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1 (a) $\nabla\times(\nabla\times\vec{V}) = \nabla(\nabla\cdot\vec{V}) - \nabla^2\vec{V}$

Componentwise

$$[\nabla\times(\nabla\times\vec{V})]_i = \varepsilon_{ijk}\partial_j \varepsilon_{klm}\partial_l V_m$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l V_m$$
$$= \partial_m\partial_i V_m - \partial_j\partial_j V_i$$
$$= \partial_i\partial_m V_m - \nabla^2 V_i$$
$$= \partial_i(\nabla\cdot\vec{V}) - \nabla^2 V_i$$
$$= [\nabla(\nabla\cdot\vec{V}) - \nabla^2\vec{V}]_i$$

(b) $\mathrm{Tr}\, T' = \sum_k T'_{kk} = \sum_k a_{ki}a_{kj}T_{ij} = a^{-1}_{ik}a_{kj}T_{ij} = \delta_{ij}T_{ij} = T_{ii} = \mathrm{Tr}\, T$

(c) $$A'_{ij} = a_{ik}a_{jl}A_{kl} = -\cancel{a_{jl}a\,a}$$
$$= -a_{jl}a_{ik}A_{lk}$$
$$= -A'_{ji}$$

$$S'_{ij} = a_{ik}a_{jl}S_{kl} = a_{jl}a_{ik}S_{lk} = S'_{ji}$$

(d) $$T_{ij} = \frac{\delta_{ij}}{3}\mathrm{Tr}\,T + A_{ij} + \left(S_{ij} - \frac{\delta_{ij}}{3}\mathrm{Tr}\,T\right)$$

~~It is obvious~~

$T_{ij} = A_{ij} + S_{ij} = \frac{1}{2}(T_{ij} - T_{ji}) + \frac{1}{2}(T_{ij} + T_{ji}) = T_{ij}$ identity holds.

First term $\frac{\delta_{ij}}{3}\mathrm{Tr}\,T$ is invariant under rotation

~~A_{ij}~~ 2nd term A_{ij} will remain anti-symmetric under rotation

3rd term $S_{ij} - \frac{\delta_{ij}}{3}\mathrm{Tr}\,T$ will remain symmetric under rotation.

Under rotation, all three parts behave independently, that is

Irreducible further.

1st term : 1 component 2nd term: $\frac{1}{2}(T_{ij} - T_{ji})$ 3 indep terms.

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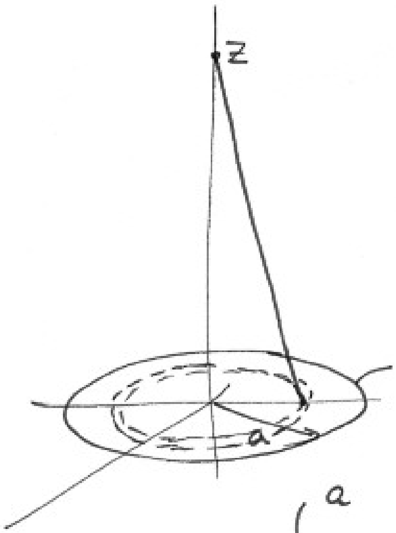
3rd term is symmetric and has 6 ~~te~~ components but there is traceless condition → 5 independent components.

2



~~at~~ potential by a ring at radius ρ and $\rho+d\rho$

4

distance to z

$\sigma = \frac{Q}{\pi a^2}$ $\qquad = \sqrt{z^2+\rho^2}$

total charge $2\pi\rho\, d\rho\, \sigma$

$$\Phi = \int_0^a \frac{1}{4\pi\varepsilon_0}\frac{2\pi\rho\, d\rho\, \sigma}{\sqrt{z^2+\rho^2}} = \frac{\cancel{2\pi}\sigma}{2\cancel{4\pi}\varepsilon_0}\int_0^a \frac{\rho\, d\rho}{\sqrt{z^2+\rho^2}}$$

$$= \frac{1}{2\varepsilon_0}\frac{Q}{\pi a^2}\int_0^a \frac{\rho\, d\rho}{\sqrt{z^2+\rho^2}}$$ let $u = z^2+\rho^2$ $\quad du = 2\rho\, d\rho$

