Supplementary Information:

Hybrid Percolation Transition in Cluster Merging Processes: Continuously Varying Exponents

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CONTENTS

I. COMPARISON BETWEEN THE ORIGINAL AND THE MODIFIED VERSIONS OF THE R-ER MODEL

A. The order parameter

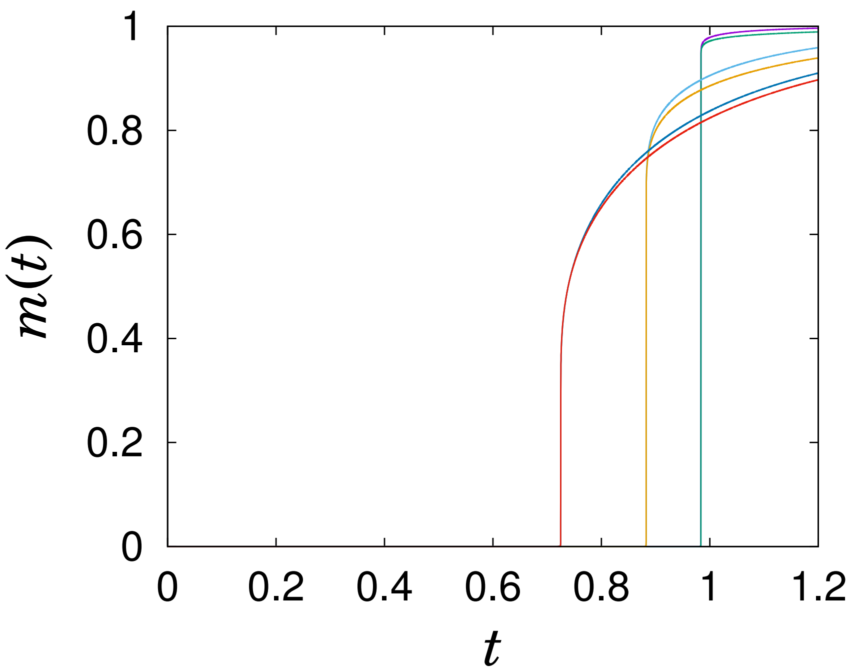

FIG. S1. Plot of the order parameter $m(t)$ vs t for the original and the modified r-ER models with $g = 0.2, 0.5$, and 0.8 from right to left. For each g, the upper (lower) curve is for the original r-ER model (the modified r-ER model).

