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BRANCH ANNIHILATION RANDOM WALKS IN ENERGETIC DISORDERED STRUCTURE AND DIRECTED POLYMER

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The problem of the branch annihilation random walks (BAW) in energetic disordered structure is introduced and studied on the checkerboard lattice of one spatial and one temporal dimensions. The energy disorder we considered is the one used in the problem of directed polymer (DPO) at zero temperature including the external field F. The evolution rule of BAW is assigned to relate to the energy disorder. It is found that depending on the value of F, a variety of phases appear, and the crossover from the DPO state to the directed percolation (DPE) state occurs. When $F > 0$, the problem is reduced to the DPO problem at zero temperature. However, for $F < 0$, there exist three states, the inactive state for $F_c < F < 0$, the active state for $-1 < F < F_c$, and the non-hopping state for $F < -1$, where F_c is the threshold value from the inactive state to the active state. The phase transition at F_c turns out to be in the DPE universality class. The robustness of the DPE universality is discussed by the Harris criterion.

The subject of nonequilibrium phase transitions is important not only in the field of physics, but also in other various fields such as chemistry, biology, and even ecology.[1,2] Recently a lot of models have been introduced in association with non-equilibrium phase transitions, such as the contact process,[3] the Schlögl's first and second models for auto-catalytic reactions,[4] and the monomer and dimer model, etc.[5–7] Even though such different models include different details of kinetic rules, most of reaction models exhibit universal features of directed percolation (DPE) universality class in $(d+1)$ dimensions.[8,9] The problem of the branch annihilation random walk (BAW) is one of models exhibiting non-equilibrium phase transition.[10,11] The original evolution rule of the BAW, which was introduced by Takayasu and Tretyakov, is defined as following. A random walker hops to one of the nearest neighbor sites with probability p. If the site is already occupied by another random walker, then the two walkers annihilate with probability 1, so that the site becomes empty. With probability $1-p$, the random walker produces n-offspring, which are

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placed on the closest neighboring places. In this model, when the probability p is relatively small, then the motion of the BAW is mainly branching instead of hopping process, so that the number of random walkers grows with increasing time, which is called the active state. However, for relatively large p, the motion of the BAW is mainly the hopping process to one of the nearest neighbor sites with few branching, so that the number of particles is bound, which is called the inactive state. Then there exists a finite threshold of probability p_c, including in general the cases of $p_c = 0$ or $p_c = 1$,[12] across which the phase transition occurs from the inactive state to the active state. In the BAW problem, it is interesting that there exist two kinds of universality classes depending on the parity of n, regardless of the magnitude of n. When n is odd, the phase transition belongs to the DPE universality class, and while when n is even, a different universality class from the DPE class appears, which is called the modulo-2 class. It is not clear yet why the universality class is classified only by the parity of n. Many modifications from the BAW model have been introduced, but the models in the modulo-2 class are rare. For example, the case of including spontaneous annihilation in the BAW with four offsprings does change the universality from the modulo-2 to the DPE class.[13] Besides this example, most of kinetic reaction models belong to the DPE class regardless of detailed kinetic rules, and accordingly the DPE universality class is robust.

The physical quantities characterizing the phase transition of the BAW are as following. The steady state concentration of particles $\bar{\rho}$, which plays a role of an order parameter, behaves as

$$\bar{\rho} \propto |p_c - p|^{\beta}, \tag{1}$$

for $p < p_c$. The averaged number of random walkers $\bar{N}(t)$ increases with increasing time before reaching the steady state as

$$\bar{N}(t) \propto t^{\eta} f_1(\Delta t^{1/\nu_{\|}}), \tag{2}$$

where $\Delta = |p - p_c|$. f_1 is a scaling function, and $\nu_{\|}$ is the correlation length exponent in temporal direction. This quantity is averaged over all trials. The surviving probability $P(t)$ means the fraction of surviving configurations out of all trials at time t, which decays as

$$P(t) \propto t^{-\delta} f_2(\Delta t^{1/\nu_{\|}}), \tag{3}$$

where f_2 is a scaling function. The mean square spreading of random walks from the starting position behaves as

