Gerasimov-Drell-Hearn Sum Rule

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Real Photon Interaction

➜ Consider the interaction between *real photon* and spin 1/2 target

➜ For *real photon*, we need *two* independent scattering amplitude.

- Virtual photon has four independent amplitudes.

$$\mathcal{M}_{1,\frac{1}{2},0,-\frac{1}{2}}, \quad \mathcal{M}_{1,\frac{1}{2},1,\frac{1}{2}}, \quad \mathcal{M}_{1,-\frac{1}{2},1,-\frac{1}{2}}, \quad \mathcal{M}_{0,\frac{1}{2},0,\frac{1}{2}}$$

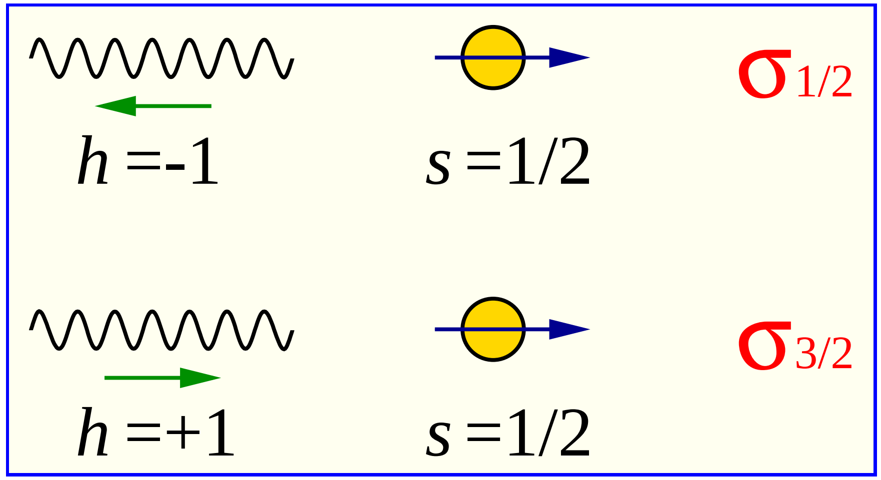
- *Real photon* does not have spin 0 state, removing them leaves *two*

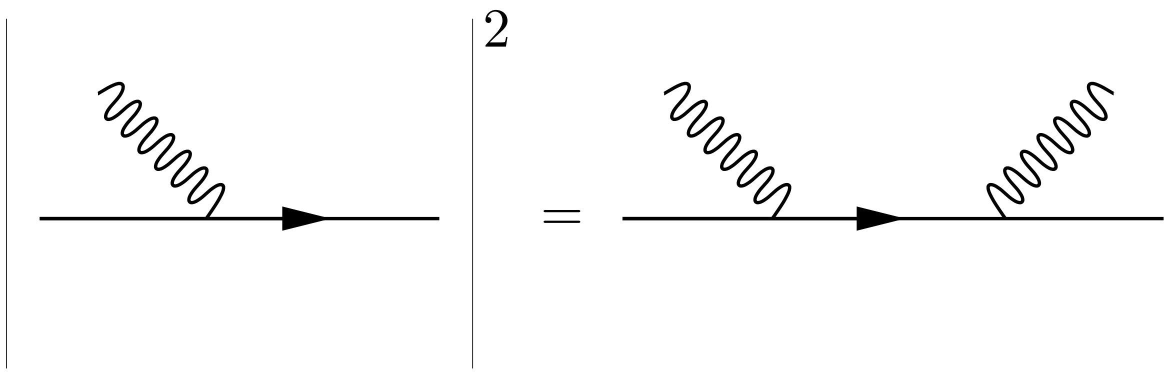
$$\mathcal{M}_{1,\frac{1}{2},1,\frac{1}{2}}, \quad \mathcal{M}_{1,-\frac{1}{2},1,-\frac{1}{2}}$$

- Or, using parity operation, we have

$$\mathcal{M}_{1,\frac{1}{2},1,\frac{1}{2}}, \quad \mathcal{M}_{-1,\frac{1}{2},-1,\frac{1}{2}}$$

