution as

$$p(s) \sim \begin{cases} s^{-3/2} & \text{for} \quad \gamma > 3, \\ s^{-3/2}(\ln s)^{-1/2} & \text{for} \quad \gamma = 3, \\ s^{-\gamma/(\gamma-1)} & \text{for} \quad 2 < \gamma < 3. \end{cases} \tag{20}$$

2.4. Crossover behaviour between the two singularities

The two singular behaviours of $\mathcal{P}(y)$ in the forms of equations (16) and (19) occurring at $y = y_*$ and $y = 1$, respectively, enable us to determine the ranges of size s where the formulae of equations (17) and (20) are valid. In particular, when $2 < \gamma \leqslant 3$, the asymptotic behaviours in equations (17) and (20) differ from each other and thus there should be a crossover behaviour in the tree-size distribution.

The ranges of ω in which equations (13) and (18) are valid are closely related to those of y for equations (16) and (19) and that of s for equations (17) and (20), respectively. Here we find those ranges of ω, y and s, and then determine the crossover in the tree-size distribution $p(s)$.

First, we study valid ranges of equations (13), (16), and (17). The coefficient $D_n(\gamma)$ in equation (13) behaves as $(1 - \omega_*)^{\gamma-1-n}$ for $n > \gamma - 1$ due to the non-analytic term $(1 - \omega)^{\gamma-1}$ in equations (8) and (10) when γ is not integer. Then, it follows that $[D_n(\gamma)(n + 1)/D_{n+1}(\gamma)] \sim 1/(1 - \omega_*) \equiv 1/\varepsilon_*$. Thus, condition (14) can be rewritten as $\omega_* - \varepsilon_c^> \ll \omega < \omega_*$, where $\varepsilon_c^> \sim \Delta$ for $\gamma > 3$, $\varepsilon_c^> \sim \Delta/\ln(1/\Delta)$ for $\gamma = 3$ and $\varepsilon_c^> \sim \Delta^{1/(\gamma-2)}$ for $2 < \gamma < 3$ from equation (11). The corresponding range of y is $y_* - \delta_c^> \ll y < y_*$, where $\delta_c^>$ is given by $\sim \Delta^2$ for $\gamma > 3$, $\sim \Delta^2/\ln(1/\Delta)$ for $\gamma = 3$ and $\sim \Delta^{(\gamma-1)/(\gamma-2)}$ for $2 < \gamma < 3$ by using equations (12) and (16).

To find valid range of s for $p(s)$ in equation (17), we use the fact that the singular functional behaviour of $\mathcal{P}(y)$ around $y = \tilde{y}$ is determined by that of $p(s)$ around $s = \tilde{s}$, where $\tilde{y}$ and $\tilde{s}$ are related as $\tilde{y}^{\tilde{s}} \sim 1$. Then, one can find that $s_c^> = \left|\ln\left(y_* - \delta_c^>\right)\right|^{-1} \sim (\delta_* - \delta_c^>)^{-1}$, so that $s_c^> \sim \Delta^{-2}$ for $\gamma > 3$, $\Delta^{-2}\ln(1/\Delta)$ for $\gamma = 3$ and $\Delta^{-(\gamma-1)/(\gamma-2)}$ for $2 < \gamma < 3$. For the range $s \gg s_c^>$, formula (17) is valid.

Second, we check the validities of equations (18), (19) and (20). Comparing the magnitude of the linear term and the next-order term in equations (8)–(10), we find that equation (18) is valid for $\omega \ll 1 - \varepsilon_c^<$, where ε_c behaves as Δ for $\gamma > 3$, $\Delta/\ln(1/\Delta)$ for $\gamma = 3$ and $\Delta^{1/(\gamma-2)}$ for $2 < \gamma < 3$. The corresponding range of y for equation (19) is given as $y \ll 1 - \delta_c^<$, where $\delta_c^< \sim \Delta^2$ for $\gamma > 3$, $\delta_c^< \sim \Delta^2/\ln(1/\Delta)$ for $\gamma = 3$ and $\delta_c^< \sim \Delta^{(\gamma-1)/(\gamma-2)}$ for $2 < \gamma < 3$. The corresponding range of s for equation (20) is $s \ll s_c^<$ with $s_c^< = \left|\ln\left(1 - \delta_c^<\right)\right|^{-1} \sim \left(\delta_c^<\right)^{-1}$ given by $s_c^< \sim \Delta^{-2}$ for $\gamma < 3$, $s_c^< \sim \Delta^{-2}\ln(1/\Delta)$ for $\gamma = 3$ and $s_c^< \sim \Delta^{-(\gamma-1)/(\gamma-2)}$ for $2 < \gamma < 3$.