B. The cluster size distribution of the original r-ER model

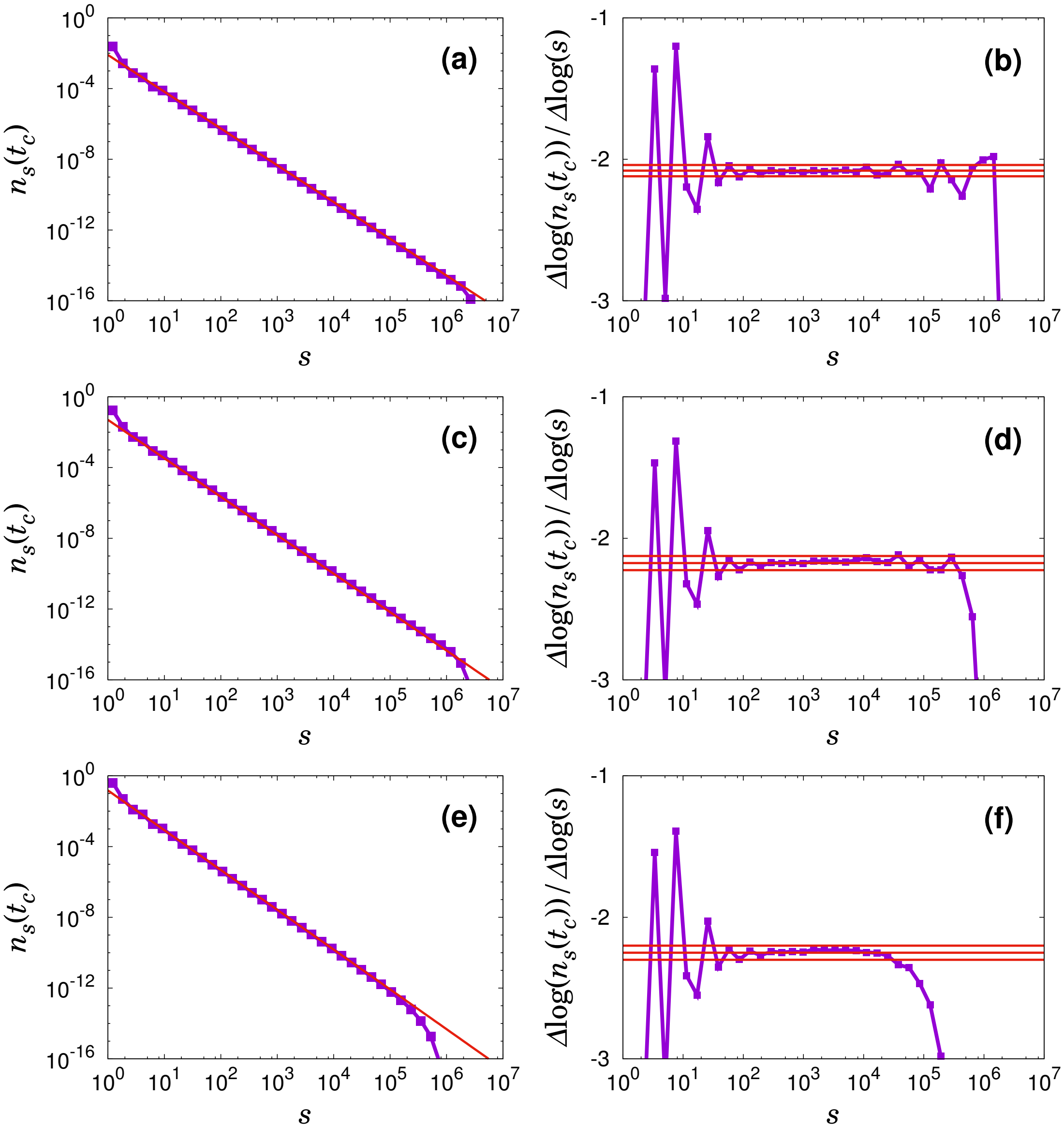

FIG. S2. Plots of the cluster size distributions at t_c for the original r-ER model with $g = 0.2$ (a), 0.5 (c) and 0.8 (e). They decay in a power-law manner as $n_s(t_c) \sim s^{-\tau}$, and the exponents τ are estimated as 2.08 ± 0.04 (b), 2.18 ± 0.05 (d) and 2.25 ± 0.05 (f). Those τ values are consistent with those obtained from the modified r-ER model. The system is taken as $N = 4.096 \times 10^7$.