Absorption of Real Photon





$$f(\nu) = f_1(\nu^2)\vec{e}^{\,\prime *} \cdot \vec{e} + \nu f_2(\nu^2) i \left(\vec{\sigma} \cdot \vec{e}^{\,\prime *} \times \vec{e}' \right)$$

Crash Course in Complex Variable

- Complex function on complex plane

$$f(z) = u(z) + iv(z)$$

- $f(z)$ is analytic at $z = z_0$ if differentiable at $z = z_0$
- Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz$$

- By requiring

$$\lim_{|z|\to\infty} |f(z)| = 0, \quad 0 \geq \arg(z) \geq \pi$$

- We get

$$f(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z_0}\, dx \quad \text{for} \quad \Im(z_0) > 0$$

To Dispersion Relation

- Letting $z_0 \longrightarrow x_0$

$$f(x_0) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx$$

- Separating real and imaginary parts

$$\begin{aligned} f(x_0) &= u(x_0) + iv(x_0) \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}\, dx - \frac{i}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}\, dx \end{aligned}$$

- Or

$$\begin{aligned} \Re[f(x_0)] &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Im[f(x_0)]}{x - x_0}\, dx \\ \Im[f(x_0)] &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Re[f(x_0)]}{x - x_0}\, dx \end{aligned}$$

Dispersion Relation

- Dispersion relation on $f_2(\nu^2)$ gives

$$\Re[f_2(\nu^2)] = \frac{1}{\pi} P \int_0^\infty \frac{\Im[f_2(\omega^2)]}{\omega^2 - \nu^2} d(\omega^2)$$

- We happen to know $\Re[f_2(0)]$ from Low Energy Theorem (LET).

$$\Re[f_2(0)] = -\frac{1}{\pi} \left(\mu - \frac{Q}{2M}\right)^2$$

- And we can express $\Im[f_2(\omega^2)]$ in terms of $\sigma_{1/2}$ and $\sigma_{2/3}$

$$\Im[f_2(\omega^2)] = \frac{1}{4\pi} \left[\sigma_{1/2}(\omega) - \sigma_{3/2}(\omega)\right]$$

- Assembling everything,

$$\frac{1}{2\pi^2} P \int_0^\infty \frac{\sigma_{1/2}(\omega) - \sigma_{3/2}(\omega)}{\omega} d\omega = -\frac{1}{\pi} \left(\mu - \frac{Q}{2M}\right)^2$$

Gerasimov-Drell-Hearn Sum Rule

$$\int_0^\infty \frac{\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)}{\nu} d\nu = -2\pi \left(\mu - \frac{Q}{2M}\right)^2$$

- Connects static observables of the target (μ,Q,M) to dynamics ($\sigma_{1/2}$,$\sigma_{3/2}$)
- Links QED with strong interactions: excitation spectrum
- Gerasimov, Drell and Hearn (1966): Dispersion theory
- Hosoda, Yamamoto (1966): Current algebra
- Iddings (1965): No explicit mention ... but already generalized version!

Fundamental Basis

- Lorentz invariance, gauge invariance $\rightarrow$ Forward Compton amplitude

$$f(\nu) = f_1(\nu^2)\vec{e}^{\,\prime *} \cdot \vec{e} + \nu f_2(\nu^2) i \left(\vec{\sigma} \cdot \vec{e}^{\,\prime *} \times \vec{e}^{\,\prime}\right)$$

- Unitarity $\rightarrow$ Optical Theorem

$$\Im[f_2(\omega^2)] = \frac{1}{4\pi}\left[\sigma_{1/2}(\omega) - \sigma_{3/2}(\omega)\right]$$

- Lorentz, gauge invariance $\rightarrow$ Low Energy Theorem

$$\Re[f_2(0)] = -\frac{1}{\pi}\left(\mu - \frac{Q}{2M}\right)^2$$

- Causality + *No subtraction* $\rightarrow$ Kramers-Krönig dispersion relation

$$\Re[f_2(\nu^2)] = \frac{1}{\pi} P \int_0^\infty \frac{\Im[f_2(\omega^2)]}{\omega^2 - \nu^2} d(\omega^2)$$

A Word on No-Subtraction

➜ ... leads to weird behavior

- Amplitude: $\lim f_2(\nu) \sim \nu$
- Differential cross section:

$$\lim_{\nu\to\infty} \frac{1}{d\Omega} \left(d\sigma_{1/2} - d\sigma_{3/2}\right)\Big|_{\theta=0} = -\infty$$

