Dynamics of a Toom Interface in Three Dimensions

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We introduce a novel three-dimensional Toom model on a bcc lattice, and study its physical properties. In the low-noise limit, the model leads to an effective solid-on-solid type model, which exhibits a stationary interface via depositions and evaporations with an avalanche process. We find that the model is described by the Edwards-Wilkinson equation for the unbiased case and the anisotropic Kardar-Parisi-Zhang equation in the *weak*-coupling limit for the biased case. Thus the square of the surface width diverges logarithmically with space and time for both unbiased and biased cases.

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In the past few years, there has been an explosion of studies in the field of nonequilibrium surface growth due to theoretical interests in the classification of universality for stochastic models and also due to the application to physical phenomena such as crystal growth, vapor deposition, electroplating, biological growth, etc. [1]. Recently Derrida, Lebowitz, Speer, and Spohn (DLSS) [2] studied physical properties of the two-dimensional Toom model. The Toom interface is formed by a rule of simple probabilistic cellular automaton. In the low-noise limit, this model leads to a $(1+1)$-dimensional solid-on-solid type (SOS) model, which is in turn much simpler for understanding the generic nature of dynamics. In the SOS model, the dynamics of spin flips may be regarded as a deposition-evaporation process of particles. Because of the nature of the Toom dynamics, the deposition-evaporation process occurs in an avalanche fashion.

Extension of the Toom model into three dimensions is an interesting problem. However, it is not obvious to define a majority rule on a *simple cubic* (sc) lattice. The majority rule applying to the group of spins consisting of itself and the three nearest-neighbor spins in each direction as in the original square lattice is ambiguous, because an even number of spins are involved. Recently Barabási, Araujo, and Stanley (BAS) [3] overcame this difficulty, and introduced a majority rule on sc structure using the five spins consisting of the next-nearest-neighbor spin in the $(-1,-1,1)$ direction in addition to the four spins mentioned above. Including the fifth spin in that particular direction leads to the anisotropic Kardar-Parisi-Zhang (AKPZ) equation [4] for the interface,

$$\partial_t h = \nu_\parallel \partial_\parallel^2 h + \nu_\perp \partial_\perp^2 h + \tfrac{1}{2}\lambda_\parallel (\partial_\parallel h)^2 + \tfrac{1}{2}\lambda_\perp (\partial_\perp h)^2 + \eta \,, \tag{1}$$

with white noise η. BAS found that their model belongs to the *strong*-coupling regime of the AKPZ universality, which is equivalent to the isotropic KPZ universality [5]. Thus the square of the surface width shows a power-law type divergence.

In this Letter, we introduce a new three-dimensional Toom model on a *body centered cubic* (bcc) lattice, in which majority rule is applied to the set of five spins comprised of itself and four *nearest* neighboring spins as depicted in Fig. 1. Our model is distinguished from that of BAS in that the four neighboring spins are of equidistance, while they are not in the BAS model. Interestingly, this difference of the local dynamic rules leads to different universality classes, which is in accordance with a recent claim that dynamic universality class depends on local dynamic rules in $(2+1)$ dimensions [6]. We studied the model for both unbiased and biased cases, and found that our model belongs to the Edwards-Wilkinson (EW) [7] universality for the unbiased case and the AKPZ universality in the *weak*-coupling limit [4] for the biased case. The latter is shown by demonstrating that the signs of the coefficients $\lambda_\parallel$ and $\lambda_\perp$ are opposite to each other. Therefore, our model is in a sense a completely opposite limit in comparison to the BAS model. Consequently, the square of the surface width exhibits a logarithmic divergence with space and time for both biased and unbiased cases. As far as we are aware of, this is the first stochastic lattice model which exhibits opposite signs for $\lambda_\parallel$ and $\lambda_\perp$.

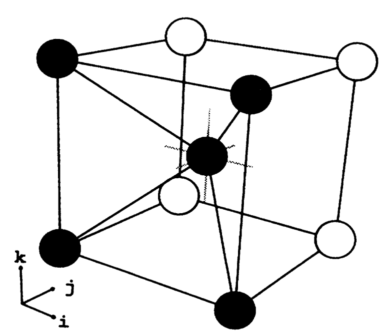

FIG. 1. The three-dimensional Toom rule on the bcc lattice used in this work. The black circled spin $\sigma_{i,j,k}$ is updated with the majority rule of itself and four nearest-neighbor spins (the black circles).