C. The cluster size distribution of the modified r-ER model for different Ns

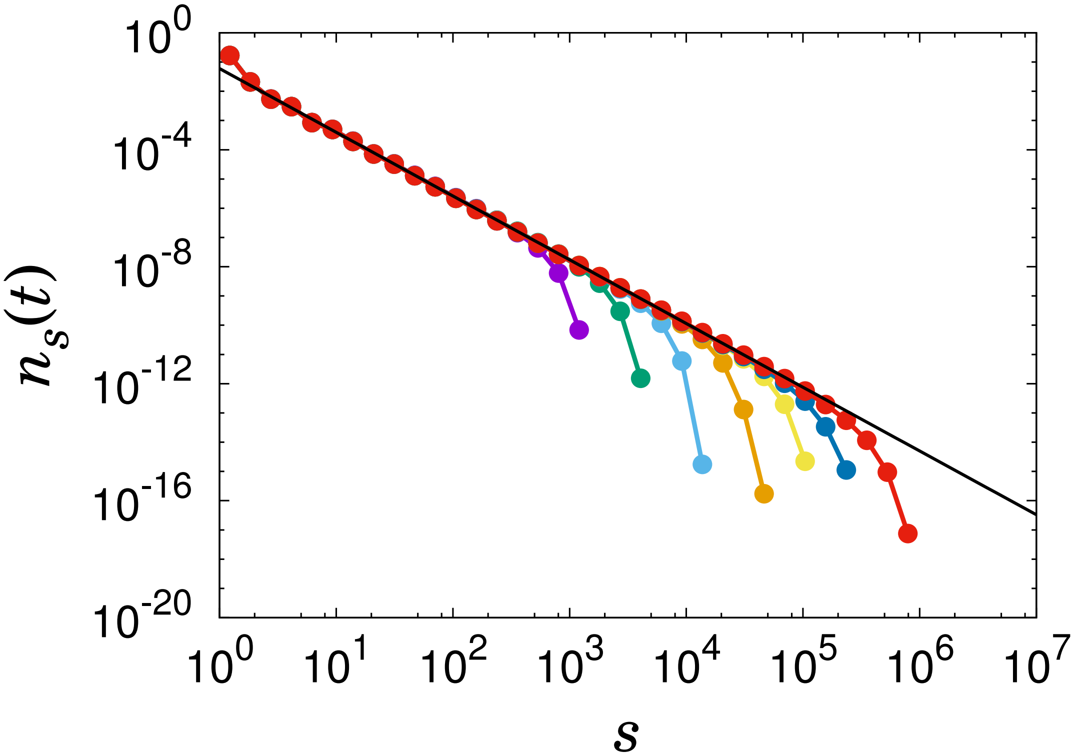

FIG. S3. Plot of the cluster size distributions $n_s(t)$ at t_c for the r-ER model with $g = 0.5$ for different system sizes $N/10^4 = 1, 4, 16, 64, 256, 1024$, and 4096 from the left. The slope is -2.18 for all system sizes, thus the exponent $\tau \approx 2.18$ is independent of N.

II. THE SMOLUCHOWSKI EQUATION FOR THE KINETICS OF THE R-ER MODEL

We set up the Smoluchowski rate equation for the kinetics of cluster aggregations in the r-ER model. At each time step, one cluster of size i is selected from the entire system and the other cluster of size j from the restricted set $R(t)$ with probabilities i/N and j/N_R, respectively, where N_R is the total number of nodes in the set R. That is, $N_R = \sum_{\alpha \in R} s_\alpha$, where s_α is denoted as the size of cluster α in the set $R(t)$. Then two nodes are selected randomly one each from the selected clusters. The pair of nodes are connected by adding an edge. The rate equation is written as follows:

$$\frac{dn_s(t)}{dt} = \sum_{i,j=1}^{\infty} \frac{in_i jn_j}{g}\delta_{i+j,s} - \Big(1+\frac{1}{g}\Big)sn_s \qquad \text{for } s < S_R, \tag{S1}$$

$$\frac{dn_s(t)}{dt} = \sum_{i,j=1}^{\infty} \frac{in_i jn_j}{g}\delta_{i+j,s} - sn_s - \Big(1-\sum_{i=1}^{S_R-1}\frac{in_i}{g}\Big) \qquad \text{for } s = S_R, \tag{S2}$$

$$\frac{dn_s(t)}{dt} = \sum_{j=1}^{\infty} \delta_{i+j,s} jn_j \sum_{i=1}^{S_R-1}\frac{in_i}{g} + \sum_{j=1}^{\infty}\delta_{S_R+j,s} jn_j\Big(1-\sum_{i=1}^{S_R-1}\frac{in_i}{g}\Big) - sn_s \qquad \text{for } s > S_R. \tag{S3}$$

The first term of the R.H.S of Eq.(S1) comes from the aggregation process of two clusters of sizes i and j, producing a cluster of size $s = i + j$. This term is obtained from $\sum_{i,j=1}^{\infty} in_i \frac{Njn_j}{N_R}\delta_{i+j,s}$, where the approximation $N_R \approx gN$ is taken. The second term is the probability that a cluster of size s is selected from the entire system and the set R. For the latter case, we used the approximation $N_R \approx gN$. The first term of the R.H.S of Eq.(S2) originates from the same process as for the case $s < S_R$. The second term is the probability that a cluster of size S_R is selected from the entire system. The third term is the probability that a cluster of size S_R is selected from the set R. This probability is precisely given as $(N_R - N\sum_{i=1}^{S_R-1} in_i)/N_R$. The second and the third terms include the case that the number of clusters of size S_R is more than one. Using the approximation $N_R \approx gN$, we obtain the form (S2). One can obtain Eq.(S3) similarly.

