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Abnormal hybrid phase transition in the passively competing Kuramoto model

Jinha Park, B. Kahng *

CCSS, CTP and Department of Physics and Astronomy, Seoul National University, Seoul 08826, Republic of Korea

HIGHLIGHTS

- A passively competing Kuramoto model with mixed signs of couplings is considered.
- Traveling wave and π phases are absent, in contrast to the actively competing models.
- Flat natural frequency distribution leads to a hybrid synchronization transition.
- Staggered order parameter exhibits an additional critical exponent.

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ABSTRACT

We consider a competing Kuramoto model (KM) with mixed signs of coupling constants that appear inside the sum, called passive interactions. It is known that the synchronization transition of this KM reduces to that of the ordinary KM with the mean of those coupling constants as a control parameter. In this type of KM, two order parameters can be introduced: the phase coherence R and the weighted phase coherence S. Here, we show that when the intrinsic frequency distribution $g(\omega)$ is unimodal, the order parameters R and S behave similarly with the same critical exponents β and β' in leading and sub-leading orders, respectively. However, when $g(\omega)$ is uniform, they behave differently, possibly because of the hybrid synchronization transition. The order parameter jumps at a transition point and then it exhibits a critical behavior.

1. Introduction

Synchronization is a collective phenomenon emerging in diverse complex systems in the real world. Examples include the flashing of fireflies, the chirp of crickets, the pacemaker cells of the heart, and synchronous neural activities [1–8]. Globally coupled phase oscillators have been used to model such synchronizations for simplicity. The conventional Kuramoto model (KM) is written as

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \tag{1}$$

where $\theta_i(t)$ is the phase of the ith oscillator at time t; ω_i is the natural frequency following the distribution function $g(\omega)$, and K is a coupling constant independent of i and j. When K is small, each oscillator rotates almost independently near its intrinsic frequency ω_i; however, as K is increased, oscillators interact with each other and a synchronized cluster forms on a macroscopic scale at a transition point K_c. The complex order parameter $Z(t)$ of the KM is defined as

$$Z(t) \equiv R(t)e^{i\Psi(t)} = \frac{1}{N}\sum_{j} e^{i\theta_j}, \tag{2}$$

which represents the coherence of the oscillators. $R(t)$ is the magnitude and $\Psi(t)$ is the average phase. In the thermodynamic limit $N \to \infty$, the order parameters R and Ψ in steady state are independent of time but depend on the coupling constant K.

The order of phase transition depends on the shape of intrinsic frequency distribution $g(\omega)$. For unimodal (bimodal) $g(\omega)$, a continuous (discontinuous) phase transition occurs [1–9]. Uniform $g(\omega)$ is an important marginal case [1,10–13], in which the order parameter jumps at a transition point K_c and then it increases continuously with criticality as K is increased. In steady state, the order parameter R behaves near the transition point K_c as follows:

$$R = \begin{cases} 0 & (K < K_c) \\ R_c + a(K - K_c)^{\beta} & (K > K_c), \end{cases} \tag{3}$$

where R_c is the size of jump of order parameter at K_c; a is a constant. Contrary to the first order transitions in which there is no criticality after the jump, this marginal case exhibits a

* Corresponding author.
E-mail addresses: jinha4815@snu.ac.kr (J. Park), bkahng@snu.ac.kr (B. Kahng).

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critical behavior with the exponent $\beta = 2/3$ [10]. Such a discontinuous transition accompanying the critical aspects of a continuous transition is called hybrid phase transition (HPT), and has been noticed in diverse systems such as synchronizations [14–16], k-core percolations [17–21], and a spin model on scale-free networks [22].

We recall that a possibility of discontinuous transitions was sought in synchronization problems [16], along with the explosive percolations studies [23,24]. To generate analogous suppressive effect, degree–frequency correlations [14,15,25,26] and frequency weighted interactions [27] were introduced to the original Kuramoto model on complex networks. Such generalizations can lead to a balance between the segregative term and integrative term of the Kuramoto model. In contrast to the explosive percolation [28–30], the explosive synchronization turns out to be discontinuous and it is characterized by an appearance of hysteresis, similar to the first-order transitions in thermodynamics [16]. However, even in complex networks, it was noticed in Ref. [25] that degree exponent $\gamma = 3$ is a marginal case showing a HPT between continuous ($\gamma > 3$) and discontinuous ($2 < \gamma < 3$) transitions. Since then, the term, *explosive synchronization*, has been used to represent the synchronization transition that exhibits a jump of the order parameter as in the first-order and the hybrid synchronization transitions, to differentiate the marginal case, we refer to the first-order synchronization with hysteresis as an explosive synchronization. In contrast, a hybrid synchronization transition is referred as a discontinuous transition which involves a singularity after the jump.

