Statistical potential in an ideal anyon gas

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We calculate the *statistical potentials* between particles in an ideal anyon gas arising from the symmetry property of the wave functions. The statistical potential is known to be purely repulsive for fermions, and to be purely attractive for bosons. In contrast, the statistical potentials for anyons are found to behave differently; the statistical potential for quasibosons is not purely attractive, displaying a minimum at finite interparticle distance. That implies there exists a stable equilibrium interparticle distance between two quasibosons. The statistical potential for quasifermions behaves similarly to that for fermions. Our result supports that the probability amplitude of finding two anyons at the same space point is zero.

Recently, two-dimensional electron systems with an external magnetic field have drawn much attention. This is mainly due to their essential role in quantum Hall effects[1] and their possible relevance to high-temperature superconductivity.[2] Moreover, these systems contain interesting possibilities for quantum statistics and fractional statistics, which interpolate continuously between bosons and fermions.[3] Anyons, which carry both electric charges and magnetic-flux tubes in two-dimensional systems,[4] are good examples of particles that follow fractional statistics.

In general, anyons have very different physical properties compared with fermions and bosons. For example, the wave function for two identical bosons is symmetric under exchange in their positions, and it is antisymmetric for two identical fermions. But when two anyons are interchanged, the wave function may acquire a complex phase factor; unlike the case of fermions and bosons, the change in phase need not be an integral multiple of π.

When we consider a quantum-statistical problem for an ideal gas, fermions or bosons, we can derive a "fictitious potential" arising from the symmetry property of the wave function of bosons and fermions, and treat the quantum mechanical problem classically.[5] The fictitious potential, which is called the statistical potential, is not a true interparticle potential, but represents the first quantum correction to the partition function obtained classically. It is known to be purely attractive for bosons, while it is purely repulsive for fermions.[5] However, the statistical potential for anyons has not been studied yet, even though it could help to understand the physical natures of anyons.

In this Brief Report, we study the statistical potential for anyons by considering the perturbations from bosons and fermions. It is found that the statistical potential perturbed from fermions, called quasifermions, exhibits behavior similar to that for ordinary fermions. But the statistical potential for quasibosons behaves very differently from that for ordinary bosons. It is not purely attractive, displaying a minimum at finite interparticle distance r. This implies there exists a stable equilibrium interparticle distance between two anyons. The stable interparticle distance r depends on the temperature and statistics determining the parameter δ. It is worth noting that we have calculated the statistical potential to leading order, which corresponds to the limit of high temperature and low density. That turns out to be possible for two-particle systems. Therefore, we shall consider a two-particle anyon system in the following.

Let us begin by defining an anyon system in two spatial dimensions. We consider a system with two electrons which interact with external magnetic-flux tubes; except for this, there is no further interaction between the two electrons. In this case the nonrelativistic Hamiltonian of the two-electron system is

$$\mathcal{H}=\sum_{i=1}^{2}\frac{1}{2m}\left[\mathbf{p}_i-\frac{e}{c}\mathbf{A}(\mathbf{r}_i)\right]^2 , \tag{1}$$

where e is the charge of the electron and $\mathbf{A}$ is the vector potential written in the form

$$\mathbf{A}(\mathbf{r})=\frac{\Phi}{2\pi}\nabla\theta . \tag{2}$$

Here $\Phi\equiv\alpha hc/e$ is the flux tube with α controlling its magnitude and θ denotes the relative angle between the two electrons.

