Phase transition in the biconnectivity of scale-free networks

P. Kim,[1] D.-S. Lee,[2,*] and B. Kahng[1,†]

[1]*Department of Physics and Astronomy, Seoul National University, Seoul 151-747, Korea*

[2]*Department of Natural Medical Sciences and Department of Physics, Inha University, Incheon 402-751, Korea*

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In information-transport and biological systems, sometimes there is more than one pathway between two nodes, so that there is a backup in case one pathway becomes defective. The size of such biconnected nodes can be an important measure of the robustness of a system. The giant biconnected components of diverse real-world networks suggest the importance of scale-free topology in biconnectivity. Thus, here, we consider the critical behavior of the largest biconnected component (BC) as links are added and form a random scale-free network. The critical exponents $\beta_{(BC)}$ and $\beta_{(SC)}$ associated with the order parameter of the percolation transition of the biconnected component and the single-connected component (SC), respectively, are compared. We obtain a ratio $\beta_{(BC)}/\beta_{(SC)} = \lambda - 1$ for $2 < \lambda < 3$ and 2 for $\lambda > 3$, where λ is the exponent of the degree distribution in scale-free networks. We also determine the finite-size scaling behavior of the order parameter analytically and numerically.

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I. INTRODUCTION

The network approach has successfully unveiled the design and working principles of various real-world systems owing to the exploration of their interconnectivity patterns [1–3]. Graph theory has been used to quantitatively characterize the structure and function of complex systems [4,5]. In particular, the investigation of single-connected components (SCs), a set of nodes connected to one another by at least one path, has allowed a variety of useful results to be obtained because SCs sustain structural robustness and form the modules facilitating information transport among nodes. The existence of a giant SC, i.e., an SC of size comparable to the system size N, is considered an indicator of structural robustness of the entire system [6,7], which is necessary for achieving collective behavior at the system level [8].

As an extension of the SC, one may consider a biconnected component (BC), in which every pair of nodes has more than one node-disjoint path. Here two paths are considered as node disjoint if they do not have any node in common. In fact, the importance of biconnected and multiconnected components in complex networks has been recognized [9,10]. In the presence of external or internal perturbations, the existence of two disjoint paths connecting two nodes has an important implication: The nodes can communicate and function even under the failure of one pathway [10]. A system with a giant BC can therefore achieve functional stability even when some parts are broken. Further, it has been shown that the size of the giant BC essentially determines the transport properties of complex networks [9]. Thus, such backup pathways are essential in the architecture of the Internet and biological systems because overall functions can operate by cooperative activities of individual parts. For example, in *Escherichia coli* metabolic networks, there exist synthetic lethal reactions that are located along two disjoint pathways [11]. When they are removed simultaneously, the biomass of the organism is significantly reduced, which results in critical damage to the organism. By using a flux-balance analysis, it was found that one pathway of the pair is never used in the wild type, but it becomes activated in case the reactions in the other pathway are blocked. Thus, biconnected pathways are essential in biological systems.

The formalism for the relation between the sizes of a giant SC and a giant BC was derived in Ref. [10], and it was applied to completely random or Erdős-Rényi (ER) networks. However, many networks in real-world systems exhibit significantly heterogeneous connectivity patterns and contain macroscopic-size BCs that are different from the giant BCs in ER networks, as shown in Fig. 1. Nevertheless, little is known about the BCs in such heterogeneous networks, and thus, it is important to investigate the BCs in scale-free networks, which follow power-law degree distributions $P_d(k) \sim k^{-\lambda}$.

In this study, we consider the formation of the BC as the number of links is increased and derive an analytic solution of the percolation transition of a giant BC as a function of the mean degree $\langle k \rangle$, which is the average number of links per node. We find that in scale-free networks, such as ER networks, the giant BC and SC emerge at the same critical point. In particular, when the degree exponent λ is smaller than 3, the critical point is $\langle k \rangle_c = 0$, and thus, a giant BC appears for all nonzero $\langle k \rangle$. However, the growth of a giant BC beyond the critical point is slower than that of a giant SC. Interestingly, this difference is reduced significantly as λ approaches 2, which is shown in terms of the critical exponent associated with the order parameter. Furthermore, our theoretical approach enables us to derive the limiting behaviors of the finite-size scaling function up to the coefficients. We show that these analytic predictions agree very well with the numerical results obtained for scale-free model networks.