Inspired by spin glass systems and neural networks with competing signs of interactions, generalizations of KM have been made [13,31–36]. One most precedent suggested by Daido [31,32] is written as

$$\dot{\theta}_i = \omega_i + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i), \tag{4}$$

where K_{ij} is of Sherrington–Kirkpatrick type, which is Gaussian distributed about zero mean, and is assumed symmetric $K_{ij} = K_{ji}$. Daido questioned the possibility of an oscillator glass under such competing interactions [31]. It was revealed that in this model the frequency entrainment occurs but the phase-locking does not. The oscillator phases show a diffusive motion and therefore the initial coherence is always lost in the long time. Discovery of non-exponential relaxation of coherence in supercritical control parameter regime suggested the presence of potential glassy oscillators; yet it still remains as an inconclusive problem [32]. A critical phenomenon, the so-called volcano transition, was found regarding the motion of local fields in the complex plane. The volcano transition point has been calculated only recently [34].

Hong and Strogatz [35,36] considered simplified KMs with competing interactions based on nodes instead of edges. One can immediately notice that two generalizations are possible: $K_{ij} = K_i$ (active) or K_j (passive). In the actively competing KM, $K_{ij} = K_i$ is given as a competing mixture of attractive and repulsive signs. K_i can be pulled outside the sum. In this case the synchronization order parameter is defined in a similar manner to that of the ordinary KM, but each oscillator i has a distinct and active response to the mean field R, depending on the value of coupling K_i. An actively competing model can be written as

$$\dot{\theta}_i = \omega_i + K_i R \sin(\Psi - \theta_i), \tag{5}$$

where K_i follows a distribution $f(K)$ and takes either a positive or negative value. For simplicity, $f(K)$ is set as a mixture of two node species $f(K) = (1-p)\delta(K-K_1)+p\delta(K-K_2)$, where p $(1-p)$ is the mixing fraction of positive (negative) coupling constant K_2 (K_1) oscillators. In the actively competing model, depending on the sign of K_i, the stability of an oscillator at the velocity-balancing position is reversed and becomes either attractive or repulsive to the mean ordering $Z = Re^{i\Psi}$. Therefore the oscillators are clustered in the phase circle into two groups that are separated roughly by an angle π. They can be either static or traveling. This leads to a rich phase transition diagram involving three different phases, namely incoherent, π and traveling wave phases [13,35].

The passively competing KM is written as

$$\dot{\theta}_i = \omega_i + \frac{1}{N} \sum_{j=1}^{N} K_j \sin(\theta_j - \theta_i), \tag{6}$$

where K_j follows the same distribution $f(K)$. Varying p from zero to unity linearly interpolates the mean coupling constant $\langle K \rangle$ between $K_1 < 0$ and $K_2 > 0$. At $p = 0$ all oscillators are repulsive, while at $p = 1$ all oscillators are attractive. The intrinsic frequency distribution $g(\omega)$ is assumed to be symmetric about zero. In this interaction form, each oscillator j interacts with the weighted (antiferromagnetic) influence of other oscillators. In this regard we call this model the passively competing KM. For the passively competing KM, a new order parameter

$$W(t) \equiv S(t)e^{i\Phi(t)} \equiv \frac{1}{N} \sum_{j=1}^{N} K_j e^{i\theta_j(t)}, \tag{7}$$

is a kind of weighted mean field order parameter [31,36]. Eq. (6) is decoupled as