$$= \frac{Q}{2\pi\varepsilon_0 a^2}\int_{z^2}^{z^2+a^2}\frac{du}{2\sqrt{u}}$$ $u^{-\frac{1}{2}} \qquad \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = 2u^{\frac{1}{2}}$

$$= \frac{Q}{4\pi\varepsilon_0 a^2}\, 2\sqrt{u}\Big|_{z^2}^{z^2+a^2}$$

$$= \frac{Q}{4\pi\varepsilon_0 a^2}\, 2\left(\sqrt{z^2+a^2} - |z|\right)$$

$$= \frac{2Q}{4\pi\varepsilon_0 a^2}\left\{\sqrt{z^2}\sqrt{\left(1+\frac{a^2}{z^2}\right)} - |z|\right\}$$

$$\frac{3\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 2^4} = \frac{5}{2^7}$$

$$= \frac{Q}{2\pi\varepsilon_0 a^2}\,|z|\left(\sqrt{1+\frac{a^2}{z^2}} - 1\right)$$

Expansion of $\left(1+\frac{a^2}{z^2}\right)^{1/2} = 1 + \left(\frac{1}{2}\right)\frac{a^2}{z^2} + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{a^2}{z^2}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{a^2}{z^2}\right)^3$

$+ \frac{1}{4!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{a^2}{z^2}\right)^4 + \cdots$

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$$\left(1+\frac{a^2}{z^2}\right)^{1/2}-1 = \frac{1}{2}\frac{a^2}{z^2} + -\frac{1}{8}\frac{a^4}{z^4} + \frac{1}{16}\frac{a^6}{z^6} + -\frac{5}{27}\left(\frac{a^2}{z^2}\right)^4 + \cdots$$

(b) ~~On the~~

$$\Phi(0,0,z) = \frac{Q}{2\pi\varepsilon_0 a^2}|z|\left\{\frac{1}{2}\frac{a^2}{z^2} - \frac{1}{8}\left(\frac{a^2}{z^2}\right)^2 + \frac{1}{16}\left(\frac{a^2}{z^2}\right)^3 - \frac{5}{27}\left(\frac{a^2}{z^2}\right)^4 + \cdots\right\}$$

(b) using $z=r$

$$\Phi(r,\theta=0) = \frac{Q}{2\pi\varepsilon_0 a^2} r\left\{\frac{1}{2}\frac{a^2}{r^2} - \frac{1}{8}\left(\frac{a^2}{r^2}\right)^2 + \frac{1}{16}\left(\frac{a^2}{r^2}\right)^3 - \frac{5}{2^7}\left(\frac{a^2}{r^2}\right)^4 + \cdots\right\}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\left\{1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 + \frac{1}{8}\left(\frac{a}{r}\right)^4 - \frac{5}{64}\left(\frac{a}{r}\right)^6 + \cdots\right\}$$

For general coordinates, it is sufficent to add $P_\ell(\cos\theta)$ to $\frac{1}{r^{\ell+1}}$ term

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\left[1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{8}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) - \frac{5}{64}\left(\frac{a}{r}\right)^6 P_6(\cos\theta) + \cdots\right]$$

3 (a) $\delta(\vec{x}-\vec{x}') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$

Check $\int \delta(\vec{x}-\vec{x}')\,dV = \int \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')\,\rho\,d\rho\,d\phi\,dz$

$$= \int\delta(\rho-\rho')d\rho\int\delta(\phi-\phi')d\phi\int\delta(z-z')dz = 1.$$

Using expansions for $\delta(\rho-\rho')$ and $\delta(\phi-\phi')$

$$\delta(\vec{x}-\vec{x}') = \frac{1}{\rho}\int_0^\infty dk\,k\rho J_m(k\rho)J_m(k\rho')\ \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\ \delta(z-z')$$

$$= \cancel{\frac{1}{2\pi}\int_0^\infty}\ \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^\infty dk\,k J_m(k\rho)J_m(k\rho')\,\delta(z-z')$$

(b) $G(\vec{x},\vec{x}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^\infty dk\,k\,J_m(k\rho)J_m(k\rho')\,g_m(k,z,z')$

$$\nabla^2 G = \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]G(\vec{x},\vec{x}')$$

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$\cancel{\nabla^2 G(\vec{x}, \vec{x}') = \frac{1}{2\pi} \sum_{m=-\infty}}$

From the Bessel eq.