As already noted, the crossover sizes $s_c^>$, $s_c^<$ and s_* are consistent for all values of γ within Δ-dependence, and thereby, we use the notation s_c for all of them. The overall behaviour of the tree-size distribution is obtained by combining equations (17) and (20). For $\gamma > 3$, there is no need to introduce a crossover. Thus, it leads to

$$p(s) \sim s^{-3/2} \exp(-s/s_c) \qquad (\gamma > 3), \tag{21}$$

for s large. Here s_c scales as Δ^{-2} for Δ close to 0 and thus the exponential-decaying pattern prevails as Δ increases.

When $\gamma = 3$, $p(s)$ is given in the scaling regime $s \to \infty$, $\Delta \to 0$, and s/s_c finite with $s_c \sim \Delta^{-2}\ln(1/\Delta)$ by

$$p(s) \sim \begin{cases} s^{-3/2}(\ln s)^{-1/2} & \text{for} \quad s/s_c \to 0, \\ s^{-3/2}\exp(-s/s_c) & \text{for} \quad s/s_c \to \infty. \end{cases} \qquad (\gamma = 3) \tag{22}$$

When $2 < \gamma < 3$, it is given by

$$p(s) \sim \begin{cases} s^{-\gamma/(\gamma-1)} & \text{for} \quad s/s_c \to 0, \\ s^{-3/2}\exp(-s/s_c) & \text{for} \quad s/s_c \to \infty, \end{cases} \qquad (2 < \gamma < 3) \tag{23}$$
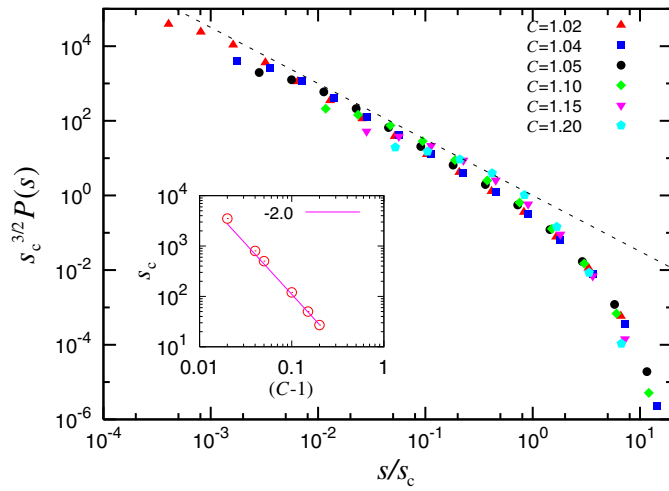
Figure 2. The tree-size distribution $p(s)$ for $\gamma = 3.3$ for various values of C in the scaling form equation (21). The dashed line is the guideline with a slope of $-3/2$. Inset: dependence of the characteristic size s_c on the mean branching number C.
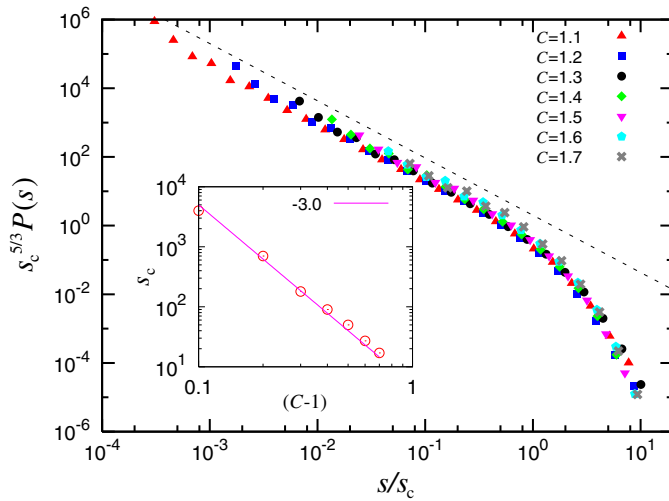

Figure 3. The tree-size distribution $p(s)$ for $\gamma = 2.5$ for various values of C in the scaling form equation (23). Dashed line is guideline with a slope of $-5/3$. Inset: dependence of the characteristic size s_c on the mean branching number C.

in the scaling regime $s \to \infty$, $\Delta \to 0$, and s/s_c finite with $s_c \sim \Delta^{-(\gamma-1)/(\gamma-2)}$. The crossover scale s_c decreases with increasing Δ and the exponential decay dominates for $s \to \infty$ and $\Delta > 0$ ($C > 1$).