$$\dot{\theta}_i = \omega_i + S(t) \sin(\Phi(t) - \theta_i), \tag{8}$$

where $S(t)$ is the magnitude of the staggered field and $\Phi(t)$ is the average phase of the order parameter $W(t)$. Here, we remark that all passively competing oscillators are under the same mean field $S(t)$. The signs of the coupling constants K_j of passively competing oscillators do not reverse the response of individual oscillators. Therefore the group separation such as π or traveling wave is absent in the passively competing KM. In case when $g(\omega)$ is unimodal, a previous study [36] showed that a continuous phase transition occurs when the mean coupling constant $\langle K \rangle \equiv \frac{1}{N}\sum_j K_j$ of the mixture reaches the value of the critical coupling strength K_c of the corresponding KM. Moreover, the onset of order $S(t)$ coincides with the onset of order $R(t)$. Therefore a phase transition from the incoherent to coherent phase in the passively competing systems is characterized solely by the order parameter S. For such reasons, it was concluded [36] that the passively competing generalization is rather a monotonous extension to the KM. However, we find that for a hybrid synchronization transition, a slight difference emerges in the passively competing KM.

In this paper, we mainly investigate the phase transition of passively competing KM with a uniform $g(\omega)$. This model exhibits a HPT with the critical exponent $\beta = 2/3$ at the transition point $\langle K \rangle_c$ which coincides with the β value obtained at the critical coupling strength K_c of the Pazó model, a conventional KM with uniform $g(\omega)$ [10]. A subleading scaling $\beta' = 1$ had been also obtained for the order parameter R, keeping finite systems in mind. That is,

$$R = \begin{cases} 0 & (K < K_c) \\ R_c + a\big(K - K_c\big)^{\beta} + b\big(K - K_c\big)^{\beta'} & (K > K_c), \end{cases} \tag{9}$$

where R_c is the jump size of the order parameter R at K_c; a and b are some constants; and $\beta = 2/3$ and $\beta' = 1$.

Next, we similarly define the leading exponent β and subleading exponent β' for the order parameter S in our model as follows:

$$S = \begin{cases} 0 & \big(\langle K \rangle < \langle K \rangle_c\big) \\ S_c + a\big(\langle K \rangle - \langle K \rangle_c\big)^{\beta} + b\big(\langle K \rangle - \langle K \rangle_c\big)^{\beta'} & \big(\langle K \rangle > \langle K \rangle_c\big), \end{cases} \tag{10}$$

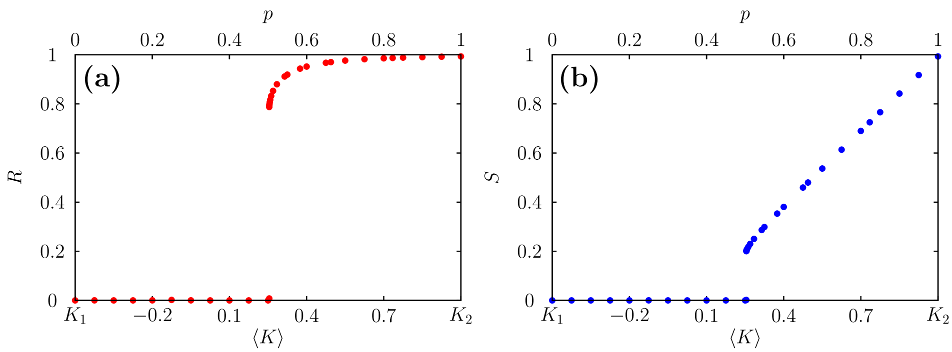

Fig. 1. Plots of the order parameters (a) R and (b) S of the KM with the passively competing coupling constants K_1 and K_2 with the probability $1-p$ and p, respectively. The intrinsic frequency distribution is uniform in the range $[-\gamma, \gamma]$. Data points are obtained by simulations for $N = 25600$; $K_1 = -0.5$ and $K_2 = 1$; and $\gamma = 0.2$. The data points are obtained by taking average over the last ten percent of the total runtime $t = 10^5$ s, corresponding to a steady state. Both order parameters show a discontinuous jump at the transition point $\langle K\rangle_c = 4\gamma/\pi \approx 0.255$.

where S_c is the jump size of the order parameter S at $\langle K\rangle_c$; a and b are again some constants; and $\beta = 2/3$. Interestingly, for the order parameter S we obtain a different subleading hybrid scaling behavior $\beta' = 4/3$.

We find that in case of the passively competing KM with uniform $g(\omega)$ the subleading scaling exponent depends on which order parameter (R or S) is in use. However, for the unimodal frequency distribution, the leading exponent β and the subleading exponent β' are the same for both order parameters R and S.