III. TESTING OF TREE-LIKE APPROXIMATION FOR $t < t_c$

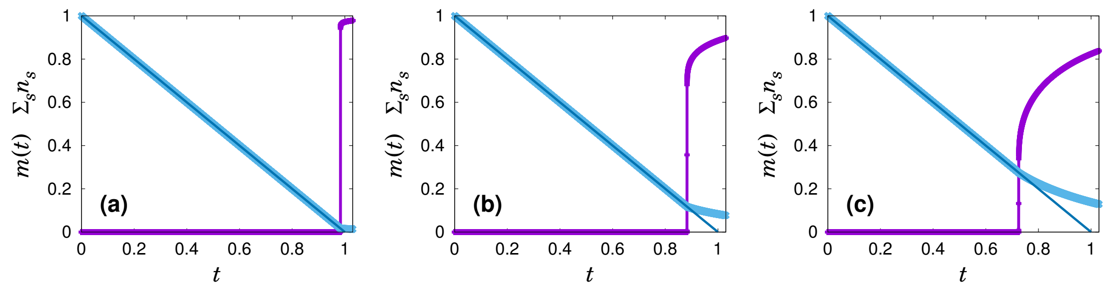

FIG. S4. Plots of the total number of clusters per node $\sum_s n_s$ at time step t and the order parameter $m(t)$ vs t for several values of $g = 0.2$ (a), 0.5 (b) and 0.8 (c). Manifestly, the relation $\sum_{s=1} n_s(t_c) = 1 - t_c$ holds for $t < t_c$.

IV. DERIVATION OF THE EXPONENT σ FOR THE CHARACTERISTIC CLUSTER SIZE

First, we expand the probability $p_s(\hat{t})$ that a randomly selected node belongs to a cluster of size s at $\hat{t}$ as

$$p_s(\hat{t}) \equiv sn_s(\hat{t}) = B_0(s) + B_1(s)\hat{t} + B_2(s)\hat{t}^2 + \dots, \tag{S4}$$

where $B_0(s) = sn_s(t_c) = a_0 s^{1-\tau}$ and $B_1(s)$ is given by

$$\begin{aligned} B_1(s) = {} & s\sum_{u=1}^{s-1}[B_0(u) - B_0(s)][B_0(s-u) - B_0(s)] - s(s-1)[B_0(s)] \\ & + 2sB_0(s)\sum_{u=s}^{\infty} B_0(u) - 2m_0 s B_0(s), \end{aligned} \tag{S5}$$

where we use the relation $\sum_{s=1}^{\infty} B_0(s) = 1 - m_0$. The last term becomes dominant for the r-ER model. Thus, $B_1(s) \approx -2m_0 a_0 s^{2-\tau}$. Similarly,

$$\begin{aligned} B_2(s) = {} & s\sum_{u=1}^{s-1}[B_1(u) - B_1(s)][B_0(s-u) - B_0(s)] - s(s-1)B_0(s)B_1(s) + sB_0(s)\sum_{u=1}^{s-1} B_1(u) \\ & - sB_1(s)\sum_{u=s}^{\infty} B_0(u) - m_0 s B_1(s). \end{aligned} \tag{S6}$$

Again the last term is dominant, and $B_2(s) \approx 2m_0^2 a_0 s^{3-\tau}$. We can obtain higher order terms similarly. Therefore, the probability $p_s(\hat{t})$ is written in a scaling form, $p_s(\hat{t}) = s^{1-\tau} f(s\hat{t}^{1/\sigma})$, where $f(x)$ is a scaling function, and therefore $\sigma = 1$. However, the numerically obtained values of σ deviate slightly from unity when g approaches unity. This discrepancy seems to result from the fact that the transition type changes from discontinuous to continuous one as $g \to 1$.