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dJ_\nu}{dx}\right) = \frac{1}{x}\left(\frac{dJ_\nu}{dx} + x\frac{d^2 J_\nu}{dx^2}\right) = -\left(1 - \frac{\nu^2}{x^2}\right) J_\nu(x)$$

using $x = k\rho$

$$\frac{1}{k\rho}\frac{d}{d(k\rho)}\left[k\rho\frac{dJ_\nu}{d(k\rho)}\right] = -\left(1 - \frac{\nu^2}{k^2\rho^2}\right) J_\nu(k\rho)$$

or $$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right] = -\left(k^2 - \frac{\nu^2}{\rho^2}\right) J_\nu(k\rho)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)G = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^\infty dk\, k\left(-k^2 + \frac{m^2}{\rho^2}\right) J_m(k\rho) J_m(k\rho') g_m(k, z, z')$$

$$\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}G = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\left(-\frac{m^2}{\rho^2}\right) e^{im(\phi-\phi')}\int_0^\infty dk\, k\left(-k^2 + \frac{m^2}{\rho^2}\right) J_m(k\rho) J_m(k\rho') g_m(k, z, z')$$

$$\frac{\partial^2}{\partial z^2}G = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \cancel{k\rho}\, e^{+im(\phi-\phi')}\int_0^\infty dk\, k\, J_m(k\rho) J_m(k\rho')\frac{d^2 g_m}{dz^2}$$

Combining all three,

$$\nabla^2 G(\vec{x}, \vec{x}') = \frac{1}{2\pi}\sum_{m=0}^{\infty} e^{im(\phi-\phi')}\int_0^\infty dk\, k\, J_m(k\rho) J_m(k\rho')\left(\frac{d^2 g_m}{dz^2} - k^2 g_m\right)$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty} e^{im(\phi-\phi')}\int_0^\infty dk\, k\, J_m(k\rho) J_m(k\rho')\,\delta(z - z') \times (-4\pi)$$

From orthogonality of $e^{im\phi}$ and $J_m(k\rho)$,

$$\frac{d^2 g_m}{dz^2} - k^2 g_m = -4\pi\,\delta(z - z')$$

(c) If $z \neq z'$,

$$\frac{d^2 g_m}{dz^2} - k^2 g_m = 0 \qquad g_m = e^{\pm kz} \quad \text{or} \quad g_m = \sinh kz, \cosh kz$$

Using the boundary condition at $z=0$ $(G(\vec{x}, \vec{x}') = 0)$, g_m should be proportional to $\sinh kz$

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Let $g_m(k, z, z') = A(z') \sinh kz \qquad z < z'$

For $z > z'$, there is another boundary condition at $z = L$, which can be satisfied by $g_m(k, z, z') = B(z') \sinh[k(L-z)]$

$$g_m(k, z, z') = \begin{cases} A(z') \sinh[kz] & z < z' \\ B(z') \sinh[k(L-z)] & z > z' \end{cases}$$

Using symmetry of $g_m(k, z, z')$ under $z \leftrightarrow z'$,

$$g_m(k, z, z') = g_m(k, z', z) = \begin{cases} A(z) \sinh kz' & z' < z \\ B(z) \sinh[k(L-z')] & z' > z \end{cases}$$

Comparison of $z < z'$ case yields $A(z') = C \sinh[k(L-z')]$ with constant C

" " $z > z'$ and $B(z') = C \sinh kz'$.