2. The self-consistency equation

The self-consistency equation of the passively competing model is written as [36]

$$S = \int dK f(K) \int d\omega g(\omega) K \cos\theta$$
$$= \int dK f(K) \int d\omega g(\omega) K \sqrt{1 - \left(\frac{\omega}{S}\right)^2}$$
$$= \langle K\rangle \int_{-S}^{S} d\omega g(\omega) \sqrt{1 - \left(\frac{\omega}{S}\right)^2}. \tag{11}$$

2.1. Uniform

Let us consider a uniform distribution of $g(\omega)$ ranging $[-\gamma, \gamma]$. In this case, the transition is hybrid. Integration in the self consistency equation can be performed exactly as follows:

$$S = \frac{\langle K\rangle}{2\gamma} \int_{-\gamma}^{\gamma} d\omega \sqrt{1 - \left(\frac{\omega}{S}\right)^2}$$
$$= \frac{\langle K\rangle S}{2\gamma} \left[\arcsin\frac{\gamma}{S} + \frac{\gamma}{S}\sqrt{1 - \left(\frac{\gamma}{S}\right)^2}\right]. \tag{12}$$

The transition point, denoted by the subscript c, occurs when $S_c = \gamma$ and is determined as

$$\langle K\rangle_c = \frac{4\gamma}{\pi} = \frac{2}{\pi g(0)}. \tag{13}$$

We remark that beyond this transition point all oscillators are phase locked with an angular spread less than or equal to π on the phase circle beyond this transition point [10]. Thus the temporal fluctuations of the order parameters are totally absent in the synchronized phase after reaching a steady state, although the value of transition point may involve some finite size corrections as in Ref. [10]. In Fig. 1, we check that both order parameters R and S capture the discontinuous synchronization transition at the same $\langle K\rangle_c$. A S-ordered state shows the coherence to R and is stable beyond the transition point.

Now for scaling analysis we take a supercritical point $\langle K\rangle = \langle K\rangle_c + \delta\langle K\rangle$ near the transition point. The angular spread of locked phases is characterized by $\theta_m \equiv \arcsin(\gamma/S)$, which becomes slightly less than $\pi/2$. Let $\theta_m \equiv \frac{\pi}{2} - \delta\theta$. From the self consistency equation $S = \frac{\langle K\rangle S}{2\gamma}\int_{-\theta_m}^{\theta_m}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$, the ordered state is given by

$$1 = \frac{\langle K\rangle_c + \delta\langle K\rangle}{2\gamma}\left[\frac{\pi}{2} - \delta\theta + \frac{1}{2}\sin(\pi - 2\delta\theta)\right]$$
$$= \left(\langle K\rangle_c + \delta\langle K\rangle\right)\left[\frac{1}{\langle K\rangle_c} - \frac{\delta\theta}{2\gamma} + \frac{1}{4\gamma}\left(2\delta\theta - \frac{(2\delta\theta)^3}{3!}\right)\right]$$
$$= \left(\langle K\rangle_c + \delta\langle K\rangle\right)\left[\frac{1}{\langle K\rangle_c} - \frac{(\delta\theta)^3}{3\gamma}\right], \tag{14}$$

where truncation is up to the lowest order in $\delta\theta$. Thus,

$$\delta\langle K\rangle = \frac{\langle K\rangle_c^2}{3\gamma}(\delta\theta)^3. \tag{15}$$

From $\gamma = S\sin\theta_m$, we have

$$\gamma = (S_c + \delta S)\sin\left(\frac{\pi}{2} - \delta\theta\right)$$
$$= (\gamma + \delta S)\left(1 - \frac{(\delta\theta)^2}{2}\right) \tag{16}$$

and therefore,

$$\delta S = \frac{\gamma}{2}(\delta\theta)^2$$
$$= \frac{\gamma}{2}\left(\frac{3\pi^2}{16\gamma}\right)^{2/3}\delta\langle K\rangle^{2/3}. \tag{17}$$

Note that the leading order calculation gives the critical exponent $\beta = 2/3$, which is also verified from the simulations as shown in Fig. 2(a). Therefore the synchronization transition of the passively competing KM with uniform frequency distribution falls into the category of the HPT of the non-competing KM with a uniform $g(\omega)$.