V. SOLUTION OF THE ORDINARY ER MODEL WITH ARBITRARY INITIAL CONDITION

First, we recall the ordinary ER dynamics in the perspective of cluster merging processes governed by the Smoluchowski coagulation equation. At each time step, two clusters of sizes i and j are selected with probabilities i/N and j/N, where N is the system size. Then two nodes are selected randomly one from each of the selected clusters. The pair of nodes are connected by adding an edge. The Smoluchowski coagulation equation is written as follows:

$$\frac{dn_s(t)}{dt} = \sum_{i+j=s} in_i jn_j - 2sn_s. \tag{S7}$$

Let $F_n(\mu,t) \equiv \sum'_{s=1} s^n n_s(t) e^{\mu s}$ ($\mu < 0$) be the generating function of the n-th moment of the cluster size distribution, where the prime represents that the sum excludes the largest cluster. Then the following relation can be obtained from the Smoluchowski coagulation equation (S7)

$$\frac{\partial F_1(\mu,t)}{\partial t} = 2\big[F_1(\mu,t) - 1\big]\frac{\partial F_1(\mu,t)}{\partial \mu}. \tag{S8}$$

The solution of the above equation is given as

$$F_1(\mu,t) = 1 - H(-\mu - 2t[F_1(\mu,t) - 1]), \tag{S9}$$

where

$$H(\mu) = 1 - F_1(-\mu, 0) = 1 - \sum_s sn_s(0)e^{-\mu s}. \tag{S10}$$

The order parameter of the ER model can be obtained from the relation, $m(t) = 1 - F_1(0^-, t)$. Then, we have the self-consistent equation for $m(t)$ as

$$m(t) = H(2tm(t)). \tag{S11}$$

The order parameter near t_c is given explicitly by

$$m(t) = \frac{2M_2^2(0)}{M_3(0)}[2M_2(0)t - 1], \tag{S12}$$

where $M_n(0) = \sum_s' s^n n_s(0)$. Thus, $t_c = 1/[2M_2(0)]$.

The susceptibility can be obtained as $\chi = F_1'(0^-, t) = F_2(0^-, t)$. Taking the derivative of the equation (S9) with respect to μ, we obtain that

$$\chi(t) = \frac{H'(2tm(t))}{1 - 2H'(2tm(t))}, \tag{S13}$$

where $H'(\mu)$ is the derivative of $H(\mu)$ with respect to μ. Explicitly, the susceptibility $\chi(t)$ near t_c is given as

$$\chi(t) = \frac{M_2(0)}{|1 - 2M_2(0)t|} \tag{S14}$$

for both $t < t_c$ and $t > t_c$.

VI. COMPARISON BETWEEN $1 - m_0$ AND g

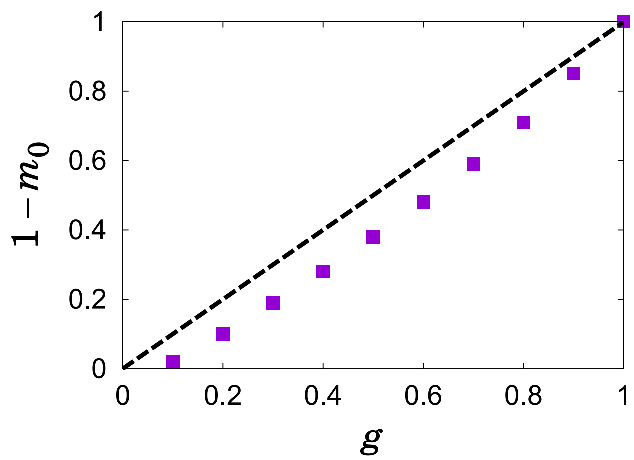

FIG. S5. Plot of $1 - m_0$ vs g. Dashed line is a guideline with relation $1 - m_0 = g$.

