Sandpile avalanche dynamics on scale-free networks

D.-S. Lee, K.-I. Goh, B. Kahng and D. Kim

School of Physics and Center for Theoretical Physics, Seoul National University NS50, Seoul 151-747, Korea

Abstract

Avalanche dynamics is an indispensable feature of complex systems. Here we study the self-organized critical dynamics of avalanches on scale-free networks with degree exponent γ through the Bak-Tang-Wiesenfeld (BTW) sandpile model. The threshold height of a node i is set as $k_i^{1-\eta}$ with $0 \le \eta < 1$, where k_i is the degree of node i. Using the branching process approach, we obtain the avalanche size and the duration distribution of sand toppling, which follow power-laws with exponents τ and δ, respectively. They are given as $\tau = (\gamma-2\eta)/(\gamma-1-\eta)$ and $\delta = (\gamma-1-\eta)/(\gamma-2)$ for $\gamma < 3-\eta$, $3/2$ and 2 for $\gamma > 3-\eta$, respectively. The power-law distributions are modified by a logarithmic correction at $\gamma = 3-\eta$.

Key words: avalanche, scale-free network, branching process
PACS: 89.70.+c, 89.75.-k, 05.10.-a

1 Introduction

Frequently, complex systems in nature as well as in human society suffer massive catastrophes triggered from only a small fraction of their constituents. Unexpected epidemic spread of diseases and the power outage in the eastern US of the last year are the examples of such avalanche phenomena. Such a cascading dynamics is not always harmful to us. The information cascades making popular hits of books, movies, and albums are good to writers, actors, and singers, respectively. Thus it is interesting to understand and predict how those cascades propagate in complex system. Recently, the network approach, by which a system is viewed as a network consisting of nodes representing its constituents and links interactions between them, simplifies complicated details of complex systems. Such a simplification unveils a hidden order such as scale-free behavior in the degree distribution. Here degree is the number of links connected to a certain node. The Internet at the autonomous system level, the World-Wide Web, social acquaintance networks, biological networks, and other many complex networks exhibit power-law degree distributions, $p_d(k) \sim k^{-\gamma}$. The networks following such power-law degree distributions are called scale-free (SF) networks [1], where non-negligible fractions of hubs, the nodes with extraneously large degrees, exist.

In this paper, we investigate the avalanche dynamics on such SF networks through the Bak-Tang-Wiesenfeld (BTW) sandpile model [2], a prototypical model exhibiting self-organized criticality (SOC). The study of sandpile dynamics has been carried out mostly on regular lattices in the Euclidean space. In the stationary state, which can be reached without tuning a parameter, the system exhibit scale-invariant features in the power-law form of the avalanche size distribution $p_a(s)$ and the duration or lifetime distribution $\ell(t)$ as

$$p_a(s) \sim s^{-\tau} \quad \text{and} \quad \ell(t) \sim t^{-\delta}. \tag{1}$$

Recently, Bonabeau has studied the sandpile dynamics on the Erdős-Rényi (ER) random networks [3] and found that the avalanche size distribution follows a power law with the exponent $\tau \simeq 1.5$, consistent with the mean-field solution [4]. Recently, Lise and Paczuski [5] studied the Olami-Feder-Christensen model [6] on regular ER networks, where degree of each node is uniform but connections are random. They found the exponent to be $\tau \approx 1.65$. However, when degree of each node is not uniform, they found no criticality in the avalanche size distribution. Note that they assumed that the threshold of each node is uniform, whereas degree is not. Here we study the BTW sandpile model on SF networks, where the threshold z_i of the node i is given as $k_i^{1-\eta}$ with k_i the degree of i and $0 \le \eta < 1$. We find that the exponents for the avalanche size and the duration distribution depend on the degree exponent γ as $\tau = (\gamma - 2\eta)/(\gamma - 1 - \eta)$ and $\delta = (\gamma - 1 - \eta)/(\gamma - 2)$ for $\gamma < 3 - \eta$ while, for $\gamma > 3 - \eta$, they show the same behaviors as the conventional mean-field solutions as observed for the ER random networks.

2 Sandpile model

We present the dynamic rule of the BTW sandpile model on a given network.