II. ANALYTIC RESULTS

In this section, we apply the branching process approach for the formation of a giant BC to the static model of scale-free networks [18,19]. It has been shown [10] that a node belongs to a giant BC if it has more than one link leading to a giant SC, as there are only BCs of size $O(1)$, except for the giant BC, if it exists. Let u be the probability for a link to *not* lead to a

*deoksun.lee@inha.ac.kr

†bkahng@snu.ac.kr

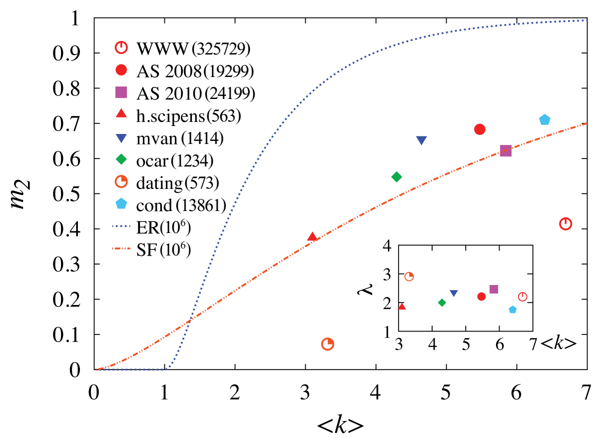

FIG. 1. (Color online) Sizes of the giant BCs of real-world scale-free (SF) networks. The relative sizes $m_2 = S_2/N$ of the World Wide Web (WWW) [12], the Internet at the autonomous system (AS) level in years 2008 and 2010 [13], the protein-protein interaction network of *Homo scipens* [14], the metabolic networks of two organisms (*Mycobacterium vanbaalenii* PYR-1 and *Oligotropha carboxidovorans* OM5) [15], the coauthorship network of the cond-mat archive [16], and the dating network from a study of US school students [17] are presented vs their mean degrees $\langle k\rangle$. The simulation results of the static model with $\lambda \to \infty$ and $\lambda = 2.2$ are also presented for comparison. The number of nodes in each network is given in parentheses. The data points of real networks are shown to be closer to the giant BCs of scale-free networks than to those of ER networks. The inset shows the estimated degree exponents of the considered real-world networks.

giant SC. Then, the relative size $m_2 = S_2/N$ of a giant BC in a network of system size N is evaluated by

$$m_2 = 1 - G(u) - (1-u)G'(u), \tag{1}$$

where $G(z) \equiv \sum_k P_d(k)z^k$ is the generating function of the degree distribution $P_d(k)$ of the considered network and $G'(z) = dG/dz$. Notice that $1 - G(u)$ corresponds to the probability for at least one link of a node to lead to a giant SC and $(1-u)G'(u)$ is the probability for exactly one link of a node to do so [10]. Thus, m_2 is the probability for a node to belong to a giant BC.

A. Probability u and the largest SC

The statistics of the SCs in the static model can be derived by investigating the ensemble of trees generated under the branching probability of k daughters, $q(k) = (k+1)P_d(k+1)/\langle k\rangle$, where $\langle k\rangle = 2L/N$ is the mean degree and L is the number of links in the system [20].

If the largest SC in a network, composed of S_1 nodes, is a tree structure, a link in the largest SC will lead to an SC of size ranging between 1 and $S_1 - 1$. The average size $\tilde{S}$ of such SCs is roughly $S_1/2$, and the number of links leading to an SC larger than $\tilde{S} = S_1/2$ is S_1, among $2L$ links, given two directions for each link. Thus we obtain

$$1 - u = \frac{S_1}{2L} = \frac{\tilde{S}}{L}. \tag{2}$$

If one considers the size distribution $R(s)$ of the SC reached by following a link in one direction, $\tilde{S}$ is determined by the condition $\sum_{s>\tilde{S}} R(s) = S_1/(2L) = \tilde{S}/L$. We will refer to $R(s)$ as the cluster-size distribution. Its generating function $\mathcal{R}(z) = \sum_{s=1}^{\infty} R(s)z^s$ satisfies the self-consistent equation [10,20,21]

$$\mathcal{R}(z) = z\frac{G'(\mathcal{R}(z))}{\langle k\rangle}. \tag{3}$$

In the static model [19], a link is assigned between nodes i and j with probability P_iP_j, where P_i is the weight of node i given by $P_i = i^{-\mu}/\zeta_N(\mu)$, with $0 \leqslant \mu < 1$, $\zeta_N(\mu) = \sum_{i=1}^{N} i^{-\mu}$, and $\mu = 1/(\lambda - 1)$. The degree distribution in the static model is analytically accessible through its generating function $G(z) = N^{-1}\sum_{i=1}^{N}\langle z^{k_i}\rangle$, which is expanded for z to be close to 1 as