VII. FINITE-SIZE SCALING ANALYSIS

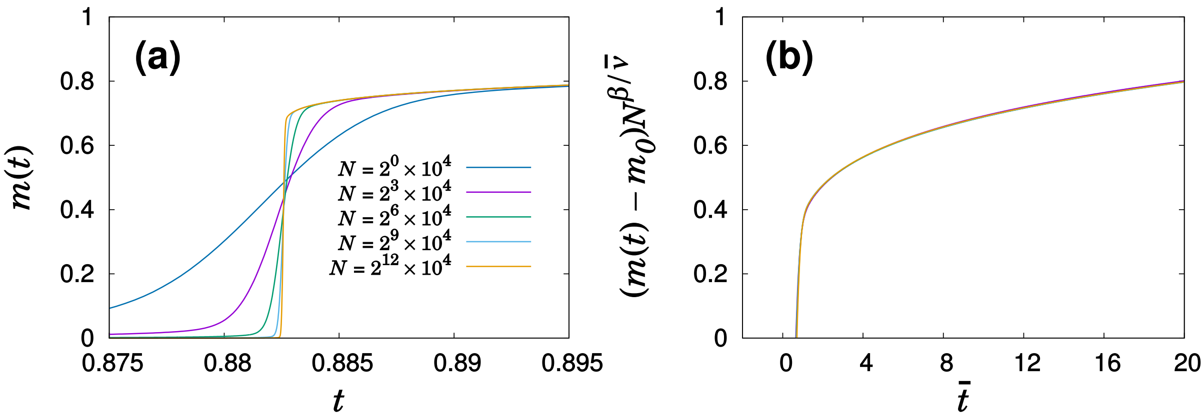

FIG. S6. (a) Plot of the order parameter $m(t)$ as a function of t for different system sizes N. (b) Scaling plot in the form of $(m(t) - m_0)N^{\beta/\bar{\nu}}$ versus $\bar{t} \equiv (t - t_c^-(N))N^{1/\bar{\nu}}$, where $N^{1/\bar{\nu}}$ is obtained from the maximum slope of the tangential line of $m(t)$ i.e. $dm(t)/dt|_{\max}$ between $t_c^-(N)$ and t_c, which behaves in a power law manner as $\sim N^{1/\bar{\nu}}$. It is estimated that $1/\bar{\nu} \approx 0.5 \pm 0.01$. As we can see, data collapse is achieved. This means that for any t larger than t_c satisfying the relation $t - t_c^-(N) \sim N^{-1/\bar{\nu}}$, $m(t) - m_0$ scales as $N^{-\beta/\bar{\nu}}$. We emphasize that m_0 is not zero. Thus, the transition is discontinuous.

