Journal of the Korean Physical Society, Vol. 52, No. 2, February 2008, pp. 350∼356

# Self-Similarity in Fractal and Non-Fractal Networks

J. S. Kim, B. Kahng* and D. Kim
*Center for Theoretical Physics & Frontier Physics Research Division,*
*Department of Physics and Astronomy, Seoul National University, Seoul 151-747*

K.-I. Goh
*Department of Physics, Korea University, Seoul 136-713*

(Received 3 August 2007)

We study the origin of scale invariance (SI) of the degree distribution in scale-free (SF) networks with a degree exponent $\gamma$ under coarse graining. A varying number of vertices belonging to a community or a box in a fractal analysis is grouped into a supernode, where the box mass $M$ follows a power-law distribution, $P_m(M) \sim M^{-\eta}$. The renormalized degree $k'$ of a supernode scales with its box mass $M$ as $k' \sim M^{\theta}$. The two exponents $\eta$ and $\theta$ can be nontrivial as $\eta \neq \gamma$ and $\theta < 1$. They act as relevant parameters in determining the self-similarity, *i.e.*, the SI of the degree distribution, as follows: The self-similarity appears either when $\gamma \leq \eta$ or under the condition $\theta = (\eta - 1)/(\gamma - 1)$ when $\gamma > \eta$, irrespective of whether the original SF network is fractal or non-fractal. Thus, fractality and self-similarity are disparate notions in SF networks.

PACS numbers: 64.60.Ak, 89.75.-k, 05.70.Jk
Keywords: Scale-free network, Scale invariance, Coarse-graining, Fractality

## I. INTRODUCTION

Kadanoff's block spin and coarse-graining (CG) picture is the cornerstone of the renormalization-group (RG) theory [1]. A system is divided into blocks of equal size and is described in terms of the block variables that represent the average behavior of each block. Scale invariance at the critical point under this CG enables one to evaluate the critical exponents. From a geometric point of view, scale invariance implies the presence of fractal structures, and their fractal dimensions are associated with the critical exponents [2]. For example, the magnetization and the singular part of the internal energy for the critical Ising model are such scale invariant quantities, which are manifested as the fractal geometric forms of the area of spin domains and the length of the spin domain interface, respectively.

The fractal dimension $d_B$ of a fractal object is measured by using the box-covering method [3], in which the number of boxes, $N_B(\ell_B)$, needed to tile the object with boxes of size $\ell_B$ follows a power law,

$$N_B(\ell_B) \sim \ell_B^{-d_B}. \tag{1}$$

This relation is referred to as *fractal scaling* hereafter. A fractal object is self-similar in the sense that it contains smaller parts, each of which is similar to the entire object [3].

While the notions of CG and self-similarity are well established when an object is embedded in Euclidean space, it is not clear how to extend and apply them usefully to objects not embedded in Euclidean space. In the former case, the number of vertices within each box, equivalent to the block size in RG terminology and referred to as the group mass for future discussion, is almost uniform, and the fractal objects are self-similar and vice versa. In the latter case, however, the group mass is extremely heterogeneous so that it follows a power-law distribution. This case can happen in scale-free (SF) networks; then, one needs to establish the way of CG and the notion of self-similarity in a new setting, which is a goal of this paper. A SF network [4] is a network whose distribution $P_d(k)$ of degree $k$, the number of edges connected to a given vertex, follows a power-law form: $P_d(k) \sim k^{-\gamma}$ with the degree exponent $\gamma$. The scale invariance of the degree distribution under CG is defined as the self-similarity in SF networks.

Most SF networks in the real world contain functional groups or communities within them [5]. In general, the distribution of the group mass $M$, the number of vertices within each group follows a power law asymptotically [6, 7]:

$$P_m(M) \sim M^{-\eta}. \tag{2}$$

When such groups are formed within networks, it would be more natural to take each group as a unit to form a supernode in CG because the vertices within each group are