$$\bar{R}^2(t) \propto t^{2z} f_3(\Delta t^{1/\nu_{\|}}), \tag{4}$$

where f_3 is other scaling function, and the quantity $\bar{R}^2(t)$ is averaged over surviving configurations. The scaling function $f_1(x)$ behaves as $f_1(x) \to$ const for $x \to 0$, and it decays exponentially with respect to the argument x for $x \gg 1$. The other scaling functions behave in a similar manner. The critical exponents β, η, δ, z are no longer

independent of each other, but there exists the scaling relation of $2\delta + \eta = dz$ with spatial dimension d. The values of the exponents for the DPE class are well known as $\beta_{\mathrm{DPE}} \approx 0.2767$, $\eta_{\mathrm{DPE}} \approx 0.3137$, $\delta_{\mathrm{DPE}} \approx 0.1596$, and $z_{\mathrm{DPE}} \approx 0.6330$ in $(1+1)$-dimensions. Numerical values for the problem of the BAW have been obtained by using the dynamic and steady Monte Carlo simulations, of which the method for analysis can be found in many references, for example, Ref. 11.

In this paper, we consider the problem of BAW in energetic disordered media by taking analogy to the problem of directed polymer (DPO) at zero temperature[14,15] with introducing branching and annihilation processes. The detailed evolution rules of the BAW will be introduced later. Instead, at this stage, it is worthwhile to recall the problem of DPO from the viewpoint of biased random walks in random potential along the direction of increasing time in $(d+1)$-dimensional phase space. For convenience, we consider the problem in $(1+1)$ dimensions. Specifically, one may consider two-dimensional checkerboard lattice, rotated 45 degrees of the square lattice, which is composed of one spatial and one time axes. At each site of the lattice, random potential $\eta(x,t)$ is assigned, which is represented by random number between 0 and 1. In the problem of DPO, directed walks are taken along the path of minimum energies, and the energy at each site is given recursively as

$$E(x,t) = \mathrm{Min}\left\{E\left(x-\frac{1}{2},t-1\right), E\left(x+\frac{1}{2},t-1\right)\right\} + \eta(x,t)\,. \tag{5}$$

Then the trajectory of the directed random walks corresponds to the chain of DPO. The diffusion equation for the biased random walks in the random potential is written as

$$\frac{\partial W}{\partial t} = [\nu\nabla^2 + \eta(x,t)]W\,, \tag{6}$$

where $W(x,t)$ represents the probability of being a random walker at site (x,t). The above equation is transformed by using the Cole–Hopf transformation, $W = \exp[h/2\nu]$, into the Kardar–Parisi–Zhang equation,[16]

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{1}{2}(\nabla h)^2 + \eta(x,t)\,, \tag{7}$$

which may be applied to various physical phenomena, for example, the problem of kinetic surface growth with lateral growing effect, etc.

We note that the energy $E(x,t)$ given by Eq. (5) increases with respect to time, because the random potential takes positive value. However, when the random potential is shifted by external field F, i.e. $\eta \to \eta + F$, the energy $E(x,t)$ does not always increase for negative F. The energy may decrease depending on the magnitude of negative field F. In particular, for $F < -1$, the energy always decreases with increasing time. For comparison, the external field F corresponds to the

external driving force in the KPZ equation for surface growth problem, and the Eq. (7) changes into

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{1}{2}(\nabla h)^2 + F + \eta(x,t)\,. \tag{8}$$

At this stage, we introduce the evolution rule of the BAW which we study in this paper. The basic principle of the evolution rule is that a biased random walker at the site (x,t) can diffuse to the nearest neighbor site(s) $(x+\frac{1}{2},t+1)$ and/or $(x-\frac{1}{2},t+1)$ as long as the energy(ies) at time $t+1$ is(are) lower than the energy $E(x,t)$. To be specific, suppose that the site (x,t) with energy $E(x,t)$ is occupied with a biased random walker. If both energies, $E(x+\frac{1}{2},t+1)$ and $E(x-\frac{1}{2},t+1)$, of the nearest neighbor sites at time $t+1$, are less than the energy $E(x,t)$, then both sites are occupied, which means the random walker branches into two pieces, itself and one offspring at $(x+\frac{1}{2},t+1)$ and $(x-\frac{1}{2},t+1)$. If one of the energies is less than $E(x,t)$, then the site of less energy is only occupied, which corresponds to the hopping process of random walks. However, if both of the nearest neighbor sites have higher energies than $E(x,t)$, then one of the sites having less energy among the two sites at time $t+1$ is taken on the assumption that the energy $E(x,t)$ is raised as was done in the problem of DPO. Due to the branching process, many biased random walkers could coexist. If two biased walkers meet at the same site and time, then they annihilate each other, and make the site empty, which is the annihilation process. For convenience, the evolution rules are depicted in Fig. 1. Since $E(x,t)$ is correlated spatio-temporally, branching and annihilation events are implicitly correlated with each other. Accordingly, the problem considered in this paper is to check how robust the BAW universality is under the correlated disorder.