VIII. COMPARISON BETWEEN NUMERICAL RESULTS FROM SIMULATIONS AND FROM DIRECT INTEGRATION OF THE SMOLUCHOWSKI EQUATION

A. the order parameter

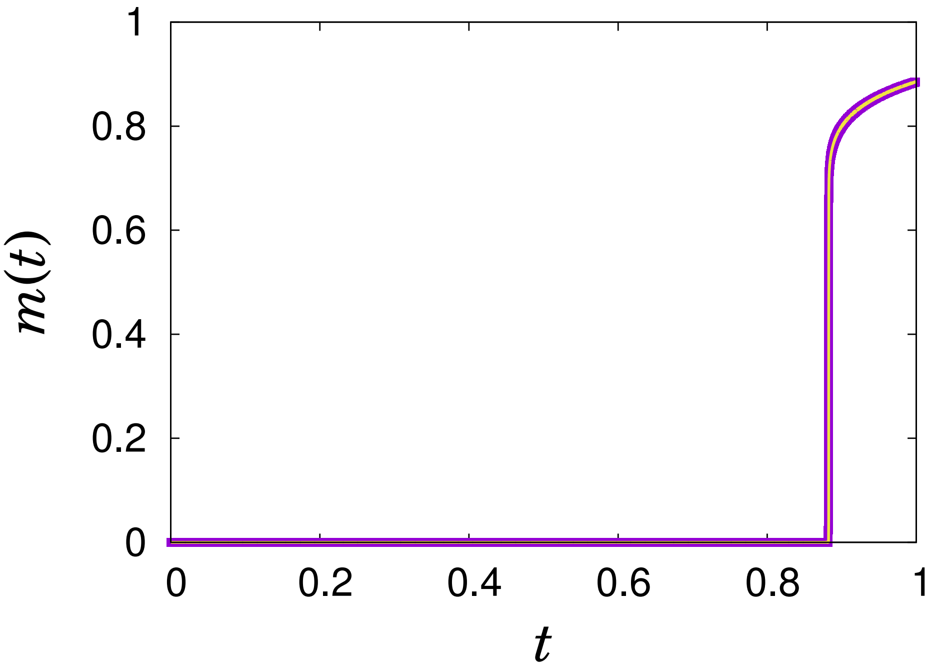

FIG. S7. Plot of $m(t)$ versus t with the data obtained from a numerical integration of the Smoluchowski equation (thick curve) and from numerical simulations (thin curve) for the system size $N = 4 \times 10^7$. The two curves cannot be distinguished. In numerical integration, Δt is chosen appropriately to take into account of finite-size effect.

B. the cluster size distribution

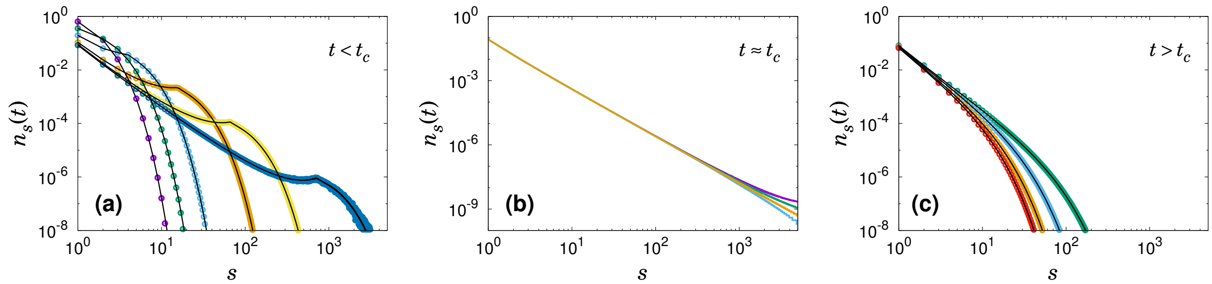

FIG. S8. Plot of $n_s(t)$ as a function of s (a) for $t < t_c$, (b) around $t = t_c$, and (c) for $t > t_c$. The data of $n_s(t)$ are obtained from numerical integration of the Smoluchowski equation (solid curve) and numerical simulations (data points). The data are for $g = 0.5$ and the simulations are carried out for $N = 4 \times 10^7$. $t_c = 0.8826$ was taken.

IX. HOW TO ESTIMATE A CRITICAL EXPONENT AND ITS ERROR BARS

We explain how to estimate the critical exponent, for instance, β associated with the order parameter and error bars for different system sizes.

i) We choose the largest feasible system size N for simulations.

ii) We determine (t_c, m_0) as the values for which the power-law region in the plot of $m - m_0$ versus $t - t_c$ in double logarithmic scales becomes the longest.

iii) Using the estimated values of (t_c, m_0), we plot $m - m_0$ versus $t - t_c$ for different system sizes, and measure local slopes. By measuring the mean value and the fluctuations of the local slopes, we estimate the values of the exponent β and its error bars.

