Classical E&M Homework 02

March 14, 2005

Due date: March 21, 2005

Each problem has 10 points.

1. Imagine a spherical shell of charge, of radius R and surface charge σ. If we remove a small circular piece of radius $a \ll R$, what is the direction and magnitude of the field at the midpoint of the aperture?

2. Let's do a small imaginary (*gedanken*) experiment as follows.

   (a) Generate an electric field using two, parallel, fixed layers of charge (of uniform surface density $\pm\sigma$, respectively).

   (b) Take a conductor and place it between the two layers of charge. The configuration of charge inside the conductor will change to a final, stable configuration.

   (c) By some magic, make the the conductor *non*-conducting, leaving the charges frozen in space.

   (d) Remove the two layers of charge used to generate the electric field.

   What does the residual electric field look like, both *inside* and *outside* of the body?

3. Use Gauss's theorem to prove that at the surface of a curved charged conductor, the normal derivative of the electric field is given by

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

   where R_1 and R_2 are the principal radii of curvature of the surface.

4. The divergence of a vector field $\mathbf{F}$ is a scalar, as we know. Suppose we try to define a vector, different from the curl, in this way:

$$G = \hat{\mathbf{x}}\frac{\partial F_x}{\partial x} + \hat{\mathbf{y}}\frac{\partial F_y}{\partial y} + \hat{\mathbf{z}}\frac{\partial F_z}{\partial z}$$

   Prove or disprove that this "G" is a vector.

5. Prove the *mean value theorem*: For charge-free space, the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered on that point.

6. In the previous problem, what would be the average potential if there is a well-localized charge distribution ρ inside the sphere?

7. Prove that no *stable* equilibrium is possible in electrostatics.

8. Prove Green's reciprocation theorem: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\Phi'\, d^3x + \int_S \sigma\Phi'\, da = \int_V \rho'\Phi\, d^3x + \int_S \sigma'\Phi\, da$$

9. Prove *Thomson's theorem*: If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors.

10. Prove the following theorem: If a number of conducting surfaces are fixed in position with a given total charge on each, the introduction of an uncharged, insulated conductor into the region bounded by the surfaces lowers the electrostatic energy.

11. Show the equation (1.81) of the textbook. (Hints are given in Problem 1.22.)

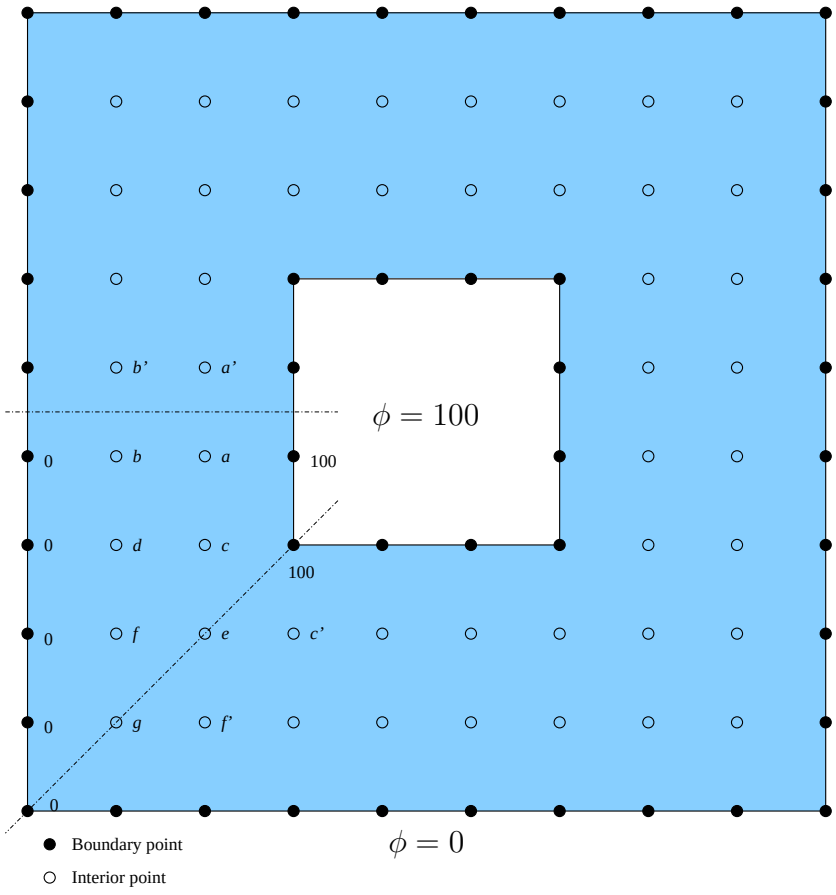
$$\langle\langle F(x,y)\rangle\rangle = F(x,y) + \frac{3}{10}h^2\nabla^2 F + \frac{1}{40}h^4\nabla^2(\nabla^2 F) + \mathcal{O}(h^6)$$

12. Let's try to apply *relaxation method* to quite a simple example. In the figure, there are two square equipotential boundaries, one inside the other. This might be a cross section through a capacitor made of two sizes of square metal tubing. The problem is to find values at an array of points which will be a good approximation to the tru potential function, $\phi(x,y)$, at these points. We adopt a fairly coarse array to keep the labor within bounds. Assign, arbitrarily, potential 100 to the inner boundary and 0 to the outer. All points on these boundaries remain at these values. In principle, we

can start with any values on the interior points; we save time however by some intelligent guesswork. You can pick any starting values for the interior points. Obviously, you should take advantage of the symmetry; only seven interior values need to be computed. Now you simply go over the lattice in some systematic manner replacing each interior-point value by the average of its four neighbors. For example,

$$c \to \frac{1}{4}(100 + a + d + e)$$

In doing so, make sure that you keep $a' = a$, $b' = b$, $c' = c$ and $f' = f$ from symmetry. Stop when tired, or when all changes resulting from a sweep over the net are sufficiently small. A good time to quit might be when no change larger than 1 unit occurs in the course of a pass.



For an answer to this problem, give starting and final values for a to g with the total number of iterations. If you are diligent enough, try to vary the starting values, and check that the final values do not depend on the choice of starting values.