*E-mail: kahng@phya.snu.ac.kr

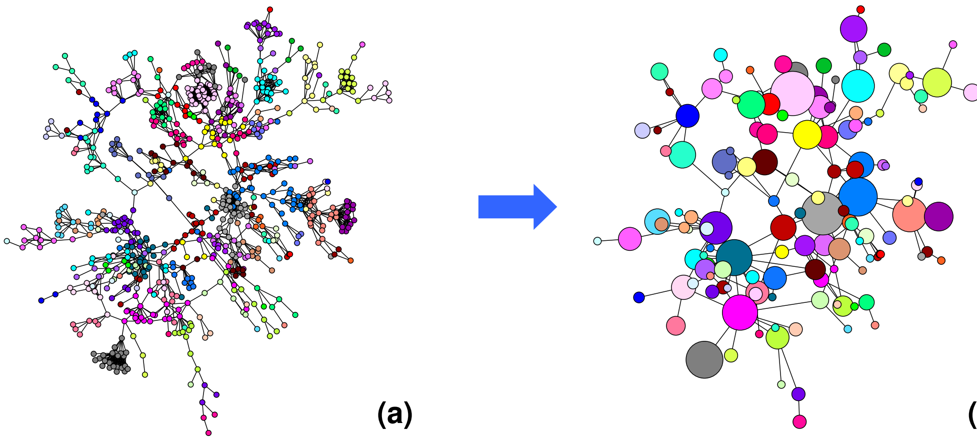


Fig. 1. (Color online) (a) The protein interaction network of the budding yeast and (b) its coarse-graining. Group masses $\{M_\alpha\}$, detected by the box covering, are heterogeneous. The node size in (b) is taken as $\sim \sqrt{M_\alpha}$ for visualization.

rather homogeneous in their characteristics, such as functionality or working division, and are connected densely. In other CG procedures, groups may not necessarily represent functional modules in bio-networks and communities in social networks; they can be taken arbitrarily in a theoretical perspective, for example, being artificially composed or boxes introduced in the fractal analysis [8–11]. Supernodes are connected if any of their merging vertices in different communities are connected. This CG method is different from the standard ones in view of extremely heterogeneous group masses as shown in Figure 1. We show below that the exponent $\eta$ plays a central role in determining the self-similarity.

So far, several endeavors to achieve a RG transformation for SF networks have been carried out. However, their methods remain in the framework of the standard RG method, ignoring the heterogeneity of the group-mass distribution. For example, Kim [12, 13] applied CG to a SF network generated on a Euclidean space [14], including long-range edges. Taking advantage of the underlying Euclidean geometry, the number of vertices within each block is uniform and increases in a power law as the block lateral size increases, so that the real-space RG method can be naturally applied. In Refs. [15] and [16], the decimation method was applied to a few deterministic SF networks, which were constructed recursively starting from each basic structure. Those deterministic models restore their shapes under CG achieved by decimating the vertices with the smallest degree at each stage. We discuss this case in detail later.

In this paper, we perform the CG of SF networks in three different ways: (i) random grouping, (ii) box covering in fractal analysis, and (iii) identifying community structure with clustering algorithms. The groups are (i) artificially composed, (ii) taken as boxes introduced in the fractal analysis, and (iii) taken as communities embedded within the network, in respective cases. For all cases, the group-mass distribution follows the power law, Eq. (2). We identify the criteria for self-similarity in terms of the exponents $\eta$ in Eq. (2) and $\theta$ which will be introduced below in Eq. (6). Such a self-similarity condition holds for non-fractal, as well as fractal real-world networks.

## II. CG BY RANDOM GROUPING

The model enables us to obtain the renormalized degree exponent $\gamma'$ analytically by using the generating function technique, showing that, indeed, $\gamma'$ depends on $\eta$. The model is constructed as follows: (i) We construct a SF network through the static model of Ref. [17]. We choose the degree exponent $\gamma = 3$ and the mean degree $\langle k \rangle = 4$. (ii) $N$ individual vertices are grouped randomly into $N'$ groups with sizes $\{M_\alpha\}$ ($\alpha = 1, \ldots, N'$), following the distribution function in Eq. (2). The exponent $\eta$ is tuned. We consider three cases of $\eta$: (a) $\eta = 2$ ($< \gamma$), (b) $\eta = 3$ ($= \gamma$) and (c) $\eta = 4$ ($> \gamma$). (iii) We perform the CG by replacing each group with a supernode and connecting them if any of their merging vertices in different groups are connected. A renormalized network is constructed. A schematic snapshot of a constructed network is shown in Figure 2. Next, the degree distribution $P'_d(k') \sim k'^{-\gamma'}$ of the renormalized network is measured. The result is as follows: $\gamma' = \eta = 2 \neq \gamma$ for $\eta = 2$ (a), $\gamma' = \gamma = 3$ for $\eta = 3$ (b) and $\gamma' = \gamma = 3$ for $\eta = 4$ (c), as shown in Figure 3.

The numerical result is understood analytically as follows: Since the vertices are grouped randomly, every vertex has an equal probability to connect to vertices in other groups per edge, so the total probability is proportional to the degree of each vertex. This leads to the relation $k'_\alpha \approx \sum_{j\in\alpha} k_j$, where $\alpha$ is the index of the group.

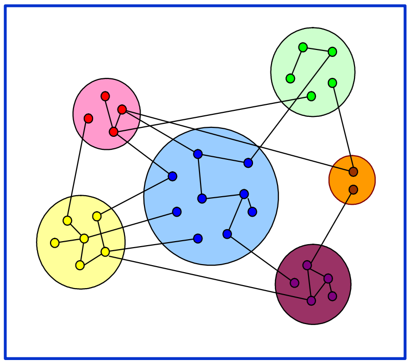

Fig. 2. (Color online) Schematic snapshot of a random SF network with random grouping.