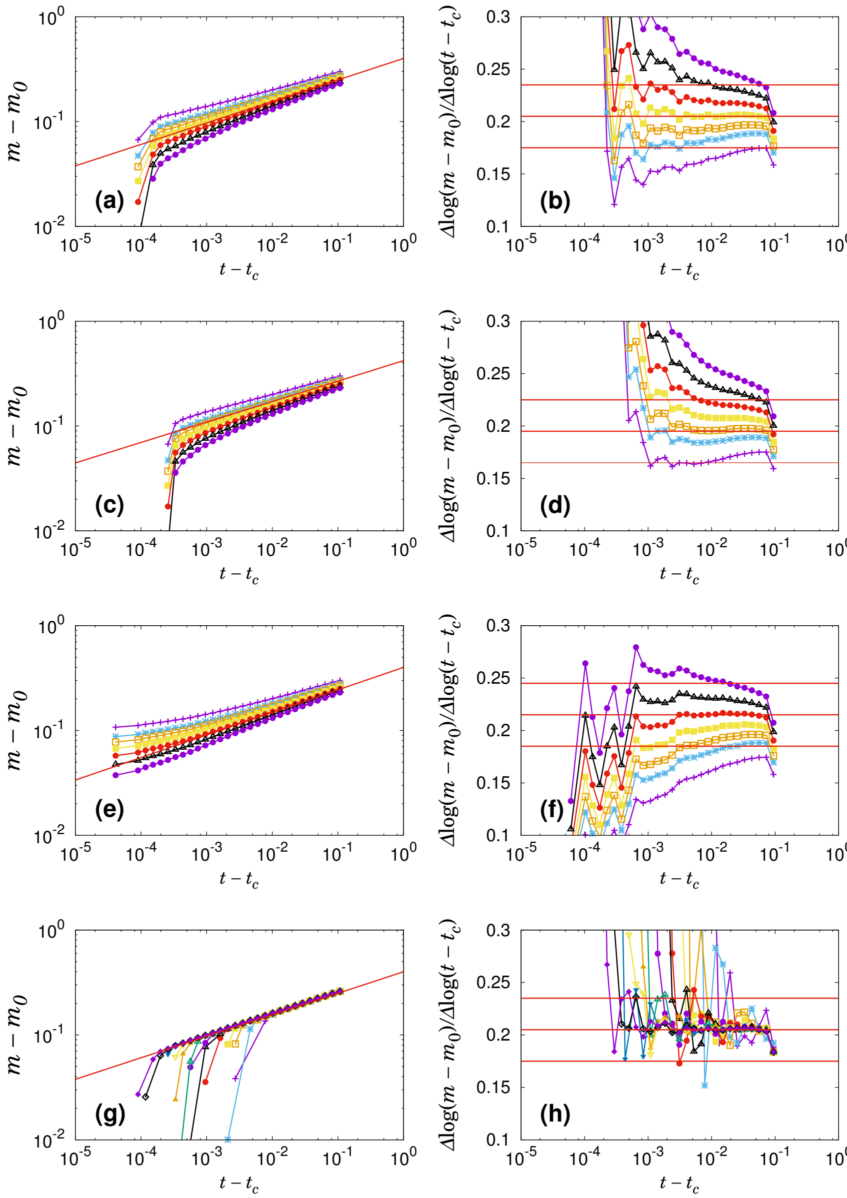

FIG. S9. Plots of $m - m_0$ vs $t - t_c$ with (a) $t_c = 0.88256$, (c) 0.8824, (e) 0.8827 and successive slopes of $m - m_0$ vs $t - t_c$ with (b) $t_c = 0.88256$, (d) 0.8824, (f) 0.8827. For each plot, $m_0 = 0.58, 0.6, 0.61, 0.62, 0.63, 0.64,$ and 0.65 are used. As m_0 increases, the successive slope increases. (g) Plots of $m - 0.62$ vs $t - 0.88256$ for different system sizes $N/10^4 = 2^0 - 2^{12}$. (h) Successive slopes of the data used in (g). The horizontal guidelines are (b) 0.205 ± 0.03, (d) 0.195 ± 0.03, (f) 0.215 ± 0.03, and (h) 0.205 ± 0.03. Based on these numerical results, $\beta = 0.21 \pm 0.05$ is estimated. $N = 4.096 \times 10^7$ and $g = 0.5$ were used.