Up to the next order we find,

$$\delta\langle K\rangle = \frac{\langle K\rangle_c^2}{3\gamma}(\delta\theta)^3 - \frac{\langle K\rangle_c^2}{15\gamma}(\delta\theta)^5 + O\left(\delta\theta^6\right) \tag{18}$$

and reversing the above series gives

$$\delta\theta = \left(\frac{3\gamma}{\langle K\rangle_c^2}\delta\langle K\rangle\right)^{1/3} + \frac{\gamma}{5\langle K\rangle_c^2}\delta\langle K\rangle. \tag{19}$$

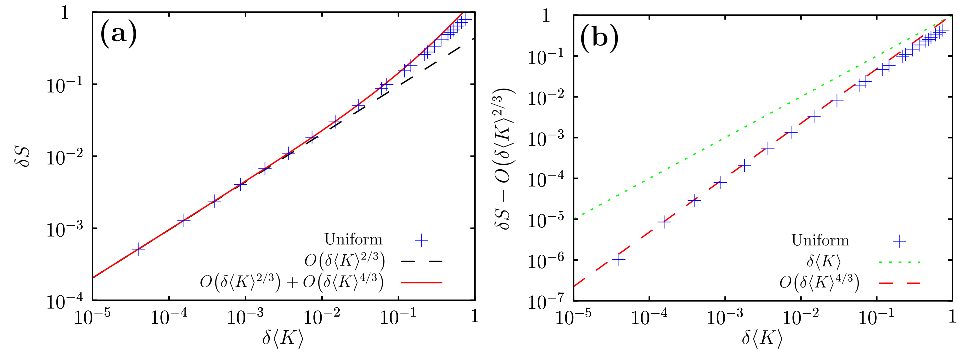

Fig. 2. (a) Plot of δS versus $\delta\langle K\rangle$ for the uniform $g(\omega)$. The black dashed line denotes the leading order δS of Eq. (20) and the critical exponent $\beta = 2/3$ is clearly noticed. The red solid line counts up to the next leading order of Eq. (20). Data points are obtained by simulations with $N = 25600, K_1 = -0.5, K_2 = 1$, and $\gamma = 0.2$. We used time averaged values during the last ten percent of the total runtime $t = 10^5$ s. For larger values of $\langle K\rangle$ beyond the critical point, a small deviation is noticed. (b) Plot of the subleading correction values versus $\delta\langle K\rangle$ to check the exponent of the subleading order $\beta' = 4/3$. Red dashed line denotes the subleading correction of Eq. (20). The dotted green line with slope one is drawn for comparison, which represents the subleading correction for the Pazó model. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

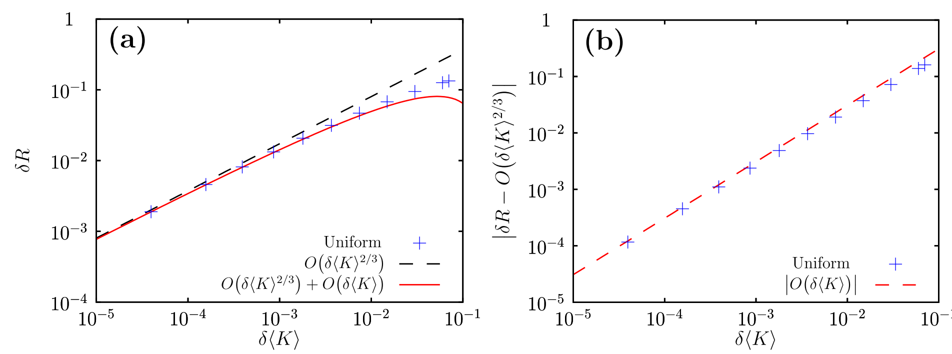

Fig. 3. (a) Plot of δR versus $\delta\langle K\rangle$ for the uniform $g(\omega)$. Scaling of δR is governed by the same hybrid critical exponent $\beta = 2/3$. The black dashed line represents the leading order of Eq. (24), while the red solid line counts up to the next leading order of Eq. (24). (b) Plot of the magnitudes of the subleading correction values versus $\delta\langle K\rangle$ to check the exponent of the subleading order $\beta' = 1$. The dashed red line denotes the absolute value of the subleading correction of Eq. (24), which is linear in $\delta\langle K\rangle$. The data points are obtained by simulations. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Therefore,

$$\delta S = \frac{\gamma(1-\cos\delta\theta)}{\cos\delta\theta} = \frac{\gamma}{2}(\delta\theta)^2 + \frac{5\gamma}{24}(\delta\theta)^4$$
$$= \left(\frac{9\pi^4\gamma}{2048}\right)^{1/3}\delta\langle K\rangle^{2/3} + \frac{289\pi^2}{5760}\left(\frac{3\pi^2}{16\gamma}\right)^{1/3}\delta\langle K\rangle^{4/3}. \qquad (20)$$

This subleading term gives a non-integer exponent $\beta' = 4/3$ for the order parameter S, which is well noticed in Fig. 2(b).