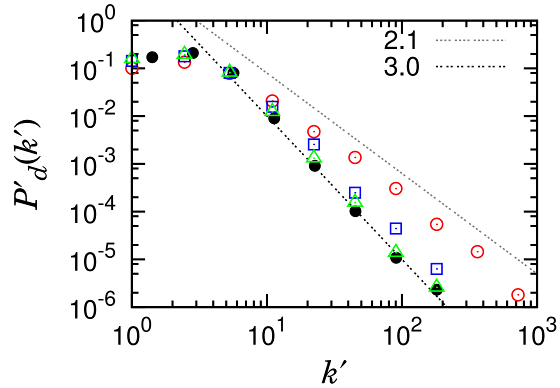


Fig. 3. (Color online) Degree distributions of a renormalized network of the static model for $\gamma = 3$ under CG. The group masses are preassigned to follow a power law $P_m(M) \sim M^{-\eta}$ with $\eta = 2(\circ)$, 3 ($\square$) and 4($\triangle$). The degree distribution of the original network is also drawn ($\bullet$). The dashed and the dotted lines with slopes of 2.1 and 3.0, respectively, are drawn as guidelines.

Then, the degree distribution $P'_d(k')$ of the renormalized network is written as

$$P'_d(k') \approx \sum_{M=1}^{\infty} P_m(M) \sum_{k_1,k_2,\ldots,k_M} \prod_{j=1}^{M} P_d(k_j) \times \delta(\sum_{j=1}^{M} k_j - k'), \quad (3)$$

$\delta$ denoting the Kronecker delta. By using the generating function technique, one can find the relation $\mathcal{P}'_d(z) = \mathcal{P}_m(\mathcal{P}_d(z))$, where $\mathcal{P}_d(z)$ is the generating function of $P_d(k)$ and so forth. The $\mathcal{P}_d(z)$ is obtained to be

$$\mathcal{P}_d(z) = 1 - \langle k\rangle(1-z) + a(1-z)^{\gamma-1} + \mathcal{O}\left((1-z)^2\right), \quad (4)$$

where $\langle k\rangle = \sum_k kP_d(k)$ and $a$ is a constant. The generating function $\mathcal{P}_m(\omega)$ is also derived in a similar form to Eq. (4). Then, one can find immediately that

$$\gamma' = \begin{cases} \gamma, & \text{for } \gamma \le \eta, \\ \eta, & \text{for } \gamma > \eta. \end{cases} \quad (5)$$

as long as both $\gamma$ and $\eta > 2$. Thus, self-similarity holds when $\gamma \le \eta$ and we can confirm that the exponent $\eta$ plays a key role in determining the exponent $\gamma'$. In the formulation, the relation $k'_\alpha \approx \sum_{j\in\alpha} k_j$ was crucial. When the relation no longer holds, the derivation of the degree exponent $\gamma'$ is more complicated. This can happen for fractal and some non-fractal networks, and also for clustered networks, which we discuss next.

## III. CG BY BOX COVERING

Recently, it was discovered [8] that fractal scaling, Eq. (1) holds in some SF networks, such as the world-wide web (WWW), the metabolic network of *Escherichia coli*, and the protein interaction network of *Homo sapiens*. Groups are formed by covering the networks by boxes that contain nodes whose mutual distances are less than a given box size. The group-mass distribution follows the power law of Eq. (2), even though the boxes' lateral sizes are fixed as $\ell_B$ for all boxes. Thus, the fractal networks are good objects for our study.

We first apply a box-covering algorithm to the WWW. Our box-covering algorithm is slightly modified from the original one introduced by Song *et al.* [8], and the details of the algorithm is presented in Ref. [9]. Both algorithms [8,9] are identical in spirit and the box size $\ell_B$ we use is related linearly to the corresponding one $\ell_S$ in Ref. [8]. Next, each box is collapsed into a supernode. Two supernodes are connected if any of their merging vertices in different boxes are connected. The degree distribution $P'_d(k')$ of the renormalized network is examined.

The distribution of box masses is measured and found to follow a power law asymptotically, $P_m(M) \sim M^{-\eta}$ [Figure 4 (a)]. The exponent $\eta$ is found to depend on the box size $\ell_B$. For small $\ell_B = 1$ or 2, $\eta \approx 2.2$ is measured, which is close to $\gamma$. On the other hand, as $\ell_B$ increases, we expect $\eta$ to approach the exponent $\tau = \gamma/(\gamma-1)$, describing the power-law behavior of the cluster-size distribution of the branching tree [18]. We find that $\eta \approx \tau \approx 1.8$ for $\ell_B = 5$. This result can be understood as follows: The WWW contains a skeleton [19], a spanning tree based on the betweenness centrality or load, which can represent the original network in the box-covering [20]. For small $\ell_B$, the lateral dimension of the box is not large enough to see the asymptotic fractal behavior of the spanning tree, so that the number of vertices in a given box scales similarly to the largest degree within that box. Thus, $\eta = \gamma$. As $\ell_B$ increases, on the other hand, the asymptotic fractal behavior of the spanning tree becomes dominant. Thus, the exponent