So $$g_m(k, z, z') = \begin{cases} C \sinh(kz) \sinh[k(L-z')] & z < z' \\ C \sinh(kz') \sinh[k(L-z)] & z > z' \end{cases}$$

The remaining constant C can be determined from the discontinuity condition at $z = z'$. Integrating diff eq for g_m from $z' - \varepsilon$ to $z' + \varepsilon$

$$\left.\frac{dg_m}{dz}\right|_{z'+\varepsilon} - \left.\frac{dg_m}{dz}\right|_{z'-\varepsilon} = -4\pi \qquad z = z' + \varepsilon \quad z > z'$$

$$\left.\frac{dg_m}{dz}\right|_{z'+\varepsilon} = \frac{d}{dz}\left\{C \sinh kz' \sinh[k(L-z)]\right\}\Big|_{z'+\varepsilon} = C \sinh kz' \cosh[k(L-z)](-k)\Big|_{z'+\varepsilon}$$

$$= C \sinh kz' \cosh k(L-z')(-k)$$

$$\left.\frac{dg_m}{dz}\right|_{z'-\varepsilon} = \frac{d}{dz} C \sinh kz \sinh k(L-z')\Big|_{z'+\varepsilon} = C \cosh kz' \sinh k(L-z')(k)$$

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So the condition reads

$$C \sinh kz' \cosh k(L-z')(-k) - C \cosh kz' \sinh k(L-z')(k) = -4\pi$$

$$-Ck\left[\sinh kz' \cosh k(L-z') + \cosh kz' \sinh k(L-z')\right] = -4\pi$$

$$Ck(\sinh kL) = 4\pi \qquad C = \frac{4\pi}{k \sinh kL}$$

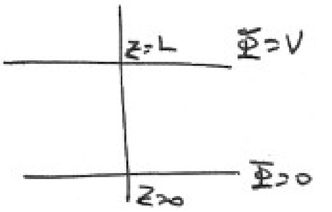
$$g_m = \frac{4\pi}{k \sinh kL}\{\sinh(kz_<)\sinh k(L-z_>)\}$$

$z_<$: smaller of z, z'

$z_>$: large of z, z'

$$G(\vec{x},\vec{x}') = \frac{2}{\cancel{k}\sinh kL}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^\infty dk\, \cancel{k}\, J_m(k\rho) J_m(k\rho') \sinh(kz_<) \sinh k(L-z_>)$$

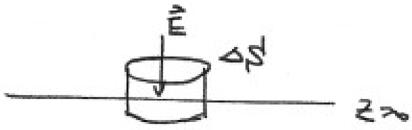
(d)



$$\Phi = \frac{z}{L}V$$

(e) Electric field $\vec{E} = -\nabla\Phi = -\frac{V}{L}\hat{z}$ is constant

Using Gauss Law on the conductor surface

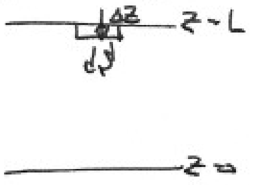


$$\vec{E}\cdot\hat{n}\,dS = \frac{\sigma\, dS}{\epsilon_0}$$

$$-\frac{V}{L}dS = \frac{\sigma}{\epsilon_0}dS \qquad \boxed{\sigma = -\frac{\epsilon_0 V}{L}}$$

σ is constant, independent of ρ, ϕ.

(f) Suppose we give a small artificial displacement of small area dS by Δz



The change of energy of the electric field is

$$\Delta W = -\frac{\epsilon_0}{2}\frac{V^2}{L^2}dS\Delta z$$

so the force on the area dS is $dF = -\frac{\Delta W}{\Delta z} = \frac{\epsilon_0}{2}\frac{V^2}{L^2}dS$

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Force per unit area is $F = \frac{\epsilon_0}{2}\frac{V^2}{L^2}$. Since ΔW decreases when the area moves toward the other conducting plate, the force is an attraction between the two plates.

4. (a) Outside of the sphere

$$\nabla \cdot \vec{E} = \frac{q}{\epsilon_0}\delta(\vec{x} - \vec{x}_0)$$

Inside of the sphere

$$\nabla \cdot \vec{D} = 0 \quad (\text{or } \nabla \cdot \vec{E} = 0)$$

All over $\nabla \times \vec{E} = 0$

(b) At the boundary, normal component of $\vec{D}$ and tangential component of $\vec{E}$ should be continuous. In this geometry, on the surface of the sphere, r component is normal and θ, ϕ component are tangential. So

$\epsilon E_r|_{inside} = \epsilon_0 E_r|_{outside}$: normal component of $\vec{D}$

$E_\theta|_{inside} = E_\theta|_{outside}$

$E_\phi|_{inside} = E_\phi|_{outside}$, which is satisfied automatically since there is no E_ϕ from the symmetry of the problem.