Finally we consider the conventional order parameter $R = \left|\frac{1}{N}\sum_j e^{i\theta_j}\right|$ in the passively competing KM. R is obtained as

$$R = \int d\omega g(\omega)\sqrt{1-\left(\frac{\omega}{S}\right)^2}. \qquad (21)$$

From Eqs. (11) and (21), we observe that the two order parameters are related to each other by

$$S = \langle K\rangle R \qquad (22)$$

as long as the coupling constant K and the intrinsic frequency ω are uncorrelated. Indeed it is checked from Fig. 1 that the jump sizes R_c and S_c at the transition point differ by the factor $\langle K\rangle_c$. This relation between the two order parameters can be expanded above the transition point $S_c = \langle K\rangle_c R_c$, giving

$$\delta S = \delta\langle K\rangle R_c + \langle K\rangle_c \delta R + \text{higher order}. \qquad (23)$$

From Eqs. (20) and (23), we find the scaling of δR as

$$\delta R = \frac{\gamma}{2\langle K\rangle_c}\left(\frac{\langle K\rangle_c^2}{3\gamma}\right)^{2/3}\delta\langle K\rangle^{2/3} - \frac{R_c}{\langle K\rangle_c}\delta\langle K\rangle$$
$$= \frac{\pi}{8}\left(\frac{3\pi^2}{16\gamma}\right)^{2/3}\delta\langle K\rangle^{2/3} - \frac{\pi^2}{16\gamma}\delta\langle K\rangle, \qquad (24)$$

where we used $R_c = \pi/4 = S_c/\langle K\rangle_c$ in the last line. This scaling of δR is the same as the one obtained by Pazó [10], except that the coupling constant K has been replaced by the mean coupling $\langle K\rangle$. The obtained scaling of δR above the critical point coincides exactly with the numerical simulation results (Fig. 3).

We notice in Eqs. (20) and (24) that δS and δR scale with the same critical exponent $\beta = 2/3$ in leading order. However, the subleading terms have different exponent values. In fact the subleading exponent $\beta' = 1$ is non-critical, for the order parameter R [10]. The exponent 4/3 will appear instead in the next order term. We remark that the linear term with exponent 1 will intervene in the scaling of δR, generally if the transition is discontinuous $R_c \neq 0$, which is derived from the r.h.s. first term of Eq. (23).

It is also noticed that if the leading scaling exponent of δS were $\beta = 1$ or $\beta = 1/2$, which is frequently the case of hybrid percolation transitions with our definitions of β in Eqs. (9) and (10), the leading or subleading exponent would have been unity anyway and therefore the difference in the scaling exponents for δR and

