Effective trapping of random walkers in complex networks

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Exploring the World Wide Web has become one of the key issues in information science, specifically in view of its application to the PageRank-like algorithms used in search engines. The random walk approach has been employed to study such a problem. The probability of return to the origin (RTO) of random walks is inversely related to how information can be accessed during random surfing. We find *analytically* that the RTO probability for a given starting node shows a crossover from a slow to a fast decay behavior with time and the crossover time increases with the degree of the starting node. We remark that the RTO probability becomes almost constant in the early-time regime as the degree exponent approaches two. This result indicates that a random surfer can be effectively trapped at the hub and supports the necessity of the random jump strategy empirically used in the Google's search engine.

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I. INTRODUCTION

In the information era, the number of Web pages increases drastically at a rate of approximately one million pages per day. Accordingly, it becomes challenging to find a page appropriate to one's interest in a short time. Thus it is desirable to develop more efficient PageRank-like algorithms. The PageRank algorithm currently used in the Google search engine [1–3] is based on the random walk (RW) approach. The PageRank algorithm assigns a score to each page, which is proportional to the number of visits to a given node by a random surfer as it continues to step on one of the neighbors indefinitely. Since the World Wide Web is a scale-free network [4], the physical properties of RWs in complex networks [5] can provide some ways to improve the PageRank algorithm.

In the Euclidean space and self-similar spaces, as RWers start from a certain node and travel to others randomly, the number of accessible sites $S(t)$ [6] in a random walk of t steps increases as

$$S(t) \sim t^{d_s/2}, \tag{1}$$

where d_s is the spectral dimension. The accessible sites are meant by the notion that the walker may find itself anywhere within the set of accessible sites after t steps. This quantity is inversely proportional to the probability of return to the origin (RTO), denoted as $R(t)$ [6–9]. That is,

$$R(t) \sim \frac{1}{S(t)} \sim t^{-d_s/2}. \tag{2}$$

When a RWer starts from one node s and travels randomly in complex networks, the RTO probability enables us to understand how much information is accessible by indefinitely clicking hyperlinks on pages via the relation (2). The RTO probability is defined conventionally as $R(t) \equiv \sum_s p_{ss}(t)/N$, where $p_{ss}(t)$ is the return probability of the RWer to the starting position s after t steps and N denotes the total number of nodes in the network. Whereas $R(t)$ and $p_{ss}(t)$ behave in the same manner in the Euclidean space, they can be different in scale-free networks. In this paper, we analytically show that when the node s has a large degree such as the hub, $p_{ss}(t)$ decays much slower than $R(t)$ in the small t regime, but behaves in the same manner as $R(t)$ in the large t regime. The crossover time increases with the degree of the starting node. Thus, random surfers may need a long time to escape from the hub.

The paper is organized as follows: In Sec. II, we introduce the effective network by RWers and find the RTO probability follows power laws with crossover behavior. In Sec. III, numerical results of the RTO probability are presented and compared with the analytical solutions. We summarize our findings and discuss their implications in Sec. IV.

II. EFFECTIVE NETWORK BY RANDOM WALKS

We begin by recalling the convention of an RW problem in uncorrelated scale-free networks. A network is composed of N nodes and L links, and the degree of each node follows a power-law distribution $D(k) \sim k^{-\gamma}$ for large k with the degree exponent γ. A RWer starts from a node s at time $t = 0$ and hops to a randomly selected neighbor node at each time step. The transition rate from node j to i is given as A_{ij}/k_j, where A_{ij} is the element of the adjacency matrix A and $k_j = \sum_i A_{ij}$.

