Classical E&M Midterm

April 26, 2006

Maximum possible score from this mid-term exam is 140 points.

1. (Total 30 points) Let us warm up ourselves on vector and tensor analysis.

   (a) (10 points) Show that
   $$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{V}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$$

   (b) (5 points) Under the rotation of the coordinate system, components of vectors or tensors transform with the matrix $\{a_{ij}\}$. The trace of a second-rank tensor is the sum of all of its diagonal elements, that is
   $$\mathrm{Tr}\,\mathbf{T} = \sum_{k=1}^{3} T_{kk}$$
   Show that the trace of a second rank tensor is invariant under rotation.

   (c) (5 points) Any second rank tensor can be decomposed into symmetric and antisymmetric parts, such as
   $$\begin{aligned} A_{ij} &= \frac{T_{ij} - T_{ji}}{2} = -A_{ji} \\ S_{ij} &= \frac{T_{ij} + T_{ji}}{2} = S_{ji} \end{aligned}$$
   Show that under the rotation, A_{ij} remains antisymmetric while S_{ji} remains symmetric.

   (d) (10 points) Now justify the following "reduction" of any second rank tensor into 3 irreducible parts:
   $$T_{ij} = \frac{\delta_{ij}}{3}\mathrm{Tr}\,\mathbf{T} + A_{ij} + \left(S_{ij} - \frac{\delta_{ij}}{3}\mathrm{Tr}\,\mathbf{T}\right)$$
   How many components are there in each irreducible part?

2. (Total 20 points) A flat circular disk of radius a has a charge Q distributed uniformly over its area. The disk is placed on xy plane with its center at the origin.

   (a) (10 points) What is the potential on the z-axis?

   (b) (10 points) Show that the potential at arbitrary point (r, θ, ϕ) where $r > a$ is given by
   $$\begin{aligned} \Phi(r,\theta,\phi) = \frac{Q}{4\pi\varepsilon_0 r}\Bigg[1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) \quad &+ \quad \frac{1}{8}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) \\ &- \quad \frac{5}{64}\left(\frac{a}{r}\right)^6 P_6(\cos\theta) + \cdots\Bigg] \end{aligned}$$

3. (Total 45 points) Let us consider two infinite planes of conductor separated by a distance L. The space between the two planes is vacuum. We want to obtain Dirichlet Green function for the region between the two conducting planes. From the symmetry of the problem, it is convenient to use cylindrical coordinate system with two planes of conductor at $z = 0$ and $z = L$. From the definition of the Dirichlet Green function, $G(\mathbf{x}, \mathbf{x}')$ should satisfy

$$\begin{aligned} \nabla^2 G(\mathbf{x}, \mathbf{x}') &= -4\pi\delta(\mathbf{x} - \mathbf{x}') \\ G(\mathbf{x}, \mathbf{x}') &= 0 \quad \text{at } z = 0, L \end{aligned}$$

From the general solution of Laplace equation in cylindrical coordinate system, $G(\mathbf{x}, \mathbf{x}')$ can be written down as

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \int_0^\infty dk\, k J_m(k\rho) J_m(k\rho') g_m(k, z, z')$$

(a) (5 points) Express $\delta(\mathbf{x}, \mathbf{x}')$ in cylindrical coordinate system in terms of ρ, ϕ and z. Be careful of normalization of the delta function and leave $\delta(z - z')$ as is.

(b) (10 points) Apply ∇^2 to $G(\mathbf{x}, \mathbf{x}')$ and show that $g_m(k, z, z')$ should satisfy

$$\frac{d^2 g_m(k, z, z')}{dz^2} - k^2 g_m(k, z, z') = -4\pi\delta(z - z')$$

(c) (15 points) Obtain $g_m(k, z, z')$ and give full expression for $G(\mathbf{x}, \mathbf{x}')$.

(d) (5 points) Suppose that the conducting plate at $z = 0$ is held at $\Phi = 0$ while the plate at $z = L$ is held at $\Phi = V$. Obtain potential $\Phi(\rho, \phi, z)$ for the region $0 \le z \le L$.