We invoke numerical simulations to confirm our analytic solutions. Figures 2 and 3 show the tree-size distributions for $\gamma = 3.3$ and $\gamma = 2.5$ in the scaling forms, equations (21) and (23), respectively. The data are well collapsed into the predicted formulae for different C values for both cases.
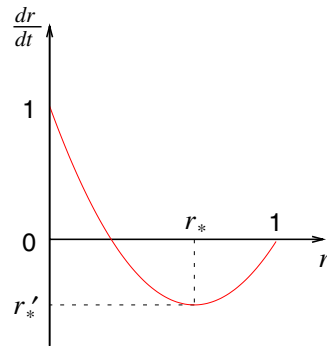
Figure 4. Schematic plot of the function $dr/dt = \mathcal{Q}(r) - r$ in the supercritical phase. $(\partial/\partial r)(dr/dt) = 0$ occurs at r_*, at which dr/dt is denoted as r_*'.

3. Lifetime distribution

Next we solve the lifetime distribution $\ell(t)$. This is defined as the probability that the branching process stops at t. To derive $\ell(t)$, we first introduce the probability that the branching process stops at or prior to time t, denoted by $r(t)$. Then $\ell(t)$ is given as $\ell(t) = r(t+1) - r(t)$. The probability distribution $r(t)$ is related to $r(t-1)$ as

$$r(t) = \sum_{k=0}^{\infty} q_k [r(t-1)]^k = \mathcal{Q}(r(t-1)). \tag{24}$$

Thus, we are given approximately a differential equation for $r(t)$,

$$\frac{dr(t)}{dt} \approx \ell(t) = \mathcal{Q}(r(t)) - r(t). \tag{25}$$

Expanding the right-hand side of equation (25) around $r = 1$, one can see its asymptotic behaviour. Using equations (5)–(7) again, we find dr/dt in the long time limit as follows:

$$\begin{aligned}
\frac{dr}{dt} = \mathcal{Q}(r) - r &= -\Delta(1-r) + \frac{B(\gamma)}{2}(1-r)^2 + \cdots \\
&+ \begin{cases} A(\gamma)(1-r)^{\gamma-1} & (\gamma \neq \text{integer}) \\ \dfrac{(-1)^\gamma}{\Gamma(\gamma)}(1-r)^{\gamma-1}\ln(1-r) & (\gamma = \text{integer}) \end{cases} + \cdots.
\end{aligned} \tag{26}$$

What we can see in this relation is that the value of r' is zero at $r = 1$. It decreases as r decreases until it reaches r_* where $(d/dr)[\mathcal{Q}(r) - r]|_{r=r_*} = 0$ holds. Passing r_*, r' increases as r decreases further, crossing the $r' = 0$ as shown in figure 4.

First, as in the case of $\omega/\mathcal{Q}(\omega)$, two singularities exist in $\mathcal{Q}(r) - r$. For r close to r_*, equation (26) is expanded as

$$r' \simeq r_*' + \sum_{n=2}^{\infty} \frac{G_n(\gamma)}{n!}(r_* - r)^n, \tag{27}$$

where $r_*' = \mathcal{Q}(r_*) - r_* < 0$ and $G_n(\gamma)$ is the nth derivative of $\mathcal{Q}(r) - r$ at r^*. When r is close to r_* such that

$$\max_{n \geqslant 2} \frac{G_{n+1}(\gamma)}{G_n(\gamma)(n+1)}(r_* - r) \ll 1, \tag{28}$$

one may neglect higher order terms, keeping only the quadratic term in $r_* - r$ as

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx r'_* + \frac{G_2(\gamma)}{2}(r_* - r)^2. \tag{29}$$