➜ ... while Regge Theory tells

- Imaginary part of the amplitude: $\displaystyle\lim_{\nu\to\infty} \frac{\Im[f_2(\nu)]}{\nu} = 0$
- Total cross section:

$$\lim_{\nu\to\infty} \left(\sigma_{1/2}^{\text{tot}} - \sigma_{3/2}^{\text{tot}}\right) = 0$$

For Nucleons

- Using *anomalous* magnetic moment κ

$$\mu = (1+\kappa)\frac{e}{2M}$$

$$I_{GDH} \equiv \int_0^{\infty} \frac{\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)}{\nu}\, d\nu = -2\pi \left(\mu - \frac{e}{2M}\right)^2$$

$$I_{GDH} = -\frac{2\pi^2\alpha}{M^2}\kappa^2$$

- Proton: $\kappa_p = 1.793$

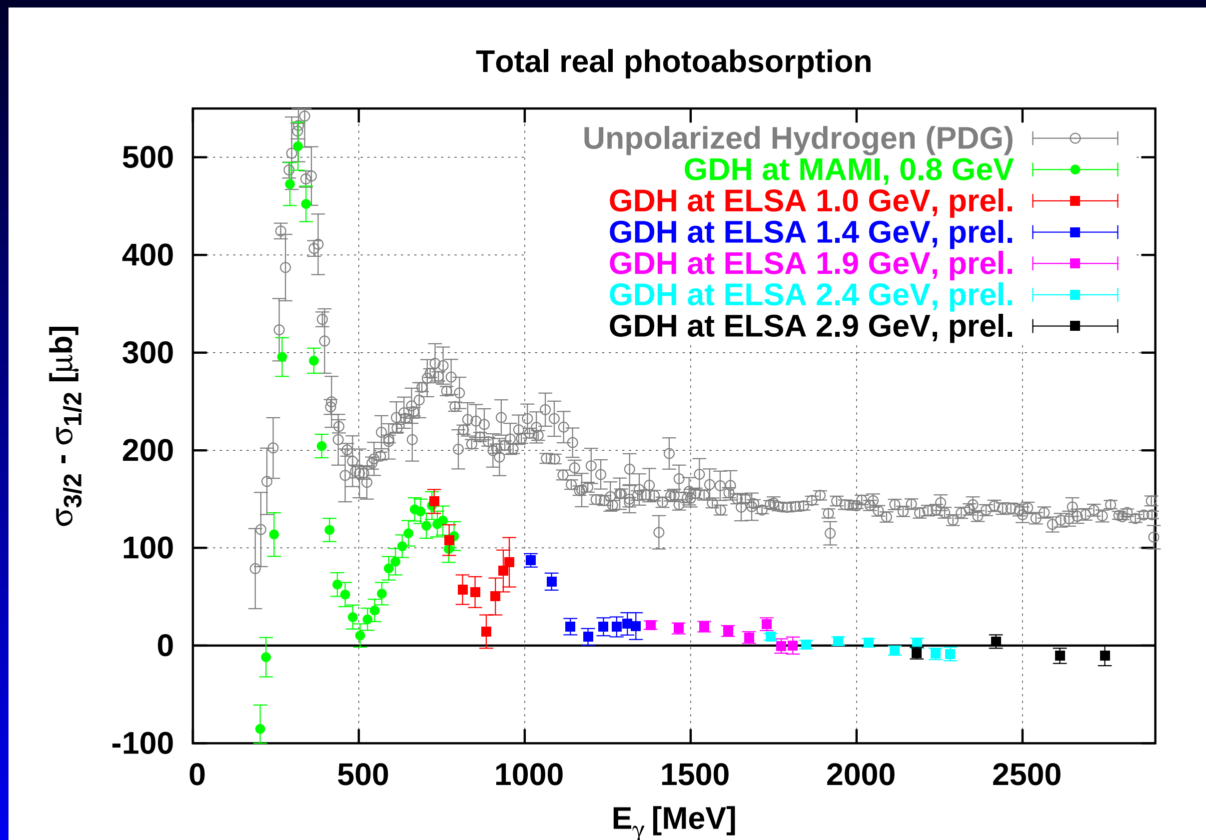
$$I_{GDH} = -204.8\mu\text{b}$$

- Neutron: $\kappa_n = -1.913$

$$I_{GDH} = -233.2\mu\text{b}$$

Experiment

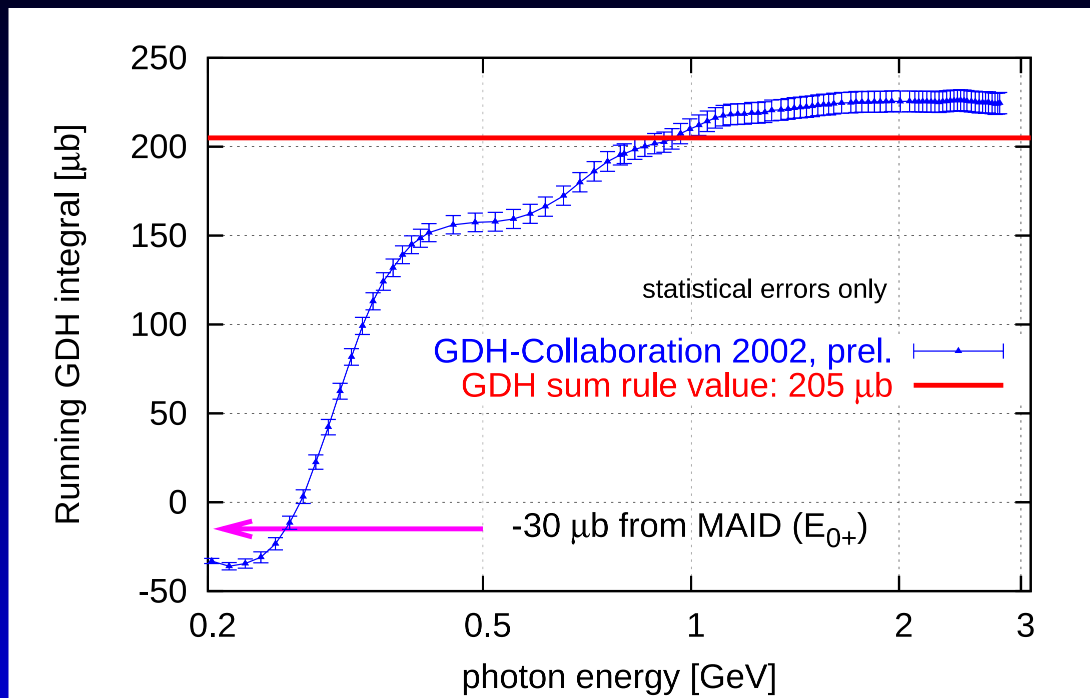
Polarized x-section difference



gdh–Collaboration

Experiment - GDH Integral

Running GDH integral



The GDH sum up to 2.8 GeV: $(\mathbf{225 \pm 5_{\text{stat}}})\mu$b (prelim.)

Contribution from $E_\gamma > 3$ GeV: -20 – -35 μb

α_{a_1} crucial $\rightarrow$ to be fixed by ELSA neutron data.

gdh–Collaboration

What about Virtual Photons?