(1) Each node i is given a prescribed threshold z_i $(\le k_i)$. The smallest integer not smaller than z_i is denoted as $\lceil z_i \rceil$ ($\lceil z_i \rceil \le k_i$).
(2) At each time step, a grain is added at a randomly chosen node i. The integer-valued height of the node i, h_i, increases by 1.
(3) If the height at the node i reaches or exceeds z_i, then it becomes unstable and the $\lceil z_i \rceil$ grains at the node topple to its $\lceil z_i \rceil$ randomly chosen adjacent nodes among k_i ones;
$h_i \to h_i - \lceil z_i \rceil$, and $h_j = h_j + 1$ for all nodes j which are chosen.
(4) If this toppling causes any of the adjacent nodes receiving grains to be unstable, subsequent topplings follow on those nodes in parallel until there is no

unstable node left. This process defines an avalanche.

(5) Repeat (2)–(4).

Here the threshold z_i of node i is given as

$$z_i = k_i^{1-\eta} \ (0 \leq \eta < 1), \tag{2}$$

which is a generalization of $z_i = k_i$ previously investigated in Ref. [7]. We concentrate on the distributions of (i) the avalanche area A, *i.e.*, the number of distinct nodes involving in a given avalanche, (ii) the avalanche size S, *i.e.*, the number of toppling events in a given avalanche, and (iii) the duration T of a given avalanche.

3 Branching process approach

The mapping of each avalanche to a tree provides a useful way of understanding the statistics of avalanche dynamics analytically. For each avalanche event, one can draw a corresponding tree: The node where an avalanche is triggered corresponds to the originator of the tree and the following nodes to descendants. In the tree structure, a descendant born at time t is located away from the originator by distance t along the shortest pathway. The tree stops to grow when no further avalanche proceeds. Then the ensemble of avalanches can be identified with that of trees grown through the branching process. In this mapping, the avalanche duration T is equal to the lifetime of the tree minus one, and the avalanche size S differs from the tree size only by the number of boundary nodes of the tree, which is relatively small when the overall tree size is very large. If one assumes that branching events at different nodes occur independently and that there is no loop in the tree, the tree size and lifetime distribution can be obtained analytically [8,9]. Those distributions are expected to share the same asymptotic behaviors with the avalanche size and duration distribution, respectively, due to the near-equivalence between an avalanche and its corresponding tree in their scales as mentioned above.

In the branching process describing an avalanche, after initial branching into k descendants with probability $q_0(k)$, successive branchings are assumed to occur independently with probability $q(k)$. $q_0(k)$ and $q(k)$ may be different in general, but the statistics of the overall size and duration of an avalanche is determined dominantly by $q(k)$. We checked also numerically the case where a new grain is added to a node with the probability proportional to the degree of that node, which gives different $q_0(k)$ from that in the case where a new grain is added randomly, and found that the nature of the avalanche dynamics is the same in both cases. Thus, for simplicity, we consider the branching process where every branching occur with probability $q(k)$. For the BTW model in the Euclidean space, where the threshold z_i of node i is equal to its degree k_i, $q(k)$ has a finite cut-off such that $q(k) = 0$ for $k > z_i = \text{const}$, because the degree of each node is uniform and finite. Consequently, the exponents of the avalanche size and the duration distributions in Eq. (1) come out to be the so-called mean-field values; $\tau = 3/2$ and $\delta = 2$ [8,9]. These results are known to hold for the BTW model on regular lattices with dimensions larger than 4 [4]. Note that when dimension is smaller than 4, the branching process approach cannot be applied, so that the values of the exponents τ and δ would not be trivial.

In SF networks, avalanches usually do not form loops, generating tree-structures: According to the numerical simulations of the BTW model for the case of $z_i = k_i$ on SF networks [7], the statistics of the two quantities A and S are nearly equal when they are large: For example, the maximum area and size ($A_{\max}$, $S_{\max}$) among avalanches are (5127, 5128), (12058, 12059) and (19692, 19692) for scale-free networks with $\gamma = 2.01$, 3.0, and ∞, respectively. The fact that A and S are almost the same implies that the avalanche structure can be treated as a tree. From now on, we shall not distinguish A and S, denoted by s. Thus it is valid to use the branching process approach to understand the avalanche dynamics on SF networks.