The occupation probability $p_{is}(t)$ of the RWer starting from a node s at time $t = 0$ to find itself at a node i after t steps is given as

$$p_{is}(t) = \sum_{j \in \text{n.n.}(i)} \frac{1}{k_j} p_{js}(t-1), \tag{3}$$

where n.n.(i) is the set of the nearest neighbor nodes of i. It is well known that

$$p_{is}(t \to \infty) = \frac{k_i}{2L} \tag{4}$$

in scale-free networks [5]. Analog to this formalism, we express $p_{is}(t)$ at finite time steps in a similar fashion,

$$p_{is}(t) = \frac{\hat{k}_i(t)}{2\hat{L}(t)}, \tag{5}$$

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where $\hat{k}_i(t)$ and $\hat{L}(t)$, called the effective degree of node i and the effective number of links, respectively, are defined as follows:

$$\hat{k}_i(t) = \sum_{j\in \text{n.n.}(i)} W_{ij}(t), \tag{6}$$

where $W_{ij}(t)$, called the link accessibility of the RWs from node j to i [6], is

$$W_{ij}(t) \propto \frac{1}{k_j} p_{js}(t-1), \tag{7}$$

and

$$\hat{L}(t) = \frac{1}{2}\sum_i \hat{k}_i(t). \tag{8}$$

As the degree k_i is the sum of A_{ij}s over j, the effective degree $\hat{k}_i(t)$ is the sum of $W_{ij}(t)$ over j [Eq. (7)]. The effective degree varies dynamically as the RWs proceed and so is the effective number of links [Eq. (8)], which is analog to the formula $L = \sum_i k_i/2$. We remark that the link accessibility W_{ij} may differ from W_{ji} even in undirected networks. Meanwhile, Eq. (3) may be rewritten in terms of $r_{is} \equiv p_{is}/k_i$ as

$$r_{is}(t) = \frac{1}{k_i}\sum_{j\in \text{n.n.}(i)} r_{js}(t-1), \tag{9}$$

which is relaxed following the diffusion equation. Thus, we call $r_{ij}(t)$ the link-crossing probability of RWs.

In the following, we propose the proportional coefficient of $W_{ij}(t)$ hypothetically and present the heuristic argument behind it.

$$W_{ij}(t) \equiv \frac{\frac{1}{k_j} p_{js}(t-1)}{\left\langle \frac{1}{k_\ell} p_{\ell\ell}(t-2)\right\rangle_{\ell\in \text{n.n.}(s)}} \text{ for } j \in \text{n.n.}(i). \tag{10}$$

The denominator of Eq. (10) was chosen to make $W_{ij}(t)$ bounded in $0 < W_{ij}(t) < 1$ for any time t and any starting node s. The proposed denominator represents the average probability for an RWer starting from a node s and visiting its neighbor node ℓ at $t = 1$ to occupy the node s at time t via the node ℓ at time $t-1$. The average is taken over all neighbor nodes of s. This quantity can be interpreted as the average probability to cross a link connected to the starting node at time $t-1$, which was crossed at the first time step. We claim that this probability is the largest among those quantities crossing any other links at time $t-1$. Our claim is based on the simple argument that the link-crossing probability over a link connected to the starting node s is relaxed from the initial value $1/k_s$ to the saturated value $1/L$ by following the diffusion equation, because the denominator can be viewed as $\langle r_{s\ell}(t-1)\rangle_{\ell\in \text{n.n.}(s)}$ under the condition that the RWer visited a node ℓ at $t = 1$. On the contrary, those quantities of other links increase from zero to the saturated value $1/2L$ as RW time steps proceed. Thus, the link-crossing probability of the first-passed link is larger than any others at any time step. Figure 1 facilitates understanding of the time evolution of the RW link accessibility of each link schematically.

Using Eqs. (6), (8), and (10) and the derivation in Appendix A, we obtain the effective number of links as

$$\hat{L}(t) \simeq \frac{\langle k\rangle}{2R(t-2)}, \tag{11}$$

for which we also used the relation $\sum_i \frac{1}{k_j}\sum_{j\in \text{n.n.}(i)} p_{js}(t-1) = \sum_i p_{is}(t) = 1$. Using the relation $R(t) \sim t^{-d_s/2}$ in the short-time regime and $R(t) = \langle k\rangle/2L$ in the long-time regime [5], we obtain the crossover behavior,

$$\hat{L}(t) \sim \begin{cases} t^{\frac{d_s}{2}} & \text{for } t \ll t_\text{x}, \\ L & \text{for } t \gg t_\text{x}, \end{cases} \tag{12}$$

where t_x scales as $t_\text{x} \sim L^{2/d_s}$ at which $\hat{L}(t_\text{x}) \simeq L$.