The solution to the above differential equation is

$$r(t) \simeq r(\infty) - \frac{2a}{\mathrm{e}^{t/t_*} - 1}, \tag{30}$$

where $r(\infty) = r_* - a$ and $a = \sqrt{2|r'_*|/G_2(\gamma)}$, and $t_* = 1/\sqrt{2|r'_*|G_2(\gamma)}$. The lifetime distribution $\ell(t) = r'(t)$ is then given by

$$\ell(t) \simeq \frac{2a\,\mathrm{e}^{t/t_*}}{t_*(\mathrm{e}^{t/t_*} - 1)^2} \sim \frac{2a}{t_*}\,\mathrm{e}^{-t/t_*}. \tag{31}$$

Second, following the same steps taken for the singularities of $\mathcal{P}(y)$, we find another approximate relation between r' and r in the region of $r(t)$ where the next order term in equation (26) is much larger than its linear term as follows:

$$\frac{\mathrm{d}r}{\mathrm{d}t} \sim \begin{cases} \dfrac{B(\gamma)}{2}(1-r)^2 & \text{for} \quad \gamma > 3, \\[2mm] -\dfrac{1}{2}(1-r)^2 \ln(1-r) & \text{for} \quad \gamma = 3, \\[2mm] A(\gamma)(1-r)^{\gamma-1} & \text{for} \quad 2 < \gamma < 3. \end{cases} \tag{32}$$

Their solutions are, in long time limit, given by

$$1 - r(t) \sim \begin{cases} t^{-1} & \text{for} \quad \gamma > 3, \\[1mm] t^{-1}(\ln t)^{-1} & \text{for} \quad \gamma = 3, \\[1mm] t^{-1/(\gamma-2)} & \text{for} \quad 2 < \gamma < 3. \end{cases} \tag{33}$$

From these results, the lifetime distributions are obtained as

$$\ell(t) \sim \begin{cases} t^{-2} & \text{for} \quad \gamma > 3, \\[1mm] t^{-2}(\ln t)^{-1} & \text{for} \quad \gamma = 3, \\[1mm] t^{-(\gamma-1)/(\gamma-2)} & \text{for} \quad 2 < \gamma < 3. \end{cases} \tag{34}$$

Different behaviours of the lifetime distribution shown in equations (31) and (34) suggest the presence of a crossover behaviour. The characteristic time that distinguishes the two behaviours for given γ can be found by considering the valid ranges of t for equations (31) and (34), respectively. When the condition of equation (28) is fulfilled, equations (30) and (31) are valid. The condition is approximately represented in different forms of $r_* - r \ll 1 - r_*$ since $G_n(\gamma) \sim (1 - r_*)^{\gamma-1-n}$. From equation (26), one can find the value of $1 - r_*$ for different γ's: $1 - r_* \sim \Delta$ for $\gamma > 3$, $\Delta/\ln(1/\Delta)$ for $\gamma = 3$ and $\Delta^{1/(\gamma-2)}$ for $2 < \gamma < 3$, respectively. Applying these conditions to equation (30), it is found that equations (30) and (31) are valid if $t \gg t_{*1}$ with $t_{*1} \sim \Delta^{-1}$ irrespective of γ as long as $\gamma > 2$.

Equations (33) and (34) are valid when the linear term is much smaller than the next order term, which is satisfied when $1 - r \gg \Delta$ for $\gamma > 3$, $1 - r \gg \Delta/\ln(1/\Delta)$ for $\gamma = 3$ and $1 - r \gg \Delta^{1/(\gamma-2)}$ for $2 < \gamma < 3$, respectively. Applying these conditions to equation (33) leads commonly to $t \ll t_{*2} \sim \Delta^{-1}$. One can find that the two characteristic times t_{*1} and t_{*2}, and t_* scale in the same manner, so that they are denoted as t_c commonly. Therefore, in the scaling regime $t \to \infty$, $\Delta \to \infty$, and t/t_c finite with $t_c \sim \Delta^{-1}$, the lifetime distribution behaves as