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} g_{n,N}\,[\langle k\rangle(1-z)]^n \\ &\simeq \sum_{n=0}^{\infty} g_n\,[\langle k\rangle(1-z)]^n + g_{1/\mu}\,\langle k\rangle^{1/\mu}(1-z)^{1/\mu}, \end{aligned} \tag{4}$$

where the expansion in the second line holds in the limit $(1-z)\langle k\rangle N^{\mu} \to \infty$. The coefficients are [20]

$$g_{n,N} = (-1)^n(1-\mu)^n\zeta_N(\mu n)N^{\mu n-1}/\,\Gamma(n+1), \tag{5}$$

$$g_n = (-1)^n(1-\mu)^n/[(1-\mu n)\Gamma(n+1)], \tag{6}$$

$$g_{1/\mu} = -\Gamma(1-1/\mu)(1-\mu)^{1/\mu}. \tag{7}$$

It is remarkable that $g_{n,N} = g_n$ for $n < 1/\mu$. The singular term $(1-z)^{1/\mu}$ is related to the asymptotic behavior $P_d(k) \sim k^{-\lambda}$ with $\lambda = 1 + 1/\mu$.

From Eq. (1), one finds that m_2 is nonzero if and only if $u < 1$. Therefore, a giant BC appears at the same critical point as a giant SC [10]. In the thermodynamic limit $N \to \infty$, $u = \lim_{z\to 1}\mathcal{R}(z) = \sum_{s<\tilde{S}} R(s)$ is smaller than 1 as long as $\tilde{S}$ is of order N; i.e., the largest SC becomes a giant SC. By setting $z = 1$ in Eq. (3), we have

$$u = \frac{G'(u)}{\langle k\rangle}. \tag{8}$$

When the coefficient of the $(1-u)$ term on the right-hand side of Eq. (8) expanded in terms of $1-u$ is smaller than -1, u is found to be smaller than 1, from which we identify the critical point given by

$$\langle k\rangle_c = \frac{1}{2g_{2,N}} = \begin{cases} \frac{1-2\mu}{(1-\mu)^2} & \text{for} \quad \lambda > 3, \\ \frac{N^{1-2\mu}}{(1-\mu)^2\zeta(2\mu)} & \text{for} \quad 2 < \lambda < 3. \end{cases} \tag{9}$$

Here $\zeta(x) = \sum_{i=1}^{\infty} i^{-x}$ is the Riemann zeta function. It should be noted that, for $2 < \lambda < 3$, $g_{2,N}$ diverges and $1-u$ is nonzero for all $\langle k\rangle > 0$ in the thermodynamic limit. For convenience, we divide the range of μ into three regions:

(i) $0 \leqslant \mu < \frac{1}{3}$ $(4 < \lambda < \infty)$,

(ii) $\frac{1}{3} < \mu < \frac{1}{2}$ $(3 < \lambda < 4)$,

(iii) $\frac{1}{2} < \mu < 1$ $(2 < \lambda < 3)$.

In the supercritical phase with $\langle k\rangle > \langle k\rangle_c$ and $u < 1$, it holds that $(1-u)\langle k\rangle N^{\mu} \to \infty$, and we use the second line of Eq. (4) to obtain the behavior of $1-u$ depending on μ for

$0 < \Delta = \langle k\rangle/\langle k\rangle_c - 1 \ll 1$ as

$$1-u \simeq \begin{cases} a_{(\mathrm{I})}\ \Delta & (\mathrm{I}),\\ a_{(\mathrm{II})}\ \Delta^{\mu/(1-2\mu)} & (\mathrm{II}),\\ a_{(\mathrm{III})}\ \langle k\rangle^{(1-\mu)/(2\mu-1)} & (\mathrm{III}),\end{cases} \quad (10)$$

where

$$a_{(\mathrm{I})} = -\frac{1}{3g_3\langle k\rangle_c^2},\quad a_{(\mathrm{II})} = \left(-\frac{\mu}{g_{1/\mu}\langle k\rangle_c^{1/\mu-1}}\right)^{\mu/(1-2\mu)},$$
$$a_{(\mathrm{III})} = \left(\frac{g_{1/\mu}}{\mu}\right)^{\mu/(2\mu-1)}. \quad (11)$$