Next, we concern $\hat{k}_i(t)$ in the numerator of Eq. (10), particularly, for the case $i = s$, because we eventually want to obtain the probability of return to the starting position $p_{ss}(t)$. To obtain $\hat{k}_s(t)$, we need to know $W_{s\ell}(t)$ for $\ell \in \text{n.n}(s)$. It is obvious that $W_{s\ell}(t=2) \sim 1/k_s$ and $W_{s\ell}(t\to\infty) \approx \text{const.}$ In the early-time regime, we show in Appendix B that $W_{s\ell}$ behaves as

$$W_{s\ell}(t) \simeq \frac{\kappa_\ell(t)}{k_s} + o\left(\frac{1}{k_s}\right), \tag{13}$$

where $\kappa_\ell(t)$ is a function independent of the starting node s. Thus, the effective degree of the node s is simply expressed as

$$\hat{k}_s(t) \simeq \frac{1}{k_s}\sum_\ell \kappa_\ell(t) = \langle\kappa_\ell\rangle_\ell \equiv \kappa(t), \tag{14}$$

which is independent of s within the leading order in the early-time regime. We summarize the behavior of the effective degree $\hat{k}_s(t)$ for different time regimes as (i) $\hat{k}_s(t) \simeq \kappa(t)$ in the early-time regime, and (ii, iii) $\hat{k}_s(t) = k_s$ in the intermediate- and the long-time regimes. This result is confirmed numerically and shown in Fig. 2.

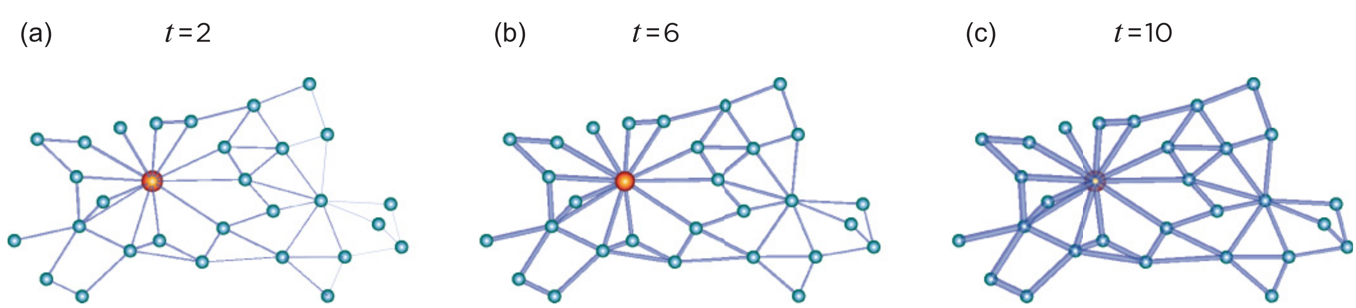

FIG. 1. (Color online) Link accessibility of random walks starting from a node (open circle) in short (a), intermediate (b), and long (c) time regimes, which is represented by the thickness of each link.