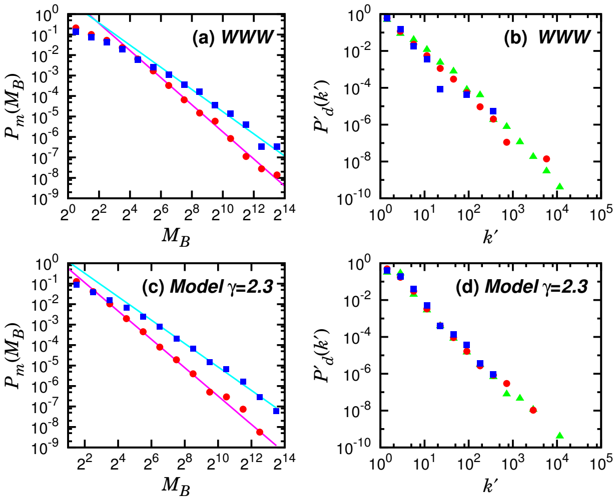


Fig. 4. (Color online) Box-mass distribution for (a) the WWW and (c) the fractal model network of Ref. [20]. Data are for $\ell_B = 2$ (●) and $\ell_B = 5$ (■). The solid lines are guidelines with slopes of –2.2 and –1.8, respectively, in both (a) and (c). The degree distributions of the original network (▲) and the renormalized networks with $\ell_B = 2$ (●) and $\ell_B = 5$ (■) for (b) the WWW and (d) the fractal model network. The fractal model has a system size $N \approx 3 \times 10^5$. (c) and (d) are adopted from Ref. [11].

$\eta$ becomes the same as the exponent $\tau$ asymptotically, which was observed in the case of $\ell_B = 5$. The case of $\eta \neq \gamma$ was not considered in Ref. [8]. The origin of the self-similarity is nontrivial as we show below.

The CG process involves two steps. The first is the vertex renormalization, *i.e.*, merging of vertices within a box into a supernode, and the second is the edge renormalization, merging of multiple edges between a pair of neighboring boxes into a single edge. The second step can yield a nonlinear relationship between the renormalized degree $k'$ and box mass $M_B$, although the total number of inter-community edges from a given box is linearly proportional to its box mass [9]. Thus, we propose that there exists a power-law relation between the average renormalized degree and the box mass:

$$\langle k' \rangle (M_B) \sim M_B^{\theta}. \tag{6}$$

The power-law relation Eq. (6) is tested numerically for the WWW, as shown in Figure 5(a). For $\ell_B = 2$, we estimate $\theta \approx 1.0 \pm 0.1$. Thus, the linear relation holds. For $\ell_B = 3$ and 5, however, $\theta \approx 0.8 \pm 0.1$ and $0.5 \pm 0.1$, respectively, implying the nonlinear relationship, Eq. (6). One may doubt the nonlinear behavior due to the scattered data shown in Figure 5(a). To confirm this result, we recall a previous study [11] for the fractal model with $\gamma = 2.3$ introduced by Goh *et al.* [20], where we can reduce the data noise by taking an ensemble average over network configurations. We obtained similar results as shown in Figure 5(b). The data for the fractal model are averaged over 10 different network configurations, so the nonlinear relationship could be seen more clearly.

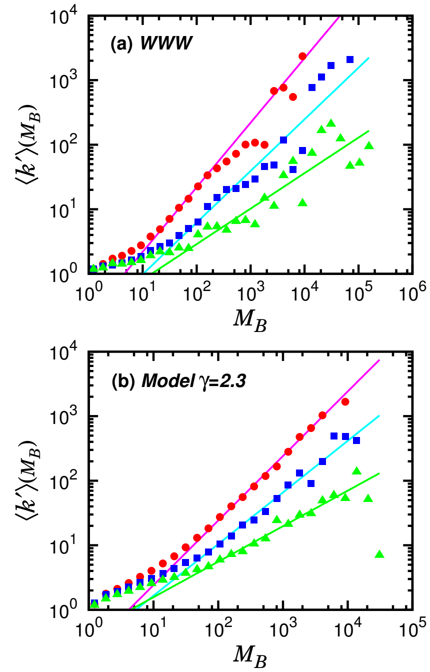


Fig. 5. (Color online) Plot of the average renormalized degree $\langle k' \rangle$ versus box mass $M_B$ for (a) the WWW and (b) the fractal model network of Ref. [20]. Data are for box sizes $\ell_B = 2$ (●), $\ell_B = 3$ (■) and $\ell_B = 5$ (▲). The solid lines, guidelines, have slopes of 1.0 (●), 0.8 (■) and 0.6 (▲), respectively, for both (a) and (b). (b) is adopted from Ref. [11].