For finite N, the value of $1-u$ may be nonzero even at the critical point, as $\tilde{S}/L$ is not zero in Eq. (2). The large-s behavior of $R(s)$ is related to the singularity of its generating function $\mathcal{R}(z)$ for z close to 1: If $\mathcal{R}(z) \simeq 1 - c(1-z)^{\tau-1}$ with τ a noninteger and c a constant, then $R(s) \simeq -c/\Gamma(-\tau+1)s^{-\tau}$ for $s \gg 1$. By expanding Eq. (3) for small $1-z$ and $1-\mathcal{R}(z)$, one finds the behavior of $\mathcal{R}(z)$ for z close to 1:

$$1-\mathcal{R}(z) \simeq \begin{cases} a_{(\mathrm{I})}^{1/2}(1-z)^{1/2} & (\mathrm{I}),\\ a_{(\mathrm{II})}^{\frac{1-2\mu}{1-\mu}}(1-z)^{\frac{\mu}{1-\mu}} & (\mathrm{II}),\\ a_{(\mathrm{III})}^{\frac{2\mu-1}{\mu}}\langle k\rangle_c^{\frac{1-\mu}{\mu}}(1-z)^{\frac{1-\mu}{\mu}} & (\mathrm{III}),\end{cases} \quad (12)$$

with $a_{(\mathrm{I})}$, $a_{(\mathrm{II})}$, and $a_{(\mathrm{III})}$ given in Eq. (11). Note that $\langle k\rangle_c$ for $\mu > 1/2$ is of the order $N^{1-2\mu}$.

Using $R(s)$ obtained from Eq. (12) in the relation $\sum_{s>\tilde{S}} R(s) = \tilde{S}/L$, one obtains $\tilde{S}$ and, in turn, the value of $1-u$ in the critical regime, given by

$$1-u \simeq \begin{cases} b_{(\mathrm{I})}\ N^{-1/3} & (\mathrm{I}),\\ b_{(\mathrm{II})}\ N^{-\mu} & (\mathrm{II}),\\ b_{(\mathrm{III})}\ N^{-(1-\mu)} & (\mathrm{III}),\end{cases} \quad (13)$$

where

$$b_{(\mathrm{I})} = \left[2a_{(\mathrm{I})}^{1/2}/|\Gamma(-1/2)|\right]^{2/3}(2/\langle k\rangle_c)^{1/3},$$
$$b_{(\mathrm{II})} = \left[\frac{(1-\mu)a_{(\mathrm{II})}^{(1-2\mu)/(1-\mu)}}{|\Gamma[-\mu/(1-\mu)]|\mu}\right]^{1-\mu}\left(\frac{2}{\langle k\rangle_c}\right)^{\mu}, \quad (14)$$
$$b_{(\mathrm{III})} = \left[\frac{\mu a_{(\mathrm{III})}^{(2\mu-1)/\mu}}{|\Gamma(1-1/\mu)|(1-\mu)}\right]^{\mu} 2^{1-\mu} = 2^{1-\mu}(1-\mu)^{1-\mu}.$$

The relative size of the largest SC is evaluated by using u in the relation $m_1 = S_1/N = \langle k\rangle(1-u)$, which has been studied in Ref. [20].

B. Finite-size scaling of the largest BC size

Although we obtained the probability u under the tree-structure assumption, it is valid around the critical point [20]. We insert Eq. (4) into Eq. (1) to obtain m_2 as

$$m_2 \simeq \begin{cases} \sum_{n=2}^{\infty} g_{n,N}\langle k\rangle^n(n-1)(1-u)^n & \text{for}\quad N^{\mu}\langle k\rangle(1-u) \ll 1,\\ \sum_{n=2}^{\infty} g_n\langle k\rangle^n(n-1)(1-u)^n + g_{1/\mu}\left(\frac{1}{\mu}-1\right)\langle k\rangle^{1/\mu}(1-u)^{1/\mu} & \text{for}\quad N^{\mu}\langle k\rangle(1-u) \gg 1.\end{cases} \quad (15)$$

We note that $g_{n,N} = g_n$ for $n < 1/\mu$ and $g_{n,N} = O(N^{\mu n-1})$ for $n > 1/\mu$.