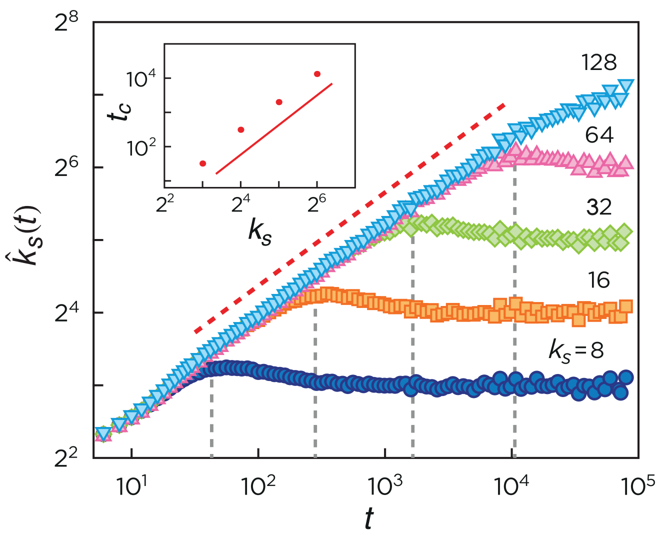

FIG. 2. (Color online) Effective degree $\hat{k}_s(t)$ as a function of time t for different k_s in the (2,3) weighted flower network composed of 11 720 nodes. The slope of the dashed line is $d_s/[2(\gamma-1)] = \ln 2/\ln 6$ [11]. The inset shows the plot of the estimated crossover time $t_c(s)$ versus k_s in double logarithmic scales, and the slope of the solid line is $2(\gamma-1)/d_s = \log 6/\log 2$.

To obtain $\kappa(t)$ and eventually $p_{ss}(t)$, we consider the effective degree distribution in the RWs starting from a node s. We first recall that in the limit $t \to \infty$, the link accessibilities of each link is constant, and thus the effective degree $\hat{k}_i(t)$ of node i reduces to its degree, the number of connections to it. Moreover, the probability to reach a node with degree k following a randomly selected link, which is denoted as $D_{\rm LN}(k)$, is given as $(k/\langle k\rangle)D(k)$. In the early time regime, however, a RWer does not visit all nodes in the system, but does some nodes around a starting node. As a result, the link accessibility of each link is no longer constant $1/L$, but can vary with time. For simplicity, we assume that the link accessibilities are uniform for the links that have ever been passed by the RWer, and thus the probability to reach a node with degree k following a link becomes $k^{1-\gamma}$ for the nodes ever visited. This assumption, a simplest way to study the time-dependent behavior of RWs, is made on the basis that the wandering pattern of RWs within the region of the links ever passed is similar to the one over the entire system in the limit $t \to \infty$. However, since the number of passed links is partial and increases with time, the proportional coefficient of $D_{\rm LN}(k;t)$ can differ from that in the limit $t \to \infty$. We consider a particular case that an RWer starts from the hub node (i.e., $s = h$), for which the maximum effective degree or the cutoff of $D_{\rm LN}(k;t)$ is the effective degree of the starting node, which is denoted as $\hat{k}_h(t)$. The natural cutoff $\hat{k}_h(t)$ of $D_{\rm LN}(k;t)$ is obtained from the relation $\int_{\hat{k}_h(t)}^{\infty} D_{\rm LN}(k;t)dk \sim \hat{k}_h(t)^{2-\gamma}$. Meanwhile, the effective degree of the hub is given as $\hat{k}_h(t)/\hat{L}(t)$. Matching those quantities, one can obtain that

$$\hat{k}_h(t) \sim \hat{L}(t)^{1/(\gamma-1)} \sim \begin{cases} t^{d_s/2(\gamma-1)} & \text{for } t \ll t_{\rm x}, \\ L^{1/(\gamma-1)} & \text{for } t \gg t_{\rm x}. \end{cases} \quad (15)$$

Both $\hat{k}_h(t)$ and $\hat{L}(t)$ increase with time and saturate to their values k_h and L at $t \simeq t_{\rm x}$, respectively. The RTO probability for the hub behaves as

$$p_{hh}(t) = \frac{\hat{k}_h(t)}{2\hat{L}(t)} \sim \begin{cases} t^{-d_s^{\rm (hub)}/2} & \text{for } t \ll t_{\rm x}, \\ \frac{k_h}{2L} & \text{for } t \gg t_{\rm x}, \end{cases} \quad (16)$$

where

$$d_s^{\rm (hub)} = d_s \frac{\gamma-2}{\gamma-1} \quad (17)$$

is called the local spectral dimension of the hub. When d_s and γ are finite, $d_s^{\rm (hub)} < d_s$, and thus the RWer wanders around the hub for a long time. Particularly, when $\gamma \to 2$, $p_{hh}(t) \sim$ const., implying that the RWer is effectively trapped at the hub.