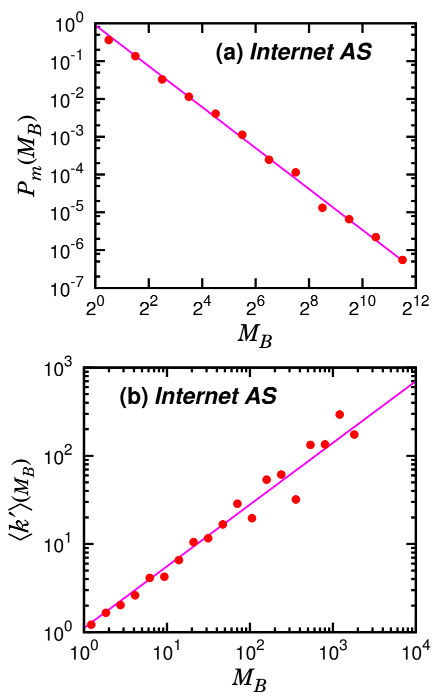


Fig. 6. (Color online) (a) Box-mass distribution and (b) average renormalized degree $\langle k' \rangle$ versus box mass $M_B$ of the Internet at the AS level. Here, the lateral box size is taken as $\ell_B = 2$ for both. The solid lines, drawn for reference, have slopes of (a) −1.8 and (b) 0.7.

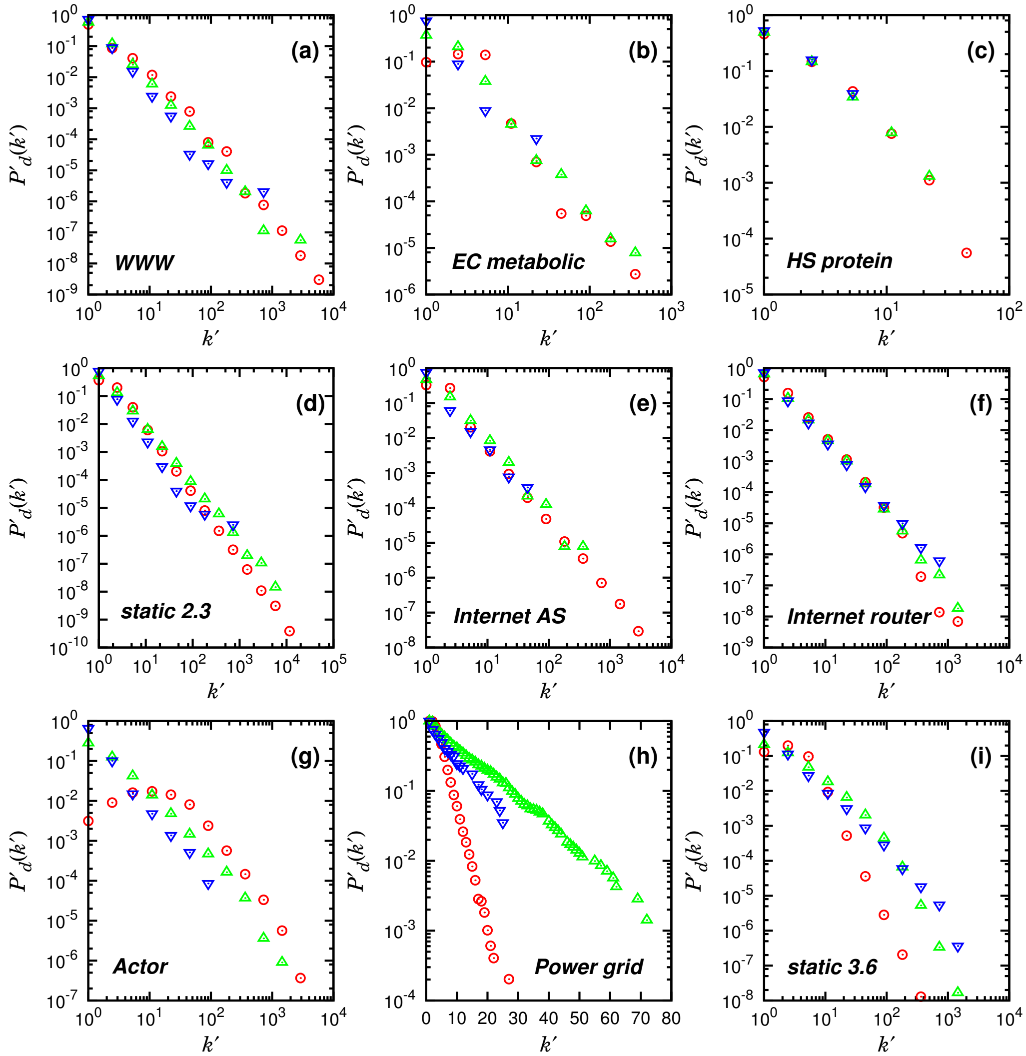


Fig. 7. (Color online) Renormalized degree distributions of real-world, (a)-(c) and (e)-(h), and model, (d) and (i), networks under successive CG transformations with a fixed box size $\ell_B = 2$. Symbols represent the original network (○), and the renormalized networks after the first (△) and the second (▽) iterations. The networks of (a)-(c) are fractals, and those of (d)-(i) are non-fractals. The networks of (a)-(f) are self-similar while those of (g)-(i) are not.