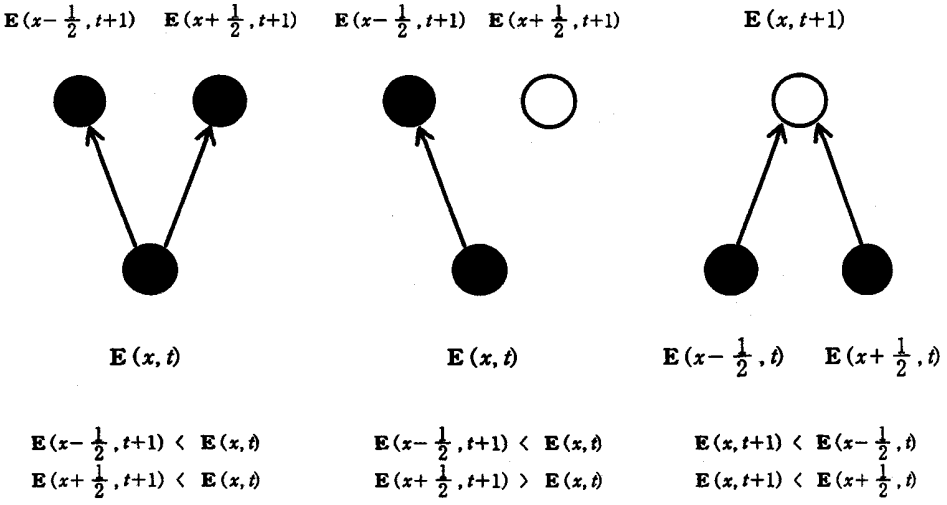

Fig. 1. The schematic evolution rule of BAW in the energetic disordered space.

We performed the dynamic Monte Carlo simulations for the evolution rule of the BAW in the energetic disorder. For the dynamic Monte Carlo simulations, we took a huge-sized array of sites, and then the maximum Monte Carlo time step is taken as the half of the array-size to account for the extreme case that random walks may take steps in one direction all the ways. Then we do not need to take care of any boundary effect. For the steady Monte Carlo simulations, the periodic boundary condition was used. Since the evolution rule depends on the embedded energy disorder, we once updated energies at each time following the recurrence relation, which is used in the problem of the DPO,

$$E(x,t) = \mathrm{Min}\left\{E\left(x-\frac{1}{2},t-1\right),E\left(x+\frac{1}{2},t-1\right)\right\}+\eta(x,t)+F\,, \tag{9}$$

and applied the evolution rule to all occupied sites simultaneously. The initial energies $\{E(x,0)\}$ were taken as zero. We believe that this choice does not yield any different result from the case of random initial energies $\{E(x,0)\}$, provided $\{E(x,0)\}$ and $\{\eta(x,0)\}$ are of same type. The synchronizing update is different from the synchronizing update normally used in the problem of BAW, but the results from both methods are expected to be same. However, we think that the synchronized simulations we used are very realistic. We have found that there exist four states depending on the magnitude of applying field F, shown in Fig. 2. When $F > 0$, the energy $E(x,t)$ increases with respect to time, so that branching and annihilation processes do not occur, and the number of random walkers remains to be one. This case corresponds to the DPO at zero temperature: The scaling exponent z has the well known value, $z_{\mathrm{DPO}} = 3/2$ from the Burgers equation. Also the energy fluctuation defined as

$$\Delta E(t) = \langle (E - \langle E\rangle)^2\rangle^{1/2} \sim t^{\omega}\,, \tag{10}$$

has the well known exponent value $\omega_{\mathrm{DPO}} = 1/3$ from the Burgers equation. The numerical data for $\Delta E(t)$ is plot in Fig. 3, and the measured value of ω is consistent to $1/3$.