Let us return to the general case of an arbitrary starting node. We know that the effective degree of the starting node s evolves with time as $\hat{k}_s(t) \sim \kappa(t)$, independent of s, until it reaches the value k_s. Therefore the behavior of $\hat{k}_s(t)$ for $t \ll t_{\rm x}$ reduces to the one similar to Eq. (15) as

$$\hat{k}_s(t) \sim \begin{cases} t^{d_s/2(\gamma-1)} & \text{for } t \ll t_c(s), \\ k_s & \text{for } t \gg t_c(s), \end{cases} \quad (18)$$

where $t_c(s)$ is the crossover time between the early- and intermediate-time regime, which depends on k_s as

$$t_c(s) \sim k_s^{2(\gamma-1)/d_s}. \quad (19)$$

From Eqs. (5), (12), and (18), we obtain the RTO probability $p_{ss}(t)$ for an arbitrary starting node s as

$$p_{ss}(t) \sim \begin{cases} t^{-d_s^{\rm (hub)}/2} & \text{for } t \ll t_c(s), \\ k_s t^{-d_s/2} & \text{for } t_c(s) \ll t \ll t_{\rm x}, \\ \frac{k_s}{2L} & \text{for } t \gg t_{\rm x}. \end{cases} \quad (20)$$

The intermediate time regime, $t_c(s) \ll t \ll t_{\rm x}$, disappears when the starting node is the hub; the crossover time t_c reduces to $t_{\rm x}$ when $k_s = k_h \sim L^{1/(\gamma-1)}$.

III. SIMULATION RESULTS

We check the analytic solution numerically in artificial scale-free networks as well as a real-world network, the World Wide Web [12]. The artificial networks are the weighted flower network [10,11,13,14] and the fractal network introduced by Song *et al.* [15]. In both networks, the RTO probability decays in a power-law manner $R(t) \sim t^{-d_s/2}$. In the weighted flower networks, the presence of shortcuts is controlled by a parameter $p = 0$ so that the network can be either a fractal or a nonfractal [11]. In the fractal network model, there are two parameters m and e which represent the branching number of each step and the hub-hub attraction probability, respectively. Those two parameters control the global spectral dimension and the degree exponent. In our simulations, a RWer starts at a node s and its trajectories are recorded up to 1000 time steps to evaluate the specific RTO probability. This simulation is repeated for all starting nodes and 10^6 independent RWers.

Figure 2 shows that the effective degree increases with time in the early time regime, and saturates to a constant value in the intermediate- and the long-time regime in the weighted flower networks. The theoretical prediction in the early time regime in Eq. (18) is represented by the dashed line, which is

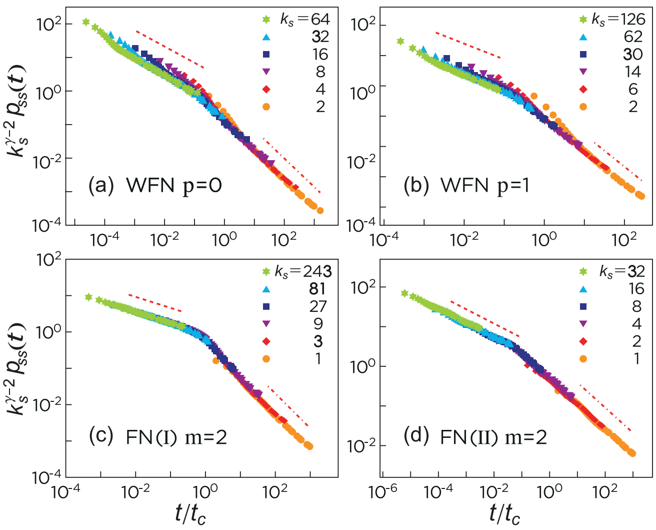

FIG. 3. (Color online) The RTO probabilities as a function of time in the scaling form for the (3,5) weighted flower networks (WFNs) with the long-range connection probability $p = 0$ (a) and $p = 1$ (b) and the fractal network (FN) with $m = 2$ and $e = 1$ (c), with $m = 2$ and $e = 0$ (d), where m represents the branching number of each step, and e does the hub-hub attraction probability [15]. The weighted flower networks in (a) are fractal and those in (b) are nonfractal. The networks with $e = 1 (e = 0)$ are fractals (not fractals). The slopes of dashed lines and the dashed-dotted lines are guidelines theoretically predicted.

in agreement with numerical data. The crossover time $t_c(s)$ is estimated for different k_s and are shown in the inset of Fig. 2. Again numerical data fit the theoretical prediction well.