Using $P'_d(k')dk' \sim P_m(M_B)dM_B$ and $k' \sim M_B^{\theta}$, we obtain the degree exponent of the renormalized network to be $\gamma' = 1 + (\eta - 1)/\theta$. In short, we argue that Eq. (5) should be generalized to

$$\gamma' = \begin{cases} \gamma, & \text{for } \gamma \leq \eta, \\ 1 + (\eta - 1)/\theta, & \text{for } \gamma > \eta, \end{cases} \quad (7)$$

where $\theta \neq 1$. Accordingly, the self-similarity holds even for $\gamma > \eta$ when

$$\theta = (\eta - 1)/(\gamma - 1). \quad (8)$$

For $\ell_B = 2$, we found that $\eta \approx \gamma$ and $\theta \approx 1$; therefore, $\gamma' \approx \gamma$. For $\ell_B = 5$, even though $\theta \neq 1$, plugging $\theta \approx 0.5 \sim 0.6$ and $\eta \approx 1.8$ into Eq. (7), we obtain $\gamma' \approx 2.3 \sim 2.6$, which is in reasonable agreement with $\gamma \approx 2.3$. Thus, self-similarity also holds [Figure 4(b)]. We recall the previous study for the fractal model with $\gamma = 2.3$ [11], finding that the numerical results are the same as those of the WWW, as shown in Figures 4(c) and (d).

The self-similarity condition in Eq. (8) is also fulfilled for some non-fractal networks. The Internet at the autonomous system (AS) level is a prototypical non-fractal SF network. However, it exhibits self-similarity. We obtain $\eta \approx 1.8$ and the nonlinear relationship in Eq. (6) with $\theta \approx 0.7$, so that $\gamma \approx \gamma' \approx 2.1(1)$ for $\ell_B = 2$, as shown in Figure 6, satisfying the condition in Eq. (8).

A few deterministic SF networks have been introduced, which were constructed recursively starting from each basic structure. The pseudofractal model [15] introduced by Dorogovtsev *et al.*, the geometric fractal model introduced by Jung *et al.* [16], and the hierarchical model [5] are such examples. These models restore their shapes under the CG achieved by decimating the vertices with the smallest degree at each stage. Thus, they are self-similar in shape. In those methods, the degree distribution is scale invariant under the CG. However, they are not fractal because fractal scaling, Eq. (1), is absent and the fractal dimension cannot be defined. The three deterministic models [5,15,16] satisfy the self-similarity condition in a trivial manner: $\eta = \gamma$ and $\theta = 1$ under the decimation. Here, the group-mass distribution is measured as the distribution of the number of deleted vertices connected to each coarse-grained vertex at each decimation stage. Thus, $\gamma' = \gamma$, and the models are self-similar, even though they do not follow the fractal scaling, Eq. (1).

We also examine the change of the degree distributions under successive CG transformations for the fractal and some non-fractal networks in Figure 7. We confirm that the self-similarity holds even for the non-fractal networks (d)–(f). It is noteworthy that the relation in Eq. (6) is linear for the first renormalization, but it becomes nonlinear for the second renormalization as $\theta \approx 0.75$ and $\eta \approx 1.8$ for the WWW. It is also interesting to note that the static model with $\gamma = 2.3$ is self-similar in Figure 7(d) while that with $\gamma = 3.6$ in Figure 7(i) is not. That is due to the topological difference of the SF networks with $\gamma > 3$ and $2 < \gamma < 3$ [9,19]. When $\gamma$ is small, the edges are compactly concentrated around the hubs while as $\gamma$ grows the edges more globally interweave the network. Consequently, when renormalization is performed on the SF network with $2 < \gamma < 3$, only the nodes around each hub in the original network are grouped into a supernode in a coarse-grained network, and the supernode again becomes a hub with a corresponding size. On the other hand, when $\gamma > 3$, the nodes far from the hubs in the original network have more chances to be connected to hubs via global edges, and the supernodes in the coarse-grained network become far bigger hubs than those in the original network. The result is more heavily-tailed degree distributions in the coarse-grained networks, as seen in Figure 7(i).