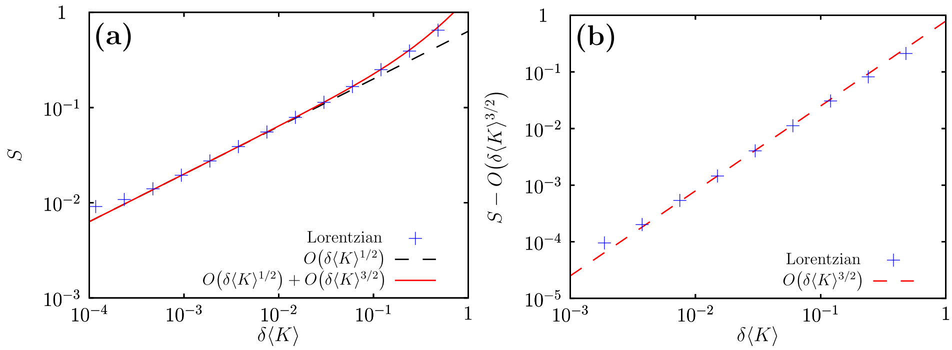

Fig. 4. (a) Plot of S versus $\delta\langle K\rangle$ for the Lorentzian $g(\omega)$. The black dashed line denotes the leading order S of Eq. (26) and the critical exponent $\beta = 1/2$ is noticed. The red solid curve includes up to the next order of Eq. (26). Simulations are performed with the Lorentzian distribution $g(\omega) = (\gamma/\pi)/(\omega^2+\gamma^2)$ for $N = 102400$, $K_1 = -0.5$, $K_2 = 1$, and $\gamma = 0.2$. Time average is taken during the last fifty percent of the total runtime $t = 10^4$ s. The error bar denotes temporal fluctuations of the order parameter S. The deviation near the critical point is due to finite-size effect. (b) Plot of the subleading correction values versus $\delta\langle K\rangle$ to check the exponent of the subleading order $\beta' = 3/2$. Red dashed line denotes the subleading correction of Eq. (26). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

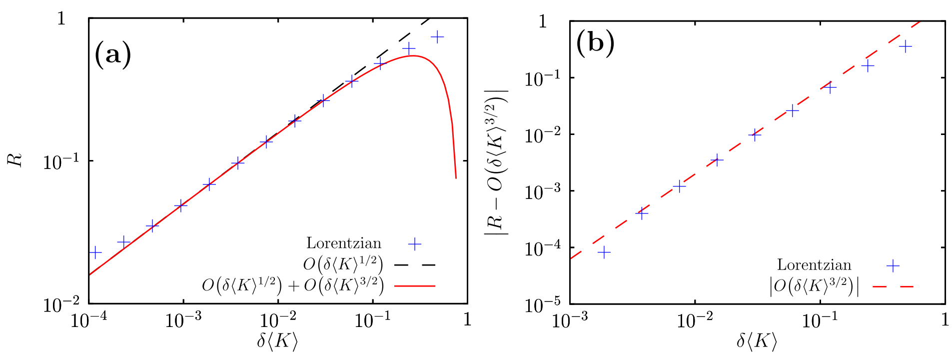

Fig. 5. (a) Plot of R versus $\delta\langle K\rangle$ for the Lorentzian $g(\omega)$. The black dashed line denotes the leading order R of Eq. (27) and the critical exponent $\beta = 1/2$ is noticed. The red solid curve includes up to the subleading order of Eq. (27). The data points are from the same simulation as that of Fig. 4. (b) Plot of the subleading correction values versus $\delta\langle K\rangle$ to check the exponent of the subleading order $\beta' = 3/2$. Red dashed line denotes the subleading correction of Eq. (27). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

δS would not have arisen. Here, the difference is owing to an abnormal exponent $\beta = 2/3$.

2.2. Lorentzian

At this stage, we need to compare our result for the subleading scaling behaviors of S and R in the case of uniform $g(\omega)$ with those in the case of unimodal $g(\omega)$. Here we take a Lorentzian distribution for $g(\omega)$ as a unimodal distribution, for which the self consistency equation is exactly integrable. The order parameter S is derived as follows:

$$S = \langle K\rangle \int_{-S}^{S} d\omega \frac{\gamma/\pi}{\omega^2+\gamma^2}\sqrt{1-\frac{\omega^2}{S^2}} = \langle K\rangle\left(\sqrt{1+\left(\frac{\gamma}{S}\right)^2}-\frac{\gamma}{S}\right). \tag{25}$$

Therefore,

$$S = \sqrt{2\gamma}\,\delta\langle K\rangle^{1/2} + \frac{1}{\sqrt{8\gamma}}\delta\langle K\rangle^{3/2} + O\big(\delta\langle K\rangle^{5/2}\big). \tag{26}$$

Comparison of the above scaling for S with the simulation results is presented in Fig. 4. For the continuous transition of Lorentzian case, jump is absent. Instead, plugging in $\langle K\rangle = \langle K\rangle_c + \delta\langle K\rangle$ directly to Eq. (22) yields

$$R = \frac{1}{\sqrt{2\gamma}}\delta\langle K\rangle^{1/2} - \frac{1}{4\gamma\sqrt{2\gamma}}\delta\langle K\rangle^{3/2}. \tag{27}$$

Notice here that the exponents $\beta = 1/2$ and $\beta' = 3/2$ are the same for both order parameters R and S (See Fig. 5). Furthermore, one can check that any symmetric unimodal $g(\omega)$ leads to the same exponents $\beta = 1/2$ and $\beta' = 3/2$, where the integral of the self consistency Eq. (25) can be solved after inserting the Taylor series of g at zero.