Next, we determine the dominant term in formulas (15) of m_2 for each regime. For regions (I) and (II) ($\lambda > 3$), the dominant contribution is made by the term $g_2\langle k\rangle_c^2(1-u)^2$ [10]. This term yields in region (I)

$$m_2 \simeq \begin{cases} g_2\langle k\rangle_c^2 a_{(\mathrm{I})}^2\Delta^2 & (\text{supercritical phase}),\\ g_2\langle k\rangle_c^2 b_{(\mathrm{I})}^2 N^{-2/3} & (\text{critical phase}),\end{cases} \quad (16)$$

and in region (II), it yields

$$m_2 \simeq \begin{cases} g_2\langle k\rangle_c^2 a_{(\mathrm{II})}^2\Delta^{2\mu/(1-2\mu)} & (\text{supercritical phase}),\\ g_2\langle k\rangle_c^2 b_{(\mathrm{II})}^2 N^{-2\mu} & (\text{critical phase}).\end{cases} \quad (17)$$

In region (III) ($2 < \lambda < 3$), the term $(1/\mu - 1)g_{1/\mu}\langle k\rangle^{1/\mu}(1-u)^{1/\mu}$ is dominant in the supercritical phase, which gives

$$m_2 \simeq g_{1/\mu}\left(\frac{1}{\mu}-1\right)a_{(\mathrm{III})}^{1/\mu}\langle k\rangle^{1/(2\mu-1)}. \quad (18)$$

In the critical regime, the value of m_2 has contributions from all the terms in the expansion of Eq. (1), as $g_{n,N}\langle k\rangle^n(1-u)^n = O(N^{-1})$ for all $n \geqslant 2$.

We summarize the behavior of m_2 in the critical and supercritical phases in the scaling form as

$$m_2 \simeq \begin{cases} N^{-\beta/\nu}\Phi(\Delta N^{1/\nu}) & (\mathrm{I})\ \text{and}\ (\mathrm{II}),\\ N^{-1}\Phi(\Delta) & (\mathrm{III}),\end{cases} \quad (19)$$

where the finite-size scaling function $\Phi(x)$ behaves as

$$\Phi(x) \sim \begin{cases} \phi & \text{for}\quad x \to 0,\\ \psi x^{\beta} & \text{for}\quad x \to \infty,\end{cases} \quad (20)$$

with ϕ and ψ as constants. The critical exponents β and ν are derived from Eqs. (10) and (13) as

$$(\beta,\nu) = \begin{cases} (2,3) & (\mathrm{I}),\\ \left(\frac{2}{\lambda-3},\frac{\lambda-1}{\lambda-3}\right) & (\mathrm{II}),\\ \left(\frac{\lambda-1}{3-\lambda},\right) & (\mathrm{III}).\end{cases} \quad (21)$$

We remark that the exponent ν is not available for $2 < \lambda < 3$. The coefficients ϕ and ψ are determined analytically by using the coefficients a and b in Eqs. (11) and (14) as

$$(\phi,\psi) \simeq \begin{cases} \frac{\langle k\rangle_c}{2}\times\left(b_{(\mathrm{I})}^2,a_{(\mathrm{I})}^2\right) & (\mathrm{I}),\\ \frac{\langle k\rangle_c}{2}\times\left(b_{(\mathrm{II})}^2,a_{(\mathrm{II})}^2\right) & (\mathrm{II}),\end{cases} \quad (22)$$

and

$$\phi = \sum_{n=2}^{\infty} \frac{(n-1)\zeta\left(\frac{n}{\lambda-1}\right)}{\Gamma(n+1)} \left(-\frac{(\lambda-1)b_{(\mathrm{III})}}{(\lambda-2)\zeta[2/(\lambda-1)]}\right)^n,$$

$$\psi = \frac{1-\mu}{\mu} g_{1/\mu} a_{(\mathrm{III})}^{1/\mu} \tag{23}$$

for region (III).