$$\ell(t) \sim \begin{cases} t^{-2} & \text{for} \quad \gamma > 3, \\[1mm] t^{-2}(\ln t)^{-1} & \text{for} \quad \gamma = 3, \\[1mm] t^{-(\gamma-1)/(\gamma-2)} & \text{for} \quad 2 < \gamma < 3, \end{cases} \tag{35}$$
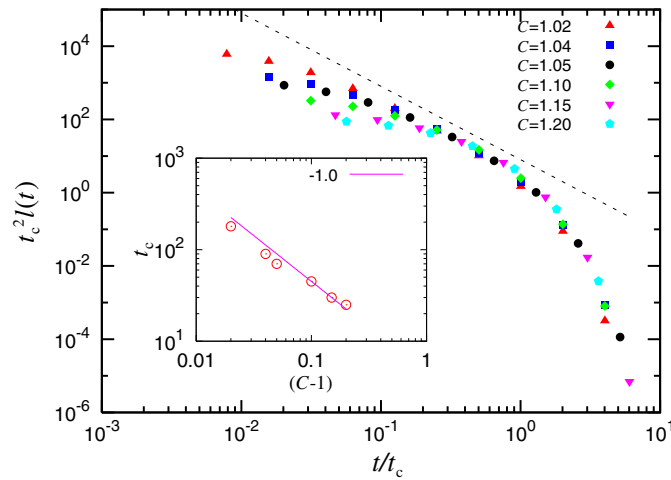
Figure 5. The lifetime distribution $\ell(t)$ for $\gamma = 3.3$ in the scaling form equations (35) and (36). Dashed line is guideline with slope -2. Data for small t are deviated from the data collapse, indicating that our solution is valid for large t only. Inset: dependence of the characteristic time t_c on the mean branching number C.
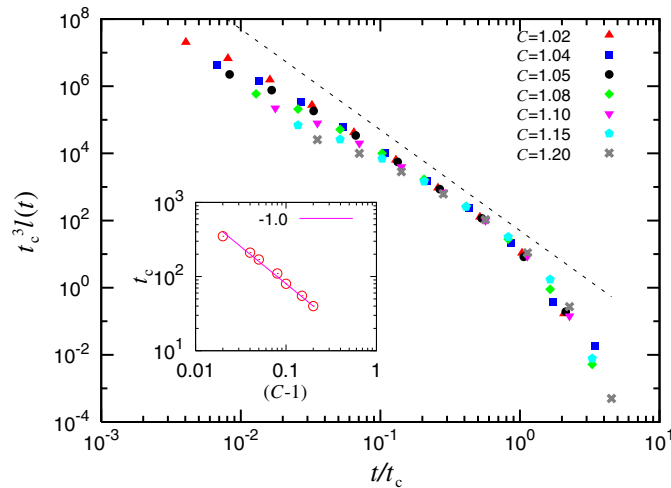

Figure 6. The lifetime distribution $\ell(t)$ for $\gamma = 2.5$ in the scaling form equations (35) and (36). Dashed line is guideline with slope -3. Data for small t are deviated from the data collapse, indicating that our solution is valid for large t only. Inset: dependence of the characteristic time t_c on the mean branching number C.

when $t/t_c \rightarrow 0$, and

$$\ell(t) \sim \mathrm{e}^{-t/t_c} \qquad \text{for} \quad \gamma > 2 \tag{36}$$

when $t/t_c \rightarrow \infty$. This exponential-decaying pattern dominates for $t \rightarrow \infty$ and $\Delta > 0$. The analytic solutions for the lifetime distribution are checked by numerical simulations in figures 5 and 6. Data in small t regime are somewhat deviated from the data-collapsed formula, indicating that our solution is valid in the large-t regime.

4. Conclusions and discussion

Our main results are equations (21), (22) and (23) for the tree-size distribution when trees are finite: contrary to the case of $\gamma > 3$ for which the tree-size distribution $p(s)$ behaves as $\sim s^{-3/2} \exp(-s/s_c)$ for all s with $s_c \sim (C-1)^{-2}$, a crossover behaviour occurs at $s_c \sim (C-1)^{-(\gamma-1)/(\gamma-2)}$ for $2 < \gamma < 3$. For $s \ll s_c$, $p(s) \sim s^{-\gamma/(\gamma-1)}$ and for $s \gg s_c$, $p(s) \sim s^{-3/2} \exp(-s/s_c)$. This result is complementary to the previous mean-field solution $p_{\mathrm{inf}}(s) \sim s^{-2}$ for infinite-size trees. From our solutions, it is noteworthy that the characteristic size s_c increases as the exponent γ approaches 2. This leads to an interesting result: a larger-size tree can be generated for a smaller value of the exponent γ. However, the probability to have such a large-size tree becomes smaller as the exponent γ approaches 2, because the exponent $\gamma/(\gamma-1)$ for the tree-size distribution $p(s)$ becomes larger.

The lifetime distribution also exhibits a crossover behaviour at $t_c \sim (C-1)^{-1}$. It follows equation (35) for $t \ll t_c$ and (36) for $t \gg t_c$.

Acknowledgments

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