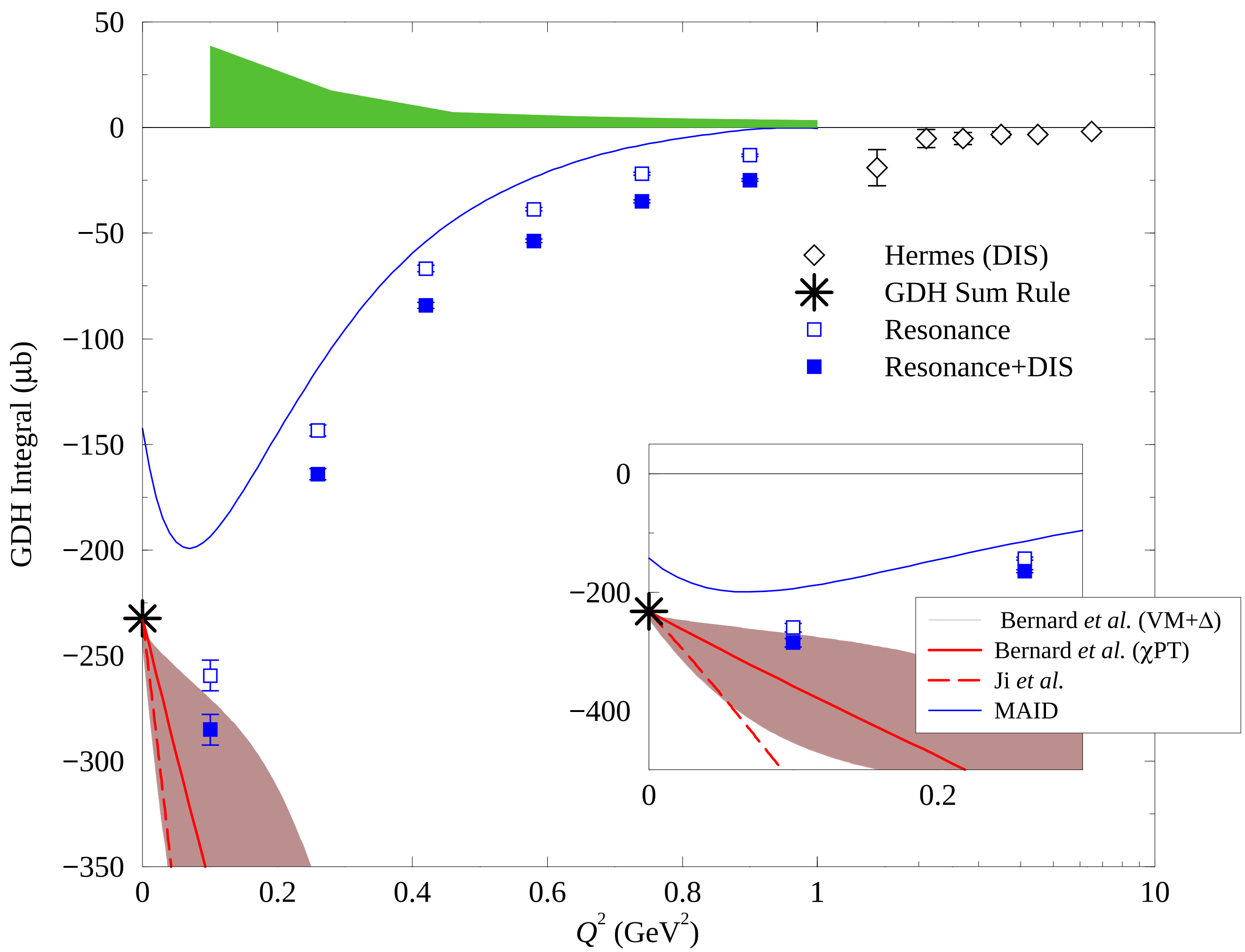
- Generalization to $Q^2 > 0$ case
- Experimentalist's favorite

$$
\begin{aligned}
I_{GDH}(Q^2) &= \int_0^\infty \frac{\sigma_{1/2}(\nu, Q^2) - \sigma_{3/2}(\nu, Q^2)}{\nu}\, d\nu \\
&= \frac{16\pi^2\alpha}{Q^2} \int_0^1 \left(g_1(x, Q^2) - \frac{4M^2x^2}{Q^2} g_2(x, Q^2) \right) dx
\end{aligned}
$$