The exponent β characterizes how fast a giant BC forms with an increasing number of links around the critical point. The fact that β values for giant BCs are larger than those for giant SCs [20] indicates the slower formation of giant BCs. The exponent β for giant BCs is twice that for giant SCs: $\beta_{(\mathrm{BC})} = 2\beta_{(\mathrm{SC})}$ for $\lambda > 3$. However, the formation of giant BCs is not as slow for $2 < \lambda < 3$ as for $\lambda > 3$. For $2 < \lambda < 3$, the relative size of giant SCs is given by $m_1 \simeq \langle k\rangle(1-u) \sim \langle k\rangle^{1/(3-\lambda)}$, where u is given by Eq. (10) [20]. Therefore, $\beta_{(\mathrm{BC})} = (\lambda-1)\beta_{(\mathrm{SC})}$, with their ratio $\beta_{(\mathrm{BC})}/\beta_{(\mathrm{SC})}$ being less than 2. The two exponents $\beta_{(\mathrm{BC})}$ and $\beta_{(\mathrm{SC})}$ become equal as $\lambda \to 2$. In contrast to such a difference between the exponents β of the giant SC and BC, the width of the critical regime, characterized by the exponent ν, is identical for both giant components, independent of λ.

III. NUMERICAL RESULTS

Numerically, the BCs of a network can be determined by applying the algorithms of, for example, Refs. [22,23]. To test the analytic results, we constructed networks for a static model with system size ranging from $N = 2\times10^6$ to 1.6×10^7 and $\lambda = 2.4$ ($\mu = 5/7$), $\lambda = 3.6$ ($\mu = 5/13$), and $\lambda \to \infty$ ($\mu = 0$). Next, we identified the largest SC and the largest BC numerically while increasing $\langle k\rangle$ [22–24]. We averaged the relative sizes m_2 and m_1 of the giant BC and the giant SC, respectively, over 10^3 different network realizations.

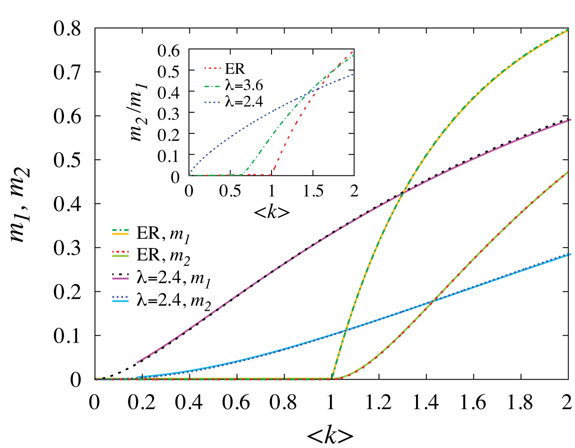

FIG. 2. (Color online) Relative sizes of a giant BC and giant SC in ER and scale-free networks. Plots of $m_1 = S_1/N$ and $m_2 = S_2/N$ as functions of the mean degree $\langle k\rangle$ in the ER networks and the scale-free networks with $\lambda = 2.4$ are presented. Dotted and solid lines represent numerical simulation and analytical results from Eqs. (1) and (8) and the relation $m_1 = 1 - G(u)$, respectively. In numerical simulations, the number of nodes N is 8×10^6 in both networks. The inset shows the ratio of m_2 to m_1 in the same networks and in SF networks with $\lambda = 3.6$.

As evident by definition, the size of the largest BC is smaller than that of the largest SC in all the considered sparse networks; the size of the largest BC is at most 60% of the largest SC up to $\langle k\rangle = 2$. The trend of the ratio m_2/m_1 displays a transition at