(c) Inside, Φ should satisfy Laplace equation since there is no charge. In spherical coordinate system, the given form is the most general solution when there is azimuthal symmetry (no ϕ dependence).

Outside, Φ should satisfy Poisson eq with charge at $\vec{x}_0$. The first term is particular solution for the charge at $\vec{x}_0$ while the 2nd term is general solution for Laplace eq whose solution can be added freely

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to particular solution of the inhomogeneous PDE.

(d) Φ_{in} should be regular at the center, $B_\ell = 0$

Φ_{out} should vanish at infinity $C_\ell = 0$

(e) $\Phi_{in} = \sum A_\ell r^\ell P_\ell(\cos\theta)$

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{x}_0|} + \sum D_\ell r^{-(\ell+1)} P_\ell(\cos\theta)$$

Near the boundary $|\vec{x}| \sim a$ and $|\vec{x}_0| = d$, so we can expand the first term of Φ_{out} as

$$\frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{x}_0|} = \frac{q}{4\pi\varepsilon_0}\sum \frac{r^\ell}{d^{\ell+1}} P_\ell(\cos\theta)$$

So $\Phi_{out} = \sum \left[\frac{q}{4\pi\varepsilon_0}\frac{r^\ell}{d^{\ell+1}} + D_\ell \frac{1}{r^{\ell+1}}\right] P_\ell(\cos\theta)$

$$E_r\Big|_{\substack{in \\ r=a}} = -\frac{\partial \Phi_{in}}{\partial r}\Big|_{\substack{in \\ r=a}} = \sum A_\ell \ell r^{\ell-1} P_\ell(\cos\theta)(-1) = -\sum A_\ell \ell a^{\ell-1} P_\ell(\cos\theta)$$

$$E_r\Big|_{\substack{out \\ r=a}} = -\frac{\partial \Phi_{out}}{\partial r}\Big|_{r=a} = \sum \left[\frac{q}{4\pi\varepsilon_0}\frac{\ell a^{\ell-1}}{d^{\ell+1}} - \frac{D_\ell(\ell+1)}{r^{\ell+2}}\right] P_\ell(\cos\theta)(-1)$$

$$= \sum \left[\frac{q}{4\pi\varepsilon_0}\frac{\ell a^{\ell-1}}{d^{\ell+1}} - \frac{D_\ell(\ell+1)}{a^{\ell+2}}\right] P_\ell(\cos\theta)(-1)$$

Equating $D_\perp$ and using orthogonality of $P_\ell(\cos\theta)$

$$\frac{\epsilon}{\epsilon_0} A_\ell \ell a^{\ell-1} = \frac{q}{4\pi\varepsilon_0}\frac{\ell a^{\ell-1}}{d^{\ell+1}} - \frac{(\ell+1)}{a^{\ell+2}} D_\ell$$

$$\frac{\epsilon}{\epsilon_0} \ell A_\ell = \frac{q}{4\pi\varepsilon_0}\frac{\ell}{d^{\ell+1}} - \frac{\ell+1}{a^{2\ell+1}} D_\ell$$

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$$-E_\theta\Big|_{r=a}^{in} = \frac{\partial \Phi_{in}}{\partial \theta}\Big|_{r=a} = \Sigma A_\ell a^\ell \frac{d}{d\theta} P_\ell(\cos\theta)$$

$$-E_\theta\Big|_{r=a}^{out} = \frac{\partial \Phi_{out}}{\partial \theta}\Big|_{r=a} = \Sigma\left(\frac{q}{4\pi\varepsilon_0}\frac{a^\ell}{d^{\ell+1}} + D_\ell \frac{1}{a^{\ell+1}}\right)\frac{d}{d\theta} P_\ell(\cos\theta)$$