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The model we introduce here consists of Ising spins ($\sigma_{i,j,k}=\pm 1$) on a bcc lattice. At each time, a randomly selected spin is updated according to the local rule that $\sigma_{i,j,k}(t+1)$ becomes, with probability $1-p-q$, equal to the majority of itself and the four nearest-neighbor spins at $(i\pm\frac{1}{2}, j-\frac{1}{2}, k\pm\frac{1}{2})$ at time t, and otherwise becomes equal to $+1$ with probability p, and -1 with probability q. Unbiased (biased) dynamics results when $p=q$ ($p\neq q$). Besides this dynamic rule, we use proper boundary conditions to generate reasonable interface between $(+)$ and $(-)$ spin domains. The boundary spins on the plane $j=0$ are fixed to the value $\sigma_{i,0,k}=+1$ (-1) if $k>L/2$ ($k<L/2$). Periodic boundary conditions are imposed on other boundary planes. With these boundary conditions, the Toom interface in three dimensions is formed between upper $(+)$ spin domain and lower $(-)$ spin domain.

In the low-noise limit ($p,q\to 0$), a spin flipped due to noise returns immediately to its original state by the majority rule, if the spin is situated away from the interface. Accordingly, we may assume that spin flips occur only at the boundary. We are thus led to study an effective SOS type model in analogy with the $(1+1)$-dimensional stair-like model studied by DLSS. In the SOS model, the dynamic rule of spin flips is mapped to a particle dynamics on a two-dimensional substrate in the form of deposition evaporation with an avalanche. If the avalanche process is not allowed, so that depositions and evaporations occur only on local valleys and mountains, respectively, then our model would be equivalent to the deposition-evaporation model proposed by Forrest and Tang [8], a generalization of the Plischke-Rácz-Liu model [9] into higher dimensions.

The SOS model is defined on the checkerboard lattice, the square lattice rotated by 45°. To each lattice point, a relative height value is assigned. The value represents the height of the interface of the original three-dimensional Toom model. Initially we begin with a flat surface characterized by the heights 0 on one sublattice and 1 on the other (see Fig. 2). The Toom dynamics is then translated to the following updating rule: At each time step, we select a random site (i,j) and start the evaporation (deposition) process with probability $\bar{p}$ (probability $1-\bar{p}$). In the evaporation (deposition) process, the height of the site is decreased (increased) by 2 if the heights of both of its two nearest neighbors at $(i+\frac{1}{2}, j-\frac{1}{2})$ and $(i-\frac{1}{2}, j-\frac{1}{2})$ are lower (higher). Otherwise there is no change. Next the avalanche process may occur on the sites $(i+\frac{1}{2}, j+\frac{1}{2})$ and $(i-\frac{1}{2}, j+\frac{1}{2})$. If the height of each site is higher (lower) by 3 than that of (i,j), then the height is decreased (increased) by 2. The avalanche rule is then applied successively to the next rows in the $\hat{\mathbf{j}}$ direction until there is no change. The avalanche process is an interesting aspect of our model, and reflects a generic feature of the Toom dynamics. The unbiased case $p=q$ in the bcc structure corresponds to the case of $\bar{p}=0.5$ in the SOS model, where deposition and evaporation occur with equal probability. The biased case $p\neq q$ corresponds to the case of $\bar{p}\neq 0.5$ in the SOS model. We impose periodic boundary conditions in both directions.

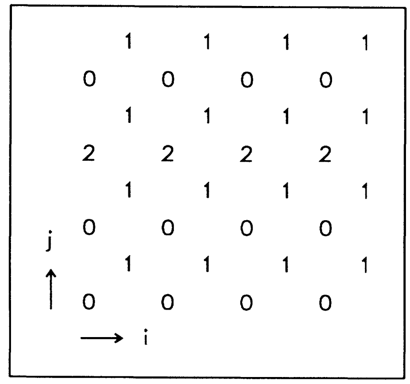

FIG. 2. The configuration of a substrate with one step along a row for linear size $L=4$.

The continuum equation for our model is also described by Eq. (1), because our model selects out a preferred direction, and because the cubic nonlinear term derived by DLSS was proved to be marginally irrelevant even in $(1+1)$ dimensions [10]. For the unbiased case, both of the nonlinear terms in Eq. (1) disappear due to the symmetry of deposition and evaporation. So the equation reduces to the EW equation, implying that the square of the surface width diverges logarithmically with space and time. For the biased case, average height grows with increasing time, so that $\lambda_\parallel$ and $\lambda_\perp$ are nonzero. In order to find out the signs of $\lambda_\parallel$ and $\lambda_\perp$, we consider the following.