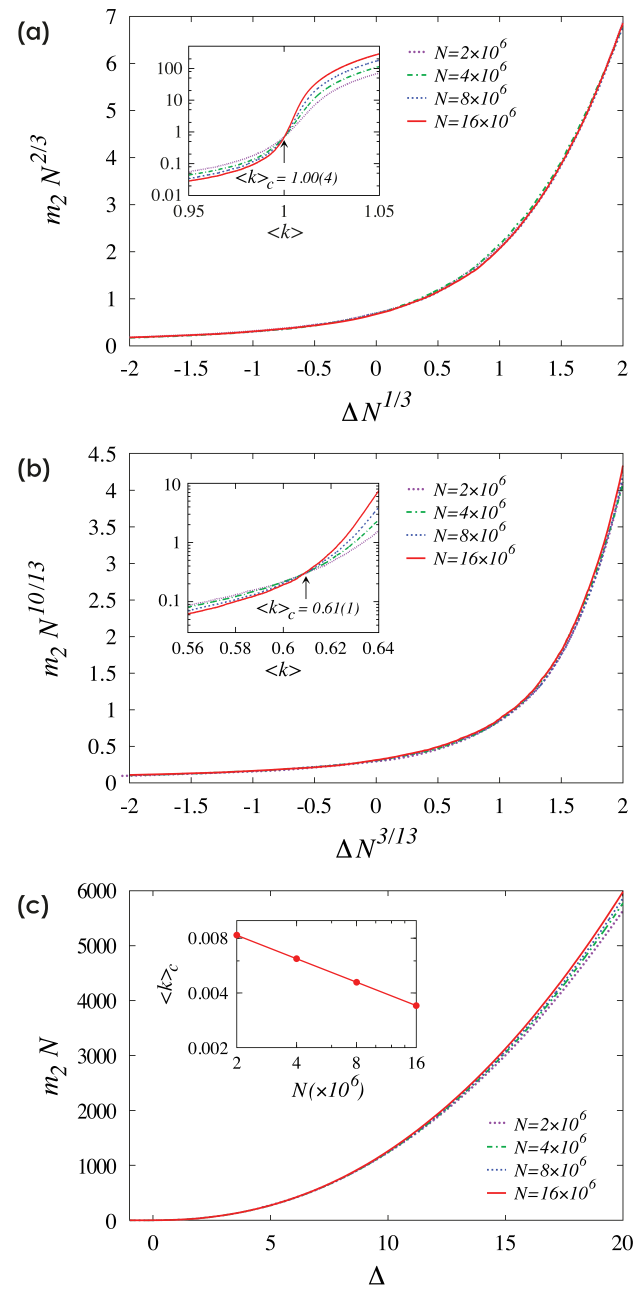

FIG. 3. (Color online) Finite-size scaling behavior of the relative size m_2 of the largest BC in scale-free networks. Plots of $m_2N^{\frac{\beta}{\nu}}$ vs $\Delta N^{1/\nu}$ with (a) $\beta = 2$, $\nu = 3$ for $\lambda \to \infty$ ($\mu = 0$) and (b) $\beta = 10/3$, $\nu = 13/3$ for $\lambda = 3.6$ ($\mu = 5/13$) are presented. (c) Plot of $m_2 N$ vs Δ. Here, $\Delta = \langle k\rangle/\langle k\rangle_c - 1$. In the insets of (a) and (b), the curves for $m_2 N^{\beta/\nu}$ vs $\langle k\rangle$ from different N values intersect at $\langle k\rangle_c = 1.00(4)$ for $\lambda \to \infty$ in (a) and at $\langle k\rangle_c = 0.61(1)$ for $\lambda = 3.6$ in (b). For the case of $\lambda = 2.4$ in (c), we used Eq. (9) to compute $\langle k\rangle_c$ given in the inset; this value was used to draw the scaled plot in the main panel. All the presented results are obtained from numerical simulations.