Using orthogonality of $\frac{d}{d\theta} P_\ell(\cos\theta)$,

$$A_\ell a^\ell = \frac{q}{4\pi\varepsilon_0}\frac{a^\ell}{d^{\ell+1}} + D_\ell \frac{1}{a^{\ell+1}}$$

$$A_\ell = \frac{q}{4\pi\varepsilon_0}\frac{1}{d^{\ell+1}} + \frac{D_\ell}{a^{2\ell+1}}$$

$$\frac{\epsilon}{\epsilon_0}\ell\left(\frac{q}{4\pi\epsilon_0}\frac{1}{d^{\ell+1}} + \frac{D_\ell}{a^{2\ell+1}}\right) = \frac{q}{4\pi\varepsilon_0}\frac{\ell}{d^{\ell+1}} - \frac{\ell+1}{a^{2\ell+1}} D_\ell$$

$$\frac{D_\ell}{a^{2\ell+1}}\left[\frac{\epsilon}{\epsilon_0}\ell + (\ell+1)\right] = \frac{q}{4\pi\varepsilon_0}\frac{1}{d^{\ell+1}}\left(\ell - \frac{\epsilon}{\epsilon_0}\ell\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{d^{\ell+1}}\ell\left(1 - \frac{\epsilon}{\epsilon_0}\right)$$

$$D_\ell = \frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}} \frac{\ell\left(1-\frac{\epsilon}{\epsilon_0}\right)}{\frac{\epsilon}{\epsilon_0}\ell + (\ell+1)}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}} \frac{\ell(\epsilon_0-\epsilon)}{\epsilon\ell + \epsilon_0(\ell+1)}$$

$$A_\ell = \frac{q}{4\pi\epsilon_0}\frac{1}{d^{\ell+1}} + \frac{q}{4\pi\epsilon_0}\frac{1}{d^{\ell+1}}\frac{\ell(\epsilon_0-\epsilon)}{\epsilon\ell+\epsilon_0(\ell+1)}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{d^{\ell+1}}\frac{\epsilon\ell + \epsilon_0(\ell+1) + \ell\epsilon_0 - \ell\epsilon}{\epsilon\ell + \epsilon_0(\ell+1)}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{d^{\ell+1}}\frac{\epsilon_0(2\ell+1)}{\epsilon\ell + \epsilon_0(\ell+1)}$$

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$$\Phi_{in} = \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{(2l+1)\epsilon_0}{l\epsilon+(l+1)\epsilon_0}\frac{r^l}{d^{l+1}}P_l(\cos\theta)$$

$$\Phi_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{x}-\vec{x}_0|} + \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{a^{2l+1}}{d^{l+1}r^{l+1}}\frac{l(\epsilon_0-\epsilon)}{l\epsilon+(l+1)\epsilon_0}P_l(\cos\theta)$$

(f) $\varepsilon \to \infty$ $\quad \dfrac{l(\epsilon_0-\epsilon)}{l\epsilon+(l+1)\epsilon_0} \to (-1)$

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{x}_0|} + \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{a^{2l+1}}{d^{l+1}r^{l+1}}(-1)P_l(\cos\theta)$$

$$(\text{2nd term}) = \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\left(-\frac{a}{d}\right)\frac{(a^2/d)^l}{r^{l+1}}P_l(\cos\theta)$$

$$= \frac{\left(-\frac{a}{d}q\right)}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{(a^2/d)^l}{r^{l+1}}P_l(\cos\theta)$$

Since $a^2/d < a$, the sum over l can be performed as

$$\sum_{l=0}^{\infty}\frac{(a^2/d)^l}{r^{l+1}}P_l(\cos\theta) = \frac{1}{|\vec{x}-\vec{x}_1|}$$ where $\vec{x}_1 = (0,0,\frac{a^2}{d})$ on the z axis and inside the sphere.

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{x}_0|} + \frac{1}{4\pi\varepsilon_0}\frac{\left(-\frac{a}{d}q\right)}{|\vec{x}-\vec{x}_1|}$$

Image charge $-\frac{a}{d}q$ is placed on $|\vec{x}_1| = \frac{a^2}{d}$ on z axis

(g) Potential due to the polarization charge of the dielectric sphere

$$\Phi_{pol} = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{a^{2l+1}}{d^{l+1}r^{l+1}}\frac{l(\epsilon_0-\epsilon)}{l\epsilon+(l+1)\epsilon_0}P_l(\cos\theta)$$