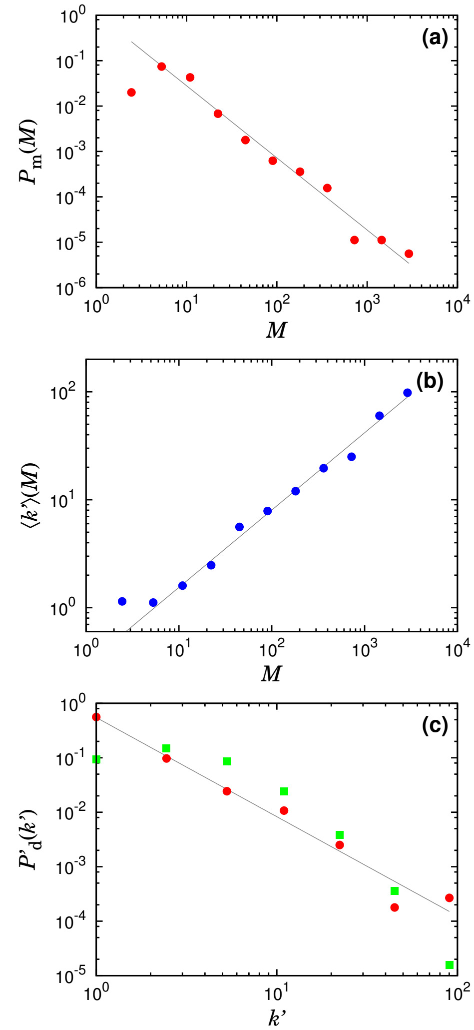


Fig. 8. (Color online) When the coauthorship network is clustered, (a) the box-mass distribution, (b) the average renormalized degree $\langle k' \rangle$ versus group mass $M$, and (c) the degree distribution of the original (■) and renormalized (●) networks are plotted. Groups are obtained by using the clustering algorithm of Ref. [23]. The solid lines, drawn for reference, have slopes of (a) $-1.6$, (b) 0.7 and (c) $-1.8$.

## IV. CG BY COMMUNITY STRUCTURE

Most SF networks in the real world contain functional modules or community structures within them, which are organized in a hierarchical manner [21, 22]. While the group-mass distributions are known to follow a power law in Eq. (2), for many cases, however, they exhibit a crossover between two distinct power-law behaviors or the power-law behavior appears only in a limited range of mass. Also, they are sensitive to various clustering algorithms [7]. Thus, it is not easy to find sufficiently good examples of clustered networks with appropriate clustering algorithms to test our argument.

Here, we choose the algorithm proposed by Clauset *et al.* [23] and apply it to the cond-mat coauthorship network [24]. The network data contain 13,861 vertices and 44,619 edges. Unfortunately, the degree distribution of this network is not a power law. Nonetheless, the data are clustered into 175 groups obtained at the point where the modularity becomes maximum in the clustering algorithm. The group-mass distribution is likely to follow the power law in Eq. (2), and the exponent is estimated to be $\eta \approx 1.6 \pm 0.2$ in Figure 8(a). Next, the CG is carried out; then, the degree distribution of the renormalized network is examined. It shows a power-law behavior with exponent $\gamma' \approx 1.8 \pm 0.2$. To check the formula of Eq. (7), we measure the relationship of Eq. (6) between the renormalized degree and the group mass in Figure 8(b). The exponent $\theta$ is measured to be $\theta \approx 0.7 \pm 0.1$. We plug the numerical values of $\eta$ and $\theta$ into the formula, and obtain $\gamma' \approx 1.9 \pm 0.4$, which is in reasonable agreement with the measured value $\gamma' \approx 1.8 \pm 0.2$ in Figure 8(c). The degree distribution of the original network is overall skewed, while that of the renormalized network follows the power law. Thus, self-similarity does not hold for this case.

## V. DISCUSSION AND CONCLUSIONS

For the fractal networks, Song *et al.* [8] showed that in the box-covering method, the renormalized degree $k'$ scales as $k' \sim s(\ell_B)k_m$, where $s(\ell_B) \sim \ell_B^{-d_k}$ with $d_k = d_B/(\gamma - 1)$ and $k_m$ being the largest degree in a given box. This form is not directly applicable to non-fractal networks, but a modified form of $k'_{\max} \sim (N'/N)^{1/(\gamma-1)}k_{\max}$ can be applied for non-fractal but self-similar SF networks, where $k'_{\max}$ ($k_{\max}$) is the largest degree in the entire renormalized (original) network and $N'(N)$ is the total number of nodes after (before) CG. The above relation can be easily derived by using the scaling of the natural cutoff of degree, $k_{\max} \sim N^{1/(\gamma-1)}$ and $k'_{\max} \sim N'^{1/(\gamma-1)}$. Thus, for SF networks, it is more general to formulate a scaling function in terms of the ratio $N'/N$ rather than the length scale $\ell_B$.