We study the BTW model on SF networks with the degree exponent γ and the threshold given as Eq. (2). The branching probability $q(k)$ consists of two factors, that is, $q(k) = q_1(k)q_2(k)$, where $q_1(k)$ is the probability that the threshold z_i of node i is in the range $k-1 < z_i \leq k$ and $q_2(k)$ is the probability that the total number of grains at the node reaches or exceeds the threshold. If $z_i = f(k_i)$ with $f(x)$ a monotonic increasing function of x satisfying $f(x) \leq x$ for all $x \geq 1$, the condition of $k-1 < z_i \leq k$ implies that $q_1(k)$ is nothing but the probability that a node i connected to the one end of a randomly chosen edge has its degree k_i in the region $(f^{-1}(k-1), f^{-1}(k)]$, and thus $q_1(k) = \sum_{k'=\lfloor f^{-1}(k-1)\rfloor+1}^{\lfloor f^{-1}(k)\rfloor} k' p_d(k')/\langle k\rangle$, where $\lfloor x\rfloor$ is the largest integer not larger than x. Notice that $\sum_{k=1}^{\infty} q_1(k) = 1$ and $q_1(k) \sim k^{(1-\gamma+\eta)/(1-\eta)}$ for large k if $f(x) \simeq x^{1-\eta}$ $(0 \leq \eta < 1)$ for large x. $q_2(k)$ is the probability that the node i has height $k-1$ at the moment of receiving a grain from one of its neighbors. We have checked numerically that a typical height of node is absent, so that all possible k values $0, 1, \ldots, k-1$ are equally likely [10]. Thus we set $q_2(k) = 1/k$. As a result, the branching probability $q(k)$ for large k is given asymptotically as

$$q(k) = \frac{1}{k} q_1(k) \sim k^{-\gamma'} \quad \left(\gamma' = \frac{\gamma - 2\eta}{1-\eta}\right). \tag{3}$$

When $z_i = k_i$ or $\eta = 0$, γ' is reduced to γ. Since we are interested in the case of $\gamma > 2$ and $0 \leq \eta < 1$, $\gamma' > 0$.

Using the independence of the branchings from different parent-nodes, one can derive the following self-consistent relation for the tree size distribution $p(s)$ as [8,9]

$$p(s) = q(0)\delta_{s,1} + \sum_{k=1}^{\infty} q(k) \sum_{s_1=1}^{\infty} \sum_{s_2=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} p(s_1)p(s_2)\ldots p(s_k)\delta_{\sum_{i=1}^{k} s_i, s-1}. \tag{4}$$

This relation can be written in a more compact form by introducing the generating functions, $\mathcal{P}(y) = \sum_{s=1}^{\infty} p(s) y^s$ and $\mathcal{Q}(\omega) = \sum_{k=0}^{\infty} q(k)\omega^k$ as

$$\mathcal{P}(y) = y\, \mathcal{Q}(\mathcal{P}(y)). \tag{5}$$

Then $\omega = \mathcal{P}(y)$ is obtained by inverting $y = \mathcal{P}^{-1}(\omega) = \omega/\mathcal{Q}(\omega)$.

The average size $\langle s \rangle$ of a finite tree can be obtained easily from the generating functions.

$$\langle s \rangle = \sum_{s=1}^{\text{finite}} sp(s) = \mathcal{P}'(1), \tag{6}$$

where $\mathcal{P}'(y) = d\mathcal{P}(y)/dy$. Using the relation, Eq. (5), we obtain

$$\langle s \rangle = \mathcal{P}'(1) = \frac{\mathcal{Q}(\mathcal{P}(1))}{1 - \mathcal{Q}'(\mathcal{P}(1))}, \tag{7}$$

where again $\mathcal{Q}'(\omega) = d\mathcal{Q}(\omega)/d\omega$.

The distribution of duration, *i.e.*, the lifetime of the tree can be evaluated similarly [8,9]. Let $r(t)$ be the probability that a branching process stops at or prior to time t. Then following the similar steps leading to Eq. (4), *i.e.*, $r(t) = \sum_{k=0}^{\infty} q_k [r(t-1)]^k$, one has

$$r(t) = \mathcal{Q}(r(t-1)). \tag{8}$$

For large t, $r(t)$ comes close to 1. One can obtain $\omega = r(t-1)$ by solving $d\omega/dt \simeq r(t) - r(t-1) = Q(\omega) - \omega$. Then the lifetime distribution $\ell(t)$ is obtained through $\ell(t) = r(t) - r(t-1) \simeq d\omega/dt$.