For $-1 < F < 0$, we have found that there exists a phase transition at F_c from the inactive state to the active state. In the inactive state ($F_c < F < 0$), the number of occupied sites is zero, while in the active state ($-1 < F < F_c$), the number of occupied sites increases without bound. In order to estimate the threshold value, we investigate a physical quantity, the surviving probability $P(t)$.

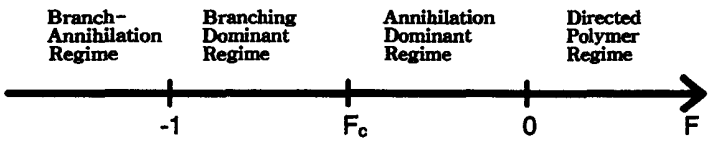

Fig. 2. The phase diagram of the BAW depending on the parameter F.

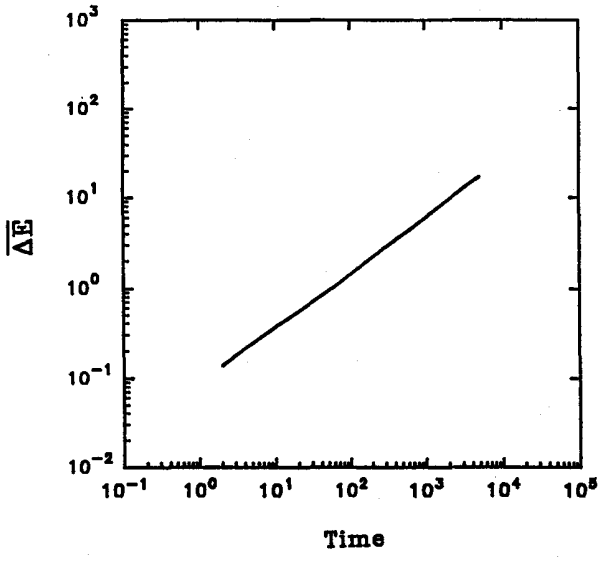

Fig. 3. The energy fluctuations $\Delta E(t)$ versus time.

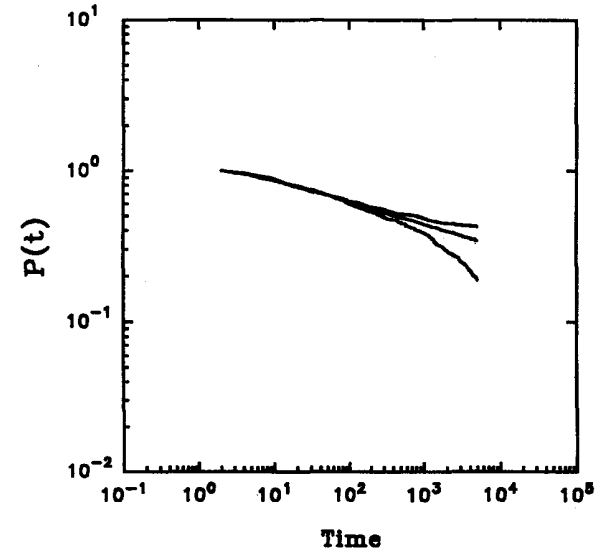

Fig. 4. The surviving probability $P(t)$ versus time for $F = -0.6500$, -0.6418, and -0.6350 from the top.

The surviving probability $P(t)$ means the fraction of configurations containing at least one random walker out of total number of configurations. At the threshold, the survival probability is expected to decay as the power law, $P(t) \sim t^{-\delta}$. However,

for the inactive (active) state, the survival probability decays faster (slower) than the power law. In Fig. 4, the survival probability is plotted against time in double logarithmic scales. As seen in this figure, the data for estimated value of F_c seem to be on the straight line, but they are on the curves vented upward (downward) for $F > F_c$ ($F < F_c$). The value of the exponent δ can be obtained by measuring the local slope,