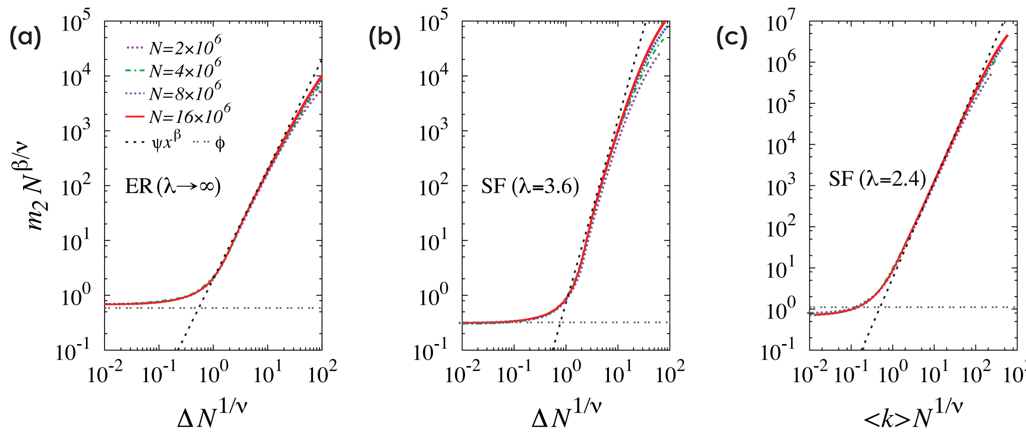

FIG. 4. (Color online) Behaviors of the finite-size scaling function $\Phi(x)$. The collapsed data are compared with the predicted behavior $\Phi(x) \to \phi$ for $x \to \infty$ and $\Phi(x) \simeq \psi x^{\beta}$ for $x \to \infty$ in Eqs. (20)–(23). (a) For $\lambda \to \infty$, the scaling variable is $x = \Delta N^{1/3}$, and the limiting behaviors of $\Phi(x)$ with $\phi = 0.587$, $\psi = 2$, and $\beta = 2$ agree with the collapsed data of $m_2 N^{2/3}$. (b) For $\lambda = 3.6$, the scaling variable is $x = \Delta N^{3/13}$, and the limiting behaviors of $\Phi(x)$ with $\phi = 0.321$, $\psi = 0.729$, and $\beta = 10/3$ agree with the collapsed data of $m_2 N^{10/13}$. (c) For $\lambda = 2.4$, the scaling variable is $x = \Delta$, and the limiting behaviors of $\Phi(x)$ with $\phi = 1.111$, $\psi = 0.203$, and $\beta = 7/3$ agree with the collapsed data of $m_2 N$.

a nonzero critical point for $\lambda > 3$, whereas it is nonzero for all nonzero $\langle k \rangle$ for $2 < \lambda < 3$, as shown in Fig. 2. In this case, even when $\langle k \rangle$ is small, the largest BC and SC are large in scale-free networks, whereas those of the ER networks are negligible. This implies that the presence of hubs is advantageous for forming large BCs and SCs, which is helpful for maintaining functional activities of the system.

The finite-size scaling behavior of m_2 is presented in Fig. 3. We used the scaling exponents β and ν of Eq. (21) to identify the critical point $\langle k \rangle_c$ and checked the collapse of the data from different N values in Figs. 3(a) and 3(b). The critical points, identified by the intersection of the curves for $m_2 N^{\beta/\nu}$ versus $\langle k \rangle$, agree with the prediction in Eq. (9). The data collapse of $m_2 N^{\beta/\nu}$ versus $\Delta N^{1/\nu}$ is almost perfect, confirming the finite-size scaling behaviors derived analytically in Sec. II. For $\lambda = 2.4$, $\langle k \rangle_c$ is zero in the limit $N \to \infty$ but nonzero for finite N, as in Eq. (9). In Fig. 3(c), we use the nonzero critical point in Eq. (9) to find the scaling behavior of m_2 that agrees with Eq. (19).

We derived analytically the limiting behaviors of the scaling function $\Phi(x)$ in Eq. (20); these are compared with the collapsed data of $m_2 N^{\beta/\nu}$ versus $\Delta N^{1/\nu}$ for $\lambda > 3$ or versus Δ for $2 < \lambda < 3$. In Fig. 4, the numerical results show good agreement with the analytic prediction of the behavior of the scaling function $\Phi(x)$ for $x \to 0$ and $x \to \infty$. Such a good agreement of even coefficients verifies the assumptions made in the analytic derivation of the largest BC in Sec. II.

IV. DISCUSSION

The biconnectivity of a network represents the ability of a system to maintain its functionality in the presence of broken pathways and is of practical importance in technological and biological networks. Motivated by the ubiquity of scale-free topology in real-world systems, we investigated the critical phenomena of the emergence of giant BCs in scale-free networks within the framework of the branching process approach. As in ER networks [10], the giant BC grows slower than the giant SC in scale-free networks; $\beta_{(\mathrm{BC})} = 2\beta_{(\mathrm{SC})}$ for $\lambda > 3$ and $\beta_{(\mathrm{BC})} = (\lambda - 1)\beta_{(\mathrm{SC})}$ for $2 < \lambda < 3$. However, as $\lambda \to 2$, the difference between the two β exponents is reduced, implying that the hub nodes facilitate the formation of the giant BC as well as the giant SC.

We derived the limiting behaviors of the finite-size scaling functions for the size of the largest BC and the critical point; our results agree very well with the numerical results. Our analytic approach can be extended to the case of $(k \geqslant 3)$ components. Given the increasing demand for maintaining stable functionality of technological, social, and biological networks, our results can be of importance for improving the structural and functional robustness of numerous complex systems.

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