4 Avalanche size and duration distribution

The growth of a tree depends on the average number of branches defined as

$$C = \sum_{k=1}^{\infty} kq(k). \tag{9}$$

When $C > 1$ ($C < 1$), a tree can (cannot) grow infinitely in a probabilistic sense. Thus the case of $C = 1$ is a critical point for the growth of a tree. One can see that for any branching process with $q(k) = (1/k)q_1(k)$ ($k \geq 1$) and $\sum_{k=1}^{\infty} q_1(k) = 1$, the average number of branches C is always 1, independent of detailed structural properties of networks. Therefore our assumption $q_2(k) = 1/k$ corresponds to the condition for the self-organized criticality (SOC) of the sandpile model.

The inverse function $\mathcal{P}^{-1}(\omega)$ satisfies $\mathcal{P}^{-1}(1) = 1$. When $C = 1$, the first-order derivative $\partial \mathcal{P}^{-1}(\omega)/\partial\omega$ at $\omega = 1$ is zero and thus $\mathcal{P}(y)$ becomes singular at $y = 1$. $\mathcal{P}(y)$ is expanded around $y = 1$ as $\mathcal{P}(y) \simeq 1 - b(1-y)^{\phi}$ with constant b and $0 < \phi <$

1. Then the asymptotic behavior of the avalanche size distribution $p(s)$ for large s is given by $p(s) \sim s^{-\phi-1}$, because if a series $\sum_{s=0}^{\infty} a_s y^s$ with the radius of convergence 1 has the asymptotic behavior

$$\sum_{s=0}^{\infty} a_s y^s \sim (1-y)^{\phi} \text{ as } y \to 1, \text{ then } a_s \sim s^{-\phi-1} \text{ as } s \to \infty. \tag{10}$$

The functional form of the branching probability $q(k)$ determines the singularity of $\mathcal{P}(y)$. To illustrate this, we first consider a simple case that

$$q(k) = \begin{cases} 1-a & (k=0), \\ a & (k=2), \\ 0 & (\text{otherwise}), \end{cases} \tag{11}$$

where $0 < a < 1$. Then the average number of branches $C = \sum kq(k) = 2a$ and the generating function $\mathcal{Q}(\omega) = \sum_{k=0}^{\infty} q(k)\omega^k = 1-a+a\omega^2$. Using the relations of $y = \omega/\mathcal{Q}(\omega)$ and $\omega = \mathcal{P}(y)$, it is obtained that

$$\mathcal{P}(y) = \frac{1-\sqrt{1-4a(1-a)y^2}}{2ay}. \tag{12}$$

The value of $\mathcal{P}(1) = \sum_{s=1}^{\text{finite}} p(s)$ is given as

$$\mathcal{P}(1) = \frac{1-|1-2a|}{2a} = \begin{cases} 1 & \text{for } 0 < a \le \frac{1}{2} \quad (C \le 1), \\ \frac{1-a}{a} & \text{for } \frac{1}{2} < a < 1 \quad (C > 1), \end{cases} \tag{13}$$

which means that when $1/2 < a < 1$ ($C > 1$), a tree can grow infinitely with probability $1 - \mathcal{P}(1) = (2a-1)/a$, and the critical point is located at $a_c = 1/2$. Near $y = 1$, $\mathcal{P}(y) \approx 1 - \sqrt{2(1-y)}$ from Eq. (12), leading to $\phi = 1/2$. Then, the avalanche size distribution $p(s)$ behaves as $p(s) \sim s^{-3/2}$. On the other hand, using Eq. (7),

$$\langle s \rangle = \mathcal{P}'(1) = \begin{cases} \frac{1}{2(a_c-a)} & (a < a_c), \\ \frac{1-a}{2a(a-a_c)} & (a > a_c). \end{cases} \tag{14}$$

Even for the case that $q(k)$ has a finite cut-off larger than 2 or decays exponentially, the above result holds. This is the conventional mean-field solution for the avalanche size distribution [4,8,9] and has been shown to hold for the BTW model on the ER random networks [3].