$$-\delta(t) = \frac{\ln[P(t)/P(t/m)]}{\ln m}, \tag{11}$$

where m means the step of measurement, and it was taken as $m = 5$. The local slope would be constant as $1/t \to 0$ at the threshold, provided that the threshold value is correctly found out. We examined the behavior of the local slope $\delta(t)$ against $1/t$ for several values of $F = -0.6500, -0.6418, -0.6350$ in Fig. 5. As seen in the figure, the local slope seems to be constant for $F = -0.6418$, but they are curved upward (downward) for $F = -0.6500$ ($F = -0.635$). We also checked with several other values between $F = -0.6415$ and -0.6420, but the behavior of $\delta(t)$ is close to the case for $F = -0.6418$ or even worse. Accordingly we took $F_c = -0.6418$ as the threshold value. The steady value of the local slope in Fig. 5, which corresponds to the exponent δ, is estimated as $\delta \approx 0.153$. The value we measured turns out to be close to the value of $\delta_{\mathrm{DPE}} = 0.1596(4)$ for the DPE class, which suggests that our model belongs to the DPE class.

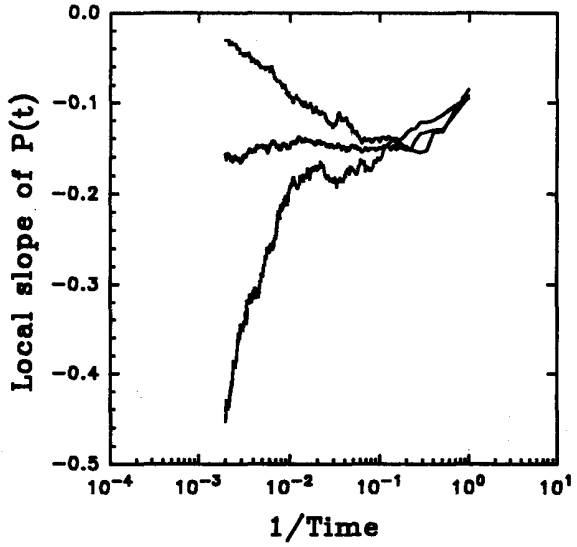

Fig. 5. The local slope $\delta(t)$ versus inverse time to determine the threshold.

We examine the behavior for the averaged spreading of occupied sites from the starting position at the threshold value F_c, which is scaled as $\bar{R}^2(t) \sim t^z$. Here

the average is taken over the surviving configurations, not over all runs. In Fig. 6, we plot the spreading of random walks against time in double logarithmic scales for several values of F. It is likely that the data for $F = F_c$ are on a straight line with slope ≈ 0.628, while the data for $F \neq F_c$ are deviated from the straight line.

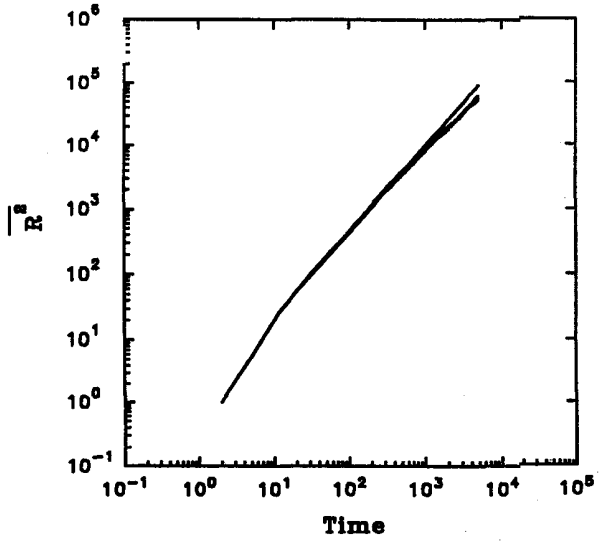

Fig. 6. The averaged spreading of random walks versus time in double logarithmic scales.