The crossover behavior of $p_{ss}(t)$ between the early- and the intermediate-time regime can be described as $p_{ss}(t) = k_s^{2-\gamma}\phi(t/t_c(s))$ with the scaling function $\phi(x)$ behaving as $x^{-d_s^{(\mathrm{hub})}/2}$ for $x \ll 1$ and $x^{-d_s/2}$ for $x \gg 1$. In Figs. 3(a)–3(d), the plots of $k_s^{\gamma-2}p_{ss}(t)$ versus $t/k_s^{2(\gamma-1)/d_s}$ show data collapse excellently for different k_s, conforming the theoretical prediction. The data in the long-time regime $t \gg t_X$ are not presented in Fig. 3, which have been already well understood and are not the main concern of this work.

For a real-world network, we simulate the RWs for the World Wide Web, which has the degree exponent $\gamma \approx 2.2$ and the spectral dimension $d_s \approx 1.8$. In our simulations, we neglect the direction of each link for simplicity. We plot the numerical results of $p_{ss}(t)$ in the scaling form and find that the numerical data do show the crossover behavior; the behavior in the early- and the intermediate-time regimes fit the theoretical predictions reasonably well, represented by the dashed and dashed-dotted lines, respectively, in Fig. 4. Since the World Wide Web contains some degree-degree correlation, the scaling plot is not as good as that obtained for artificial networks. Nevertheless, the slow decay $\sim t^{-0.16}$ behavior of the specific RTO probability in the early-time regime implies that a random surfer on the World Wide Web is effectively trapped at hub pages. This result may be related to why the PageRank algorithm needs to include random jumps for efficient searching.

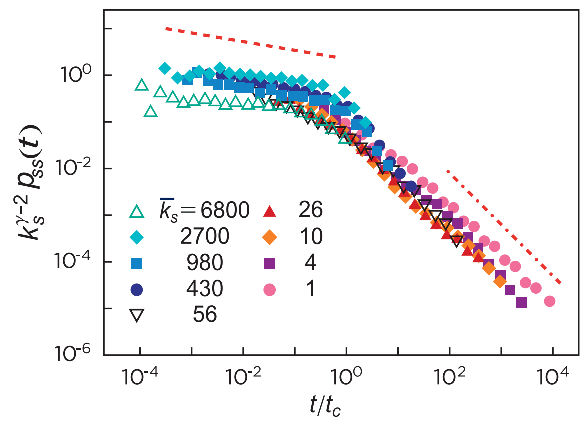

FIG. 4. (Color online) The RTO probabilities in the scaling form for the World Wide Web. The dashed and dashed-dotted lines represent theoretical formula $p_{ss}(t) \sim t^{-d_s^{(\mathrm{hub})}/2}$ and $p_{ss}(t) \sim t^{-d_s/2}$ with $d_s^{(\mathrm{hub})} \approx 0.33$ evaluated by Eq. (17) using the measured values $\gamma \approx 2.2$ and $d_s \approx 1.8$.

IV. DISCUSSION

In summary, we have studied the time-dependent behavior of the RTO probability of RWs in scale-free networks in relation to information accessibility during random surfing in the World Wide Web. It was found that the specific RTO probability $p_{ss}(t)$ exhibited the crossover between a slow and a fast decay behavior. The crossover time increases with the degree of the starting node. Thus, an RWer starting from a hub takes a long time to escape from it. This result implies that it is undesirable for a random surfer to start from a portal site in the World Wide Web that contains a great number of hyperlinked pages, and random jumps are needed to escape from it when the RWer reaches there during travels.