When $q(k)$ decays slowly as in Eq. (3), however, its generating function $\mathcal{Q}(\omega)$ is singular at $\omega = 1$. For $q(k)$ in Eq. (3), the expansion of $\mathcal{Q}(\omega)$ around $\omega = 1$ is given as

$$\mathcal{Q}(\omega) \simeq 1-(1-\omega)+\begin{cases} A_1\,(1-\omega)^{\gamma'-1} & (2<\gamma<\gamma_c), \\ -A_2\,(1-\omega)^2\ln(1-\omega) & (\gamma=\gamma_c), \\ A_3\,(1-\omega)^2 & (\gamma>\gamma_c), \end{cases} \tag{15}$$

where A_i's are constants, γ' is given in Eq. (3), and $\gamma_c = 3-\eta$. The derivation of the logarithmic correction for the case of $\gamma=\gamma_c$ can be found in [11]. Note that the singular term $(1-\omega)^{\gamma'-1}$ is the second leading term of $1-\mathcal{Q}(\omega)$ for $\gamma<\gamma_c$. Using the relation $\mathcal{P}^{-1}(\omega) = \omega/\mathcal{Q}(\omega)$ in Eq. (5), the behavior of $\mathcal{P}(y)$ around $y=1$ is obtained for each region of γ from Eq. (15), and in turn, using Eq. (10), $p(s)$ for $s\to\infty$. We find that

$$p(s)\sim\begin{cases} s^{-(\gamma-2\eta)/(\gamma-1-\eta)} & (2<\gamma<\gamma_c), \\ s^{-3/2}(\ln s)^{-1/2} & (\gamma=\gamma_c), \\ s^{-3/2} & (\gamma>\gamma_c). \end{cases} \tag{16}$$

Thus, the exponent τ is given as $\tau=(\gamma-2\eta)/(\gamma-1-\eta)$ for $2<\gamma<\gamma_c$ and $\tau=3/2$ for $\gamma\geq\gamma_c$.

Also obtained is $r(t)$ from Eq. (15) by using Eq. (8). The duration distribution $\ell(t)$, which is the derivative of $r(t)$, is found to be

$$\ell(t)\sim\begin{cases} t^{-(\gamma-1-\eta)/(\gamma-2)} & (2<\gamma<\gamma_c), \\ t^{-2}(\ln t)^{-1} & (\gamma=\gamma_c), \\ t^{-2} & (\gamma>\gamma_c). \end{cases} \tag{17}$$

That is, the exponent δ is given as $\delta=(\gamma-1-\eta)/(\gamma-2)$ for $2<\gamma<\gamma_c$ and $\delta=2$ for $\gamma\geq\gamma_c$.

5 Conclusion

We have studied the BTW sandpile model on SF networks with the degree exponent γ to understand the avalanche dynamics in complex systems. The main results are the avalanche size and duration distribution. The exponents τ and δ increase with increasing γ, implying that the hubs play a role of reservoir, that is, sustain large amount of grains to make the SF network resilient under avalanche dynamics. This is reminiscent of the structural resilience of the SF network under random removal of nodes for $\gamma \leq 3$ [12,13,14]. We also checked the case where the threshold z_i contains noise in the way that $z_i = \zeta_i k_i$ with ζ_i being distributed uniformly in [0,1]. We find that such a variation does not change the nature of the avalanche dynamics. However, when the threshold is given in terms of a quantity other than degree, e.g., load, the corresponding avalanche dynamics has no reason to follow the same statistics as studied in this paper, which remains further works.

This work is supported by the KOSEF Grant No. R14-2002-059-01000-0 in the ABRL program.

References

[1] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).

[2] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).

[3] E. Bonabeau, J. Phys. Soc. Japan **64**, 327 (1995).

[4] P. Alstrøm, Phys. Rev. A **38**, 4905 (1988).

[5] S. Lise and M. Paczuski, Phys. Rev. Lett. **88**, 228301 (2002).

[6] Z. Olami, H.J.S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992); K. Christensen and Z. Olami, Phys. Rev. A **46**, 1829 (1992).

[7] K.-I. Goh, D.-S. Lee, B. Kahng, and D. Kim, Phys. Rev. Lett. **91**, 148701 (2003).

[8] T.E. Harris, *The Theory of Branching Processes* (Springer-Verlag, Berlin, 1963).

[9] R. Otter, Ann. Math. Statist. **20**, 206 (1949).

[10] D.-S. Lee, K.-I. Goh, B. Kahng, and D. Kim, to appear in J. Korean Phys. Soc. (2004).

[11] J.E. Robinson, Phys. Rev. **83,** 678 (1951).

[12] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **406**, 378 (2000).

[13] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).

[14] D.S. Callaway, M.E.J. Newman, S.H. Strogatz, and D.J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).