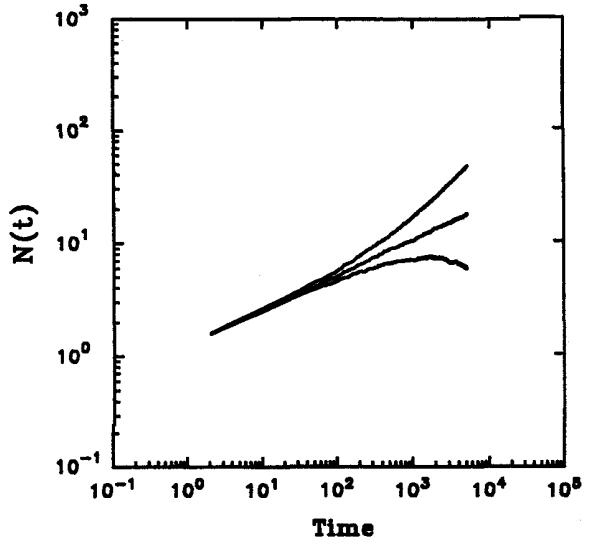

Fig. 7. The averaged number of occupied sites $N(t)$ versus time in double logarithmic scales.

The estimated value of $z \approx 0.628$ is close to the known value of the DPE class, $z_{\mathrm{DPE}} \approx 0.633$. Next, we examine the averaged number of occupied sites $N(t)$, which is averaged over total configurations. The averaged number of occupied sites means the averaged number of random walkers. In Fig. 7, the quantity $N(t)$ is plotted against time t in double logarithmic scales. The straight line is formed in Fig. 7 for the data at $F = F_{\mathrm{c}}$, which has the slope, $\eta \approx 0.3127$, which is close to the known value $\eta_{\mathrm{DPE}} \approx 0.3137$ for the DPE. The numerical values obtained from the dynamic Monte Carlo simulations are in good agreement with the known values of the DPE universality class. Accordingly we conclude that the phase transition at F_{c} belongs to the DPE universality.

For $F < -1$, the energy decreases with increasing time, so that the branching occurs at each time. Since a site occupied by random walks makes the site vacant by the annihilation process, the trajectories of random walks appear in the form of the Sierpinski tree, as shown in Fig. 8. For this case, the number of occupied sites $N(t)$ is given as $N(t) = t$ when $t = 2^k$ for integer k, and by the recursion relation $N(2^k + i) = 2N(i)$ when $t = 2^k + i$.

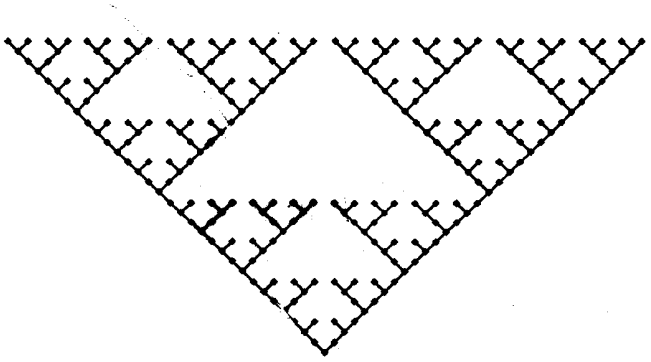

Fig. 8. The trajectory of the BAW, the Sierpinski tree structure, for $F < -1$.

In summary, we have considered the problem of the branch-annihilation random walks in the energetic-disordered structure, in which the evolution rule is assigned as related to the energy disorder. A biased random walker can diffuse to the nearest neighbor sites at advanced time as long as the energies are less than the one in previous time. Depending on the magnitude of the applied field F, a variety of states can occur. When $F > 0$, the hopping process of directed random walks is dominant, so that the trajectory of directed random walks is described by the directed polymer. When $F_{\mathrm{c}} < F < 0$, the annihilation process is dominant, and the state becomes inactive. However, when $-1 < F < F_{\mathrm{c}}$, the branching process is dominant, and the state becomes active. When $F < -1$, the branching and annihilation processes are dominant, and the state becomes trivial. The phase transition occurring at F_{c} turned out to belong to the directed percolation universality class. Accordingly, the energy disorder does not give any effect to break the robustness of the directed

percolation universality. The robustness of the DPE universality class under the energy disorder may be explained by the Harris criterion, even though it is not justified yet to apply the Harris criterion to dynamic systems including the BAW problem. The arguments for applying the Harris criterion are following.[17] In the DPE of two dimensions, the sum of the correlation length exponents $\nu_\perp + \nu_\parallel$ is greater than 2. Therefore, the exponent $\alpha = 2 - (\nu_\perp + \nu_\parallel)$ becomes less than zero. Then by the Harris criterion, the energy disorder does not lead to the breakdown of the universality class in homogeneous system, the DPE universality class.

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