(e) (5 points) Obtain surface charge density $\sigma(\rho, \phi)$ for the conducting plate at $z = 0$.

(f) (5 points) What's the force per unit area on the surface of the conductor? Obtain the magnitude and the direction.

4. (Total 65 points) Consider a dielectric sphere of radius a and a point charge q at a distance d from the center of the dielectric sphere. Let us assume $a < d$. The sphere has dielectric constant ε and both the sphere and the charge are placed in vacuum. Based on the symmetry of the problem, it would be convenient to use the spherical coordinate system. Furthermore, let us assume that the charge q is on the z axis with the center of the sphere at the origin. For future convenience, let's denote the position of the charge as $\mathbf{x}_0$.

(a) (5 points) Write down Maxwell equations to be satisfied by $\mathbf{E}$ and/or $\mathbf{D}$ for inside and outside the sphere.

(b) (5 points) What are the boundary conditions to be satisfied on the surface of the dielectric sphere in terms of E_r, E_θ and E_ϕ?

(c) (10 points) Justify the following expression for the solution of the problem.

$$\begin{aligned}\Phi_{\text{in}} &= \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(\cos\theta) \\ \Phi_{\text{out}} &= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{x}-\mathbf{x}_0|} + \sum_{l=0}^{\infty}(C_l r^l + D_l r^{-(l+1)})P_l(\cos\theta)\end{aligned}$$

where Φ_{in} and Φ_{out} denote potential inside and outside of the dielectric sphere, respectively. Why there is no ϕ dependence in Φ?

(d) (5 points) Using simple arguments, show that $B_l = 0$ and $C_l = 0$ for all l.

(e) (10 points) From the boundary conditions at the surface of the sphere, obtain A_l and D_l and give full expression for Φ_{in} and Φ_{out}.

(f) (10 points) When ε goes to ∞, the dielectric sphere becomes conducting sphere. Show that Φ_{out} becomes the potential for the case of a point charge and a conducting sphere.

(g) (10 points) What is the force on the point charge?

(h) (10 points) When $d \gg a$, what is the leading term in the force on the point charge? Give a physics interpretation on this leading term. Conversely, you can solve this problem from a simple physics argument even though you are not able to solve the previous problems.

Hopefully useful formulae

1. A few key properties of rotation matrix $\{a_{ij}\}$
   (a) Inverse is equal to transpose: $a_{ij}^{-1} = a_{ji}$
   (b) Transformation of vectors: $V_i' = \sum_{j=1}^{3} a_{ij} V_j$
   (c) Transformation of second rand tensor: $T_{ij}' = \sum_{k=1}^{3}\sum_{l=1}^{3} a_{ik}a_{jl}T_{kl}$

2. Laplacian in cylindrical coordinate system

$$\nabla^2\Psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{\partial^2\Psi}{\partial z^2}$$

3. Bessel function satisfies the following differential equation.

$$\frac{d^2 J_\nu(x)}{dx^2} + \frac{1}{x}\frac{dJ_\nu(x)}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)J_\nu(x) = 0$$

4. Solution of boundary value problem using Dirichlet Green function

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D}{\partial n'}da'$$

5. Induced dipole moment in a dielectric sphere of radius a under constant electric field.

$$\mathbf{p} = 4\pi\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)a^3\mathbf{E}_0$$

6. Potential and field by a dipole.

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{\mathbf{p}\cdot\mathbf{x}}{4\pi\varepsilon_0 r^3}\\ \mathbf{E}(\mathbf{x}) &= \frac{3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}}) - \mathbf{p}}{4\pi\varepsilon_0 r^3}\end{aligned}$$

7. Miscellaneous formulae

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)\\ \delta(\rho-\rho') &= \int_0^\infty dk\, k\rho J_m(k\rho)J_m(k\rho')\\ \delta(\phi-\phi') &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}\end{aligned}$$