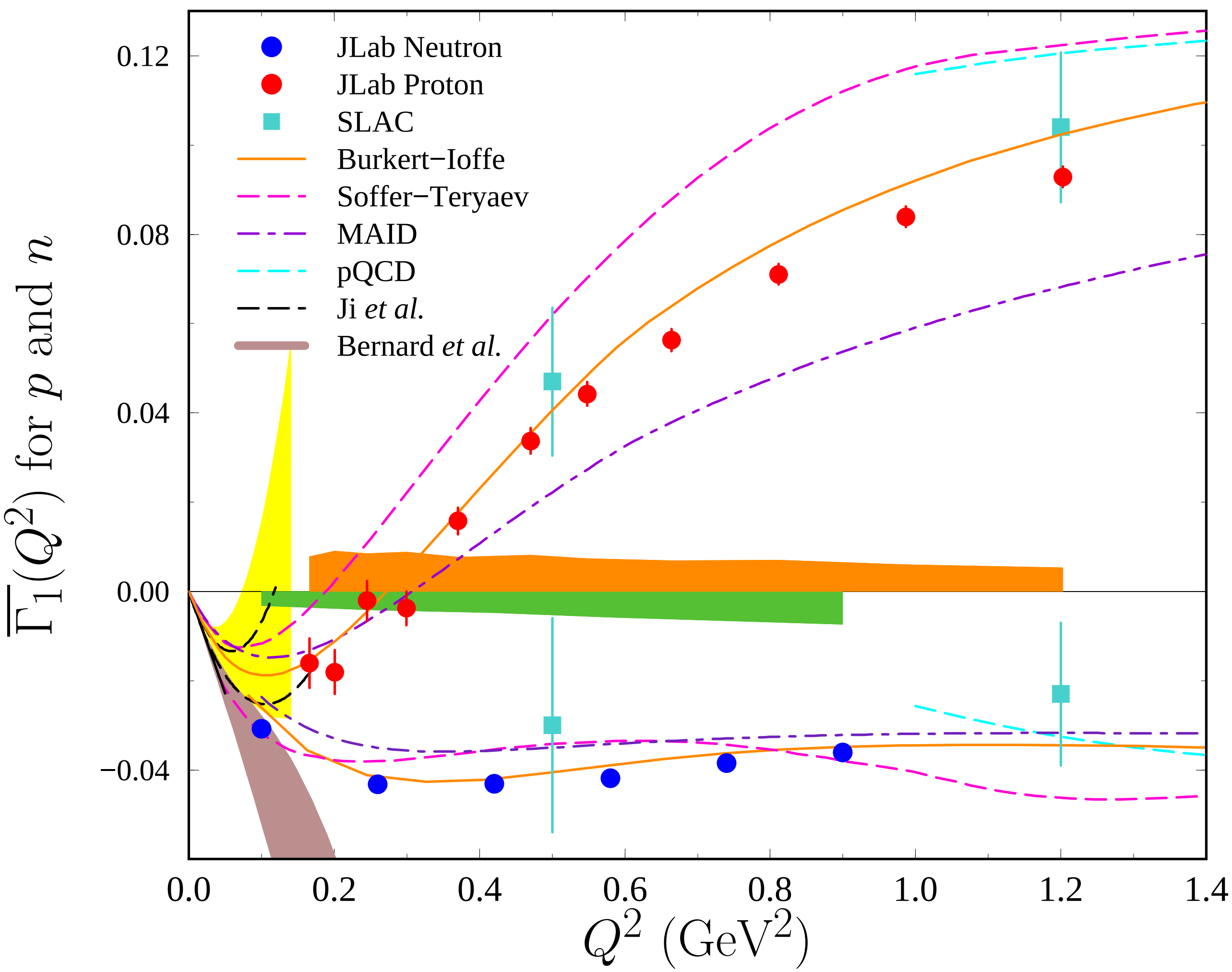
- Theorist's favorite

$$
\begin{aligned}
I_{GDH}(Q^2) &= \frac{16\pi^2\alpha}{Q^2} \int_0^1 g_1(x, Q^2)\, dx \\
&= \frac{16\pi^2\alpha}{Q^2} \Gamma_1(Q^2)
\end{aligned}
$$

GDH Integral for the Neutron at $Q^2 > 0$



$\Gamma_1(Q^2)$ for the Nucleons at $Q^2 > 0$



Bjorken Sum at Small Q^2

- Bjorken sum at $Q^2 = 0$

$$
\begin{aligned}
\Gamma_1^p(Q^2) - \Gamma_1^n(Q^2)\big|_{Q^2=0} &= \frac{Q^2}{16\pi^2\alpha}\left[I_{GDH}^p(Q^2) - I_{GDH}^n(Q^2)\right]_{Q^2=0} \\
&= \frac{Q^2}{16\pi^2\alpha}[-204.8 - (-233.2)][\mu\text{b}] \\
&= 0.06365 Q^2 \quad (Q^2 \text{in GeV}^2)
\end{aligned}
$$

Bjorken Sum at Small Q^2