Therefore for the case of unimodal $g(\omega)$, β and β' are not changed, regardless of R or S. In contrast, the subleading exponents β' for the uniform $g(\omega)$ depends on which of the order parameter, R or S, is in use. This is owing to the jump of the order parameters at the transition point and the abnormal exponent β.

3. Discussion

The Ott–Antonsen (OA) method has been quite successful in the bifurcation analysis of KM including the Lorentzian and bi-Lorentzian $g(\omega)$, where $N \to \infty$ degrees of freedom of continuum oscillators are reduced to a few coupled modes [37–39]. An exact low dimensional system is also known for the Kuramoto oscillators with identical frequency [40–46].

Especially in the hybrid synchronization case of uniform $g(\omega)$, the previous reduction methods lead to an integral that is difficult to evaluate analytically and therefore the bifurcation analysis is not feasible. For such a reason, the exact hybrid synchronization dynamics remains veiled. Instead, one can investigate the phase transition from the stationary self-consistency solutions. This method is applicable to the current passively competing KMs. It turned out that, in contrast to the actively competing

KMs, which can exhibit π clustering patterns, traveling wave dynamics [35], hysteresis, and metastability [13], the passively competing KMs exhibit a rather simple transition pattern from incoherent to coherent phase. We remark however that this method is applicable to a general intrinsic frequency distribution $g(\omega)$. We also remark that the self-consistency equation is obtained just by assuming stationarity, regardless of the use of the OA ansatz. Stability of the self consistency solutions can be checked by numerical simulations, or one can use an empirical linear stability criterion [47] instead. However, the validity of the empirical linear stability criterion remains subtle; for example our previous analysis has shown that the linear stability becomes incomplete for uniform $g(\omega)$, a case which gives metastable solutions with neutral or weakly stable linear stability [13].

Kuramoto's self consistency method [2] is still a general powerful tool for studying phase transition of the Kuramoto oscillator system. This mean field theory is applicable to many other types of intrinsic frequency distributions $g(\omega)$ other than Lorentzian or uniform distributions. It is also useful when analyzing phase transitions of finite size systems. On the contrary, the OA method provides a bifurcation for limited types of distributions, and it is not suitable for calculations on finite size systems. Finite size effects are important. For example, Daido's discovery on the divergence of temporal fluctuations at the transition point is an important critical phenomenon, an aspect which is analogous to the divergence of susceptibility in second order phase transitions in thermodynamics [9]. Therefore the self consistency method can be used complementarily to the Ott–Antonsen method.

Finally, we remark that an avalanche collapse of synchronization [10,25] also arises at the hybrid critical point of the passively competing model. This work will be published elsewhere. A slight change of mean coupling constant results in a complete breakdown of synchrony. The giant frequency cluster completely dissolves into smaller clusters with independent running frequencies. Such a feature is thought to be an important aspect which characterizes the hybrid synchronization transition [25].

4. Conclusion

We investigated the synchronization transition of the Kuramoto model (KM) with passively competing interactions and a uniform intrinsic frequency distribution, which was compared with that of the corresponding KM with a unimodal intrinsic frequency distribution. A common feature between the two results is that the transition point $\langle K\rangle_c$ of the mean coupling constant of the competing mixture plays a similar role as K_c of the ordinary KM. The order parameter S representing the weighted coherence was useful in analyzing the phase transition of passively competing KMs. We have unexpectedly found a novel exponent $\beta' = 4/3$ for δS in the subleading order for the passively competing KM with uniform frequency distribution, although $\beta' = 1$ for δR remains the same as in the Pazó model. We found that this difference is attributed by the jump of the order parameter at the hybrid synchronization transition with an abnormal exponent $\beta = 2/3$ in the leading order. This suggests that further interesting features such as multicritical behavior could occur from the hybrid synchronization transition of the passively competing generalization of the KM beyond the model we considered. Although the subleading correction itself is not a finite size effect, a correct understanding of the subleading order in the scaling of the order parameter is requested in finite systems, because the leading order term governs only near the transition point, which is generally not accessible with small-size systems.

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