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APPENDIX A: DERIVATION OF THE DENOMINATOR OF EQ. (10)

To calculate the denominator of Eq. (10) explicitly, we first present a general framework how to calculate the average of a general function $f(\ell)$ in which ℓ is the node index, but actually the function depends on its degree k_ℓ in the form of $\hat{f}(k_\ell)$. Then,

$$\langle f(\ell)\rangle_{\ell\in \mathrm{n.n.}(s)} \equiv \frac{1}{k_s}\sum_{\ell\in \mathrm{n.n.}(s)} f(\ell) \tag{A1}$$

$$\approx \sum_k \frac{k}{\langle k\rangle} D(k)\hat{f}(k). \tag{A2}$$

The step from (A1) to (A2) is obtained by using $D_{\mathrm{LN}}(k)$ which is given as $\frac{k}{\langle k\rangle}D(k)$. We also remark that the degrees of neighbor nodes $\ell \in$ n.n.(s) are a randomly composed subset of $D_{\rm LN}(k)$, and if their numbers are large, one may regard their distribution as the same as $D_{\rm LN}(k)$. Using the above formalism, we calculate the denominator as follows:

$$\left\langle \frac{1}{k_\ell} p_{\ell\ell}(t-2) \right\rangle_{\ell\in{\rm n.n.}(s)} \approx \sum_k \frac{k}{\langle k\rangle} D(k) \frac{\hat{p}_k(t-2)}{k} = \frac{1}{\langle k\rangle}\sum_k D(k)\hat{p}_k(t-2) = \frac{1}{\langle k\rangle} R(t-2), \quad \text{(A3)}$$

where we denote $\hat{p}_{k_\ell}(t) \equiv p_{\ell\ell}(t)$.

APPENDIX B: LINK ACCESSIBILITY IN THE EARLY-TIME REGIME

To explore $W_{s\ell}(t)$ in the early-time regime, we decompose the occupation probability $p_{\ell s}(t-1)$ in Eq. (10) into two parts as

$$p_{\ell s}(t-1) = p_{\ell\ell}(t-2)p_{\ell s}(1) + \sum_{\substack{m\in{\rm n.n.}(s)\\ m\neq\ell}} p_{\ell m}(t-2)p_{ms}(1). \quad \text{(B1)}$$

The RWer at node $m \neq \ell$ can reach the node ℓ, which is also a neighbor of s, mainly via the starting node s. This leads to $p_{\ell m}(t-2) \simeq (1/k_s)q_{\ell m}(t-3)$ within the leading order of $(1/k_s)$, in which $q_{\ell m}(t-3)$ is regarded as being independent of k_s. For instance, $p_{\ell m}(2) = 1/(k_s k_m)$ and $q_{\ell m}(1) = 1/k_m$. On the other hand, $p_{\ell\ell}(t-2)$ is not proportional to $1/k_s$ because the RWer does not necessarily pass through the starting node to return to ℓ. Therefore, by using $p_{\ell s}(1) = 1/k_s$ for $\ell \in$ n.n.(s), we find that

$$W_{s\ell}(t) = \frac{\kappa_\ell(t)}{k_s} + o\left(\frac{1}{k_s}\right), \quad \text{(B2)}$$

with

$$\kappa_\ell(t) \simeq \frac{\frac{1}{k_\ell}\left(p_{\ell\ell}(t-2) + \frac{1}{k_s}\sum_{\substack{m\in{\rm n.n.}(s)\\ m\neq\ell}} q_{\ell m}(t-3)\right)}{\left\langle \frac{1}{k_{\ell'}} p_{\ell'\ell'}(t-2)\right\rangle_{\ell'\in{\rm n.n.}(s)}}. \quad \text{(B3)}$$

This argument is relevant when the network structure is locally treelike, since the RWer starting from m cannot reach ℓ without visiting s in the tree network. The treelike structure actually appears in a random graph, because the probability to form a link between ℓ and m is $k_\ell k_m/2L$, which is as small as $\langle k\rangle/N \ll 1$. As t increases, the difference between $p_{\ell\ell}(t-2)$ and $p_{\ell m}(t-2)$ becomes reduced, since the RWer can exploit various pathways from m to ℓ. In the stationary state, $p_{\ell m}(t-2)$ does not depend on m whether m is equal to ℓ or not, leading to $W_{s\ell} = 1$.

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