For convenience, let us consider the pure deposition case ($\bar{p}=0$) on three kinds of substrates with linear size L: (a) a flat substrate, (b) a substrate with one step along a row, and (c) a substrate with one step along a column. The case of (b) is shown in Fig. 2. For the case of (a), the average number of particles deposited in one time step is $L^2/2L^2$, since only local valleys are available for deposition. Thus the average height change is $\langle\Delta h\rangle_0 = (L^2/2L^2)2=1$. For the case of (b), the average height changes by $\langle\Delta h\rangle_\parallel = [L(L-1)/2L^2]2+(L/2L^2)6=1+2/L$, where the first term results from deposition at local valleys and the second one comes from the avalanche on the downward hill. Therefore, $\langle\Delta h\rangle_\parallel > \langle\Delta h\rangle_0$, which implies $\lambda_\parallel>0$. Next, for the case of (c), the average height change is $\langle\Delta h\rangle_\perp=[L(L-1)/2L^2]2=1-1/L$, because there is one less column of valleys for deposition. Therefore, $\langle\Delta h\rangle_\perp<\langle\Delta h\rangle_0$, which implies $\lambda_\perp<0$. The above considerations can be generalized to initial substrates with an arbitrary slope and to arbitrary $\bar{p}\neq 0$. Since the signs of $\lambda_\parallel$ and $\lambda_\perp$ are opposite, the AKPZ equation renormalizes to the weak-coupling limit as shown by Wolf

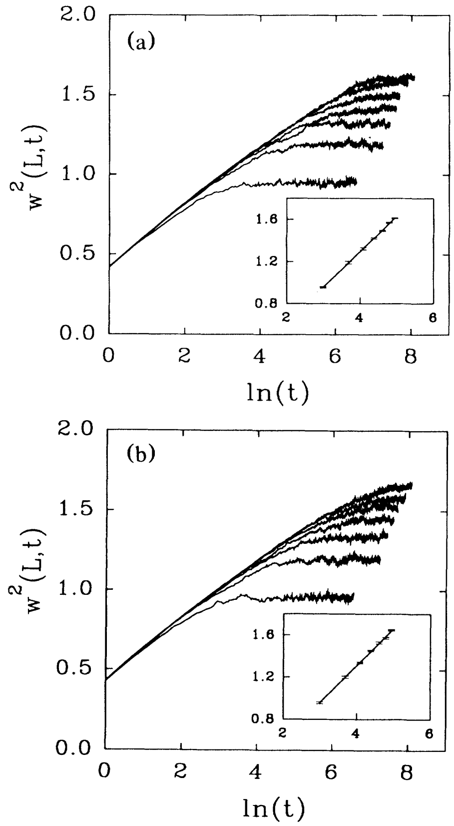

FIG. 3. w^2 vs $\ln t$ for unbiased case $\bar{p}=0.5$ (a) and for biased case $\bar{p}=0.3$ (b). Each curve is for system sizes $L=20$, 40, 60, 80, 100, 120, and 140, respectively, from bottom to top, and is averaged over 300 configurations. Insets: w^2 vs $\ln L$ after saturation for each case.

[4]. Consequently, our model belongs to the weak-coupling regime of the AKPZ universality. Therefore, the square of the surface width is logarithmic for both unbiased and biased cases.

We performed numerical simulations of the three-dimensional Toom model on the bcc lattice, and compared the result with that of the (2+1)-dimensional SOS model. The results for both cases run on small sizes are in complete agreement with each other in the low-noise limit. Accordingly, we performed simulations intensively for larger systems using the SOS model. The simulations are done in the range of system size $L=20-140$ for several values of $\bar{p}$. For all cases, we expect that the square of the surface width w^2 diverges logarithmically with space and time as $w^2 \sim \ln t$ before saturation and $w^2 \sim \ln L$ after saturation. We present the result for $\bar{p}=0.5$ (unbiased) and $\bar{p}=0.3$ (biased) in Fig. 3. The numerical data are in good agreement with the theoretical predictions. Besides the square of the surface width, we also measured the height-height correlation functions in $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ directions, respectively, and found that they behave in the same manner as w^2. The detailed numerical data will be presented elsewhere [11].

Finally, we examined the avalanche size distribution $D(s)$, which is defined as the number of successive spin flips by a single noise process. $D(s)$ was measured in two different manners. In the first case, it is measured in the critical state (after saturation), while in the second case, it is measured during the whole time steps. In both cases, the distribution function $D(s)$ is found to be exponential, $D(s) \sim \exp(-s/s^*)$, with a constant value $s^*=1.097 \pm 0.057$ [11]. The exponential type functional form seems to reflect indirectly the validity of the collective-variable approximation used by DLSS [2].

In conclusion, we have introduced a new three-dimensional Toom model, and its associated SOS type model in (2+1) dimensions. The spin dynamics in three dimensions is mapped into particle dynamics via the deposition and evaporation process with an avalanche on the checkerboard lattice. We have found that for the unbiased case, the interface is described by the EW equation, and for the biased case, it is described by the AKPZ equation with the opposite signs of $\lambda_\parallel$ and $\lambda_\perp$. Consequently, the square of the surface width diverges logarithmically with space and time. This result is confirmed by numerical simulations.

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