Although, in this paper, we limited the notion of self-similarity to the scale invariance of the degree distribution, one may wonder if other quantities, such as $C(k)$ and $\langle k_{\rm nn}\rangle(k)$, are scale invariant under the CG. We find that such quantities tend to obey the scale invariance for the WWW in the box-covering method, but the statistics from real-world networks are not sufficiently good to support this conclusion. That means, the self-similarity thus defined does not imply any recursive topological identity nor does it even guarantee that the degree distribution within each group is identical from group to group. On the other hand, the origin of fractality is understood by the power-law relation between the length scale and the size of the skeleton underlying the original network [9]. In such senses, it seems that the fractality describes an important topological feature of SF networks at a fundamental level while the self-similarity does not. Moreover, the results we obtained till now empirically show that all the fractal networks are self-similar, but the converse is not true. Thus, we conjecture that fractality implies self-similarity.

In summary, we have studied the renormalization-group transformation of the degree distribution, in particular, when the number of vertices within each block follows a power-law distribution, $P_m(M) \sim M^{-\eta}$. We found that the average renormalized degree scales with the box mass as $\langle k'\rangle(M_B) \sim M_B^{\theta}$. The two exponents $\eta$ and $\theta$ can be nontrivial as $\eta \neq \gamma$ and $\theta \neq 1$. They act as relevant parameters and are analogous to the scaling exponents associated with the magnetization and the singular part of the internal energy in the renormalization group theory. We obtained the degree exponent $\gamma'$ of a remormalized network in terms of $\eta$ and $\theta$. Many non-fractal networks are self-similar. The notions of fractality and self-similarity are disparate in SF networks, which is counterintuitive in view of their equivalence in Euclidean space.

## ACKNOWLEDGMENTS

This work was supported by a Korean Science and Engineering Foundation (KOSEF) grant funded by the Korean Ministry of Science and Tecnology (MOST) (No. R17-2007-073-01000-0). J. S. K. is supported by the Seoul Science Foundation. K.-I.G. is supported by a Korea University grant.

## REFERENCES

[1] L. P. Kadanoff, Physics **2**, 263 (1966).
[2] M. den Nijs, Physica A **251**, 52 (1998).
[3] J. Feder, *Fractals* (Plenum, New York, 1988).
[4] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[5] E. Ravasz and A.-L. Barabási, Phys. Rev. E **67**, 026112 (2003).
[6] M. E. J. Newman, Phys. Rev. E **69**, 066133 (2004).
[7] L. Danon, A. Díaz-Guilera, J. Duch and A. Arenas, J. Stat. Mech. P09008 (2005).
[8] C. Song, S. Havlin and H. A. Makse, Nature (London) **433**, 392 (2005).
[9] J. S. Kim, K.-I. Goh, G. Salvi, E. Oh, B. Kahng and D. Kim, Phys. Rev. E **75**, 016110 (2007).
[10] J. S. Kim, K.-I. Goh, B. Kahng and D. Kim, Chaos **17**, 026116 (2007).
[11] J. S. Kim, K.-I. Goh, B. Kahng and D. Kim, New J. Phys. **9**, 177 (2007).
[12] B. J. Kim, Phys. Rev. Lett. **93**, 168701 (2004).
[13] B. J. Kim, J. Korean Phys. Soc. **46**, 722 (2005).
[14] A. F. Rozenfeld, R. Cohen, D. Ben-Avraham and S. Havlin, Phys. Rev. Lett. **89**, 218701 (2002).
[15] S. N. Dorogovtsev, A. V. Goltsev and J. F. F. Mendes, Phys. Rev. E **65**, 066122 (2002).
[16] S. Jung, S. Kim and B. Kahng, Phys Rev. E **65**, 056101 (2002).
[17] K.-I. Goh, B. Kahng and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).
[18] K.-I. Goh, D.-S. Lee, B. Kahng and D. Kim, Phys. Rev. Lett. **91**, 148701 (2003).
[19] D.-H. Kim, J. D. Noh and H. Jeong, Phys. Rev. E **70**, 046126 (2004).
[20] K.-I. Goh, G. Salvi, B. Kahng and D. Kim, Phys. Rev. Lett. **96**, 018701 (2006).
[21] H. Y. Kim, H. Y. Kang, J. W. Rhy, C. N. Yoon and S. K. Han, J. Korean Phys. Soc. **50**, 290 (2007).
[22] J. W. Rhy, H. Y. Kim, T. H. Kang, J. S. Yoo and J. S. Chung, J. Korean Phys. Soc. **51**, 1805 (2007).
[23] A. Clauset, M. E. J. Newman and C. Moore, Phys. Rev. E **70**, 066111 (2004).
[24] M. E. J. Newman, Proc. Natl. Acad. Sci. U.S.A. **98**, 404 (2001).