Electric field at $\vec{x}_0$ would be radial only from symmetry.

$$\vec{E}(\vec{x}_0) = E_r\hat{r} = E_r\hat{z} \quad (\text{on the z axis } \hat{r}=\hat{z})$$

$$= -\frac{\partial\Phi}{\partial r}\hat{z}\Big|_{r=d} = \frac{-q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{a^{2l+1}}{d^{l+1}}(-(l+1))\frac{1}{d^{l+2}}\frac{l(\epsilon_0-\epsilon)}{l\epsilon+(l+1)\epsilon_0}P_l(\cos\theta)$$

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And at $\theta = 0$, $P_\ell(\cos\theta) = P_\ell(1) = 1$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{\ell(\ell+1)(\epsilon_0-\epsilon)}{\ell\epsilon+(\ell+1)\epsilon_0} \left(\frac{a}{d}\right)^{2\ell+1} \frac{1}{d^2}\hat{z}$$

If we assume $\varepsilon > \varepsilon_0$,

$$\vec{E} = -\frac{q}{4\pi\varepsilon_0}\frac{1}{d^2} \sum_{\ell=0}^{\infty} \frac{\ell(\ell+1)(\epsilon-\epsilon_0)}{\ell\epsilon+(\ell+1)\epsilon_0} \left(\frac{a}{d}\right)^{2\ell+1}$$

$$\vec{F} = q\vec{E} = -\frac{q^2}{4\pi\varepsilon_0}\frac{1}{d^2} \sum_{\ell=0}^{\infty} \frac{\ell(\ell+1)(\epsilon-\epsilon_0)}{\ell\epsilon+(\ell+1)\epsilon_0} \left(\frac{a}{d}\right)^{2\ell+1}$$

(b) If $d \gg a$, the first non-vanishing term will be dominant, which is $\ell=1$ term

$$\vec{F} \simeq -\frac{q^2}{4\pi\varepsilon_0}\frac{1}{d^2} \frac{2(\epsilon-\epsilon_0)}{\epsilon+2\epsilon_0}\left(\frac{a}{d}\right)^3 \hat{z}$$

At such a big distance, we can approximate the field generated by the point charge at the sphere as constant electric field. ~~with~~ n

$$\vec{E}_{sphere} \simeq -\frac{q}{4\pi\varepsilon_0}\frac{1}{d^2}\hat{z}$$

When dielectric sphere is placed on this constant field, it will produce dipole ~~ele~~ moment

$$\vec{p} = 4\pi\varepsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0+2}\right) a^3 \vec{E}_{sphere}$$

$$= \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0+2}\right) a^3 (-q) \frac{1}{d^2}\hat{z}$$

$$= (-q)\frac{a^3}{d^2}\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\hat{z}$$

Then this dipole moment will produce electric field at the position of the point charge.

$$\vec{E}_q = \frac{3\hat{n}(\vec{p}\cdot\hat{n})-\vec{p}}{4\pi\varepsilon_0 d^3} \quad \text{with} \quad \hat{n} = \hat{z}$$

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$$\vec{E}_q = \frac{3\hat{z}\,(\vec{p}\cdot\hat{z}) - \vec{p}}{4\pi\epsilon_0 d^3} = \frac{3\hat{z}\,p_z - p_z\hat{z}}{4\pi\epsilon_0 d^3}$$

$$= \frac{2p_z}{4\pi\epsilon_0 d^3}\,\hat{z}$$

$$= \frac{2}{4\pi\epsilon_0 d^3}(-q)\,\frac{a^3}{d^2}\,\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\,\hat{z}$$

$$= -\frac{q}{4\pi\epsilon_0}\,\frac{1}{d^2}\,\frac{2(\epsilon-\epsilon_0)}{\epsilon+2\epsilon_0}\left(\frac{a}{d}\right)^3\hat{z}$$

So the force on the point charge q will be

$\vec{F}_q \simeq -\frac{q^2}{4\pi\epsilon_0}\,\frac{1}{d^2}\,\frac{2(\epsilon-\epsilon_0)}{\epsilon+2\epsilon_0}\left(\frac{a}{d}\right)^3\hat{z}$ same as the result obtained ~~from~~ from the expression of (g).