The Schrödinger equation for the Hamiltonian (1) can be solved easily by decomposing it into the center-of-mass and relative-coordinate parts, and by using the gauge transformation $\mathbf{A}(r)\rightarrow\mathbf{A}(r)-\nabla\theta$. Then the wave function is obtained as

$$\Psi_\alpha=\sum_n e^{i\mathbf{K}\cdot\mathbf{R}}e^{i(n+\alpha)\theta}J_{|n|}(kr) ,$$
$$\equiv e^{i\mathbf{K}\cdot\mathbf{R}}\psi_\alpha , \tag{3}$$

with the corresponding eigenvalue $E=\hbar^2K^2/4m+\hbar^2k^2/m$. $J_m(x)$ is the Bessel function. Here Bose (Fermi) statistics requires that $n+\alpha\equiv l$ be even (odd), and the wave function is rewritten as

$$\Psi_\alpha=\sum_l e^{i\mathbf{K}\cdot\mathbf{R}}e^{il\theta}J_{|l-\alpha|}(kr) . \tag{4}$$

The partition function defined as $Z=\mathrm{Tr}e^{-\beta\mathcal{H}}$ simply takes the form $Z=2A\lambda_T^{-2}\tilde{Z}$, where $\tilde{Z}$ is the partition function of a single particle in the relative-coordinate problem, λ_T is the thermal wavelength, and A is the area of the system. The new partition function $\tilde{Z}$ is explicitly given by

$$\tilde{Z}=\int d^2p\int d^2r\, e^{-\beta p^2/m}|\psi_\alpha|^2$$
$$=\sum_{l=-\infty}^{\infty}\int d^2p\int d^2r\, e^{-\beta p^2/m}|J_{|l-\alpha|}(kr)|^2\,, \tag{5}$$

with $p=\hbar k$. We first integrate out the momentum part to obtain the statistical potential. The integration of the momentum part can be performed through the use of relation[6]

$$\int_0^{\infty}e^{-\alpha x}J_\nu(2\beta\sqrt{x}\,)J_\nu(2\gamma\sqrt{x}\,)dx$$
$$=\frac{1}{\alpha}I_\nu\left[\frac{2\beta\gamma}{\alpha}x\right]\exp\left[-\frac{\beta^2+\gamma^2}{\alpha}\right]. \tag{6}$$

Then the partition function is reduced to the form

$$\tilde{Z}=\tfrac{1}{2}\sum_{l=-\infty}^{\infty}\int_0^{\infty}dx\, e^{-x}I_{|l-\alpha|}(x)\,, \tag{7}$$

where $x\equiv mr^2/2\beta\hbar^2=\pi r^2/\lambda_T^2$, and $I_{|l-\alpha|}(x)$ is the modified Bessel function. Equation (7) was also obtained by Arovas *et al.* by using the path-integral technique.[3]

Next, by simple mathematical manipulations, we can rewrite Eq. (7) as

$$\tilde{Z}=\frac{1}{2h^2}\int d^2p\, e^{-\beta p^2/m}\int d^2r\sum_{l=-\infty}^{\infty}2e^{-mr^2/2\beta\hbar^2}I_{|l-\alpha|}\times\left[\frac{mr^2}{2\beta\hbar^2}\right], \tag{8}$$

which is to be compared with the partition function for classical systems:

$$\tilde{Z}=\frac{1}{2h^2}\int d^2p\, e^{-\beta p^2/m}\int d^2r\, e^{-\beta v(r)}\,. \tag{9}$$

Therefore, the statistical potential $v(r)$ is given by the

$$e^{-\beta v(r)}=2\sum_{l=-\infty}^{\infty}e^{-x}I_{|l-\alpha|}(x)\,, \tag{10}$$

again with $x=\pi r^2/\lambda_T^2$.

The next step is to find the allowed values of $|l-\alpha|$. We take our original particles to have Bose statistics, so that l is an even number. For quasifermions we take $\alpha=2j+1+\delta$ with $|\delta|<1$. Then the allowed values of $|l-\alpha|$ are $1\pm\delta$, $3\pm\delta$, etc., and Eq. (10) becomes

$$e^{-\beta v(x)}=2e^{-x}\left[\sum_{n=0}^{\infty}I_{2n+1+\delta}(x)+I_{2n+1-\delta}(x)\right]. \tag{11}$$

From Eq. (11) we compute the statistical potential numerically. As shown in Fig. 1(a), the overall behavior is the same as that of fermions, except that the repulsive potential becomes weaker as δ increases. When $\delta=0$, Eq. (11) reduces to the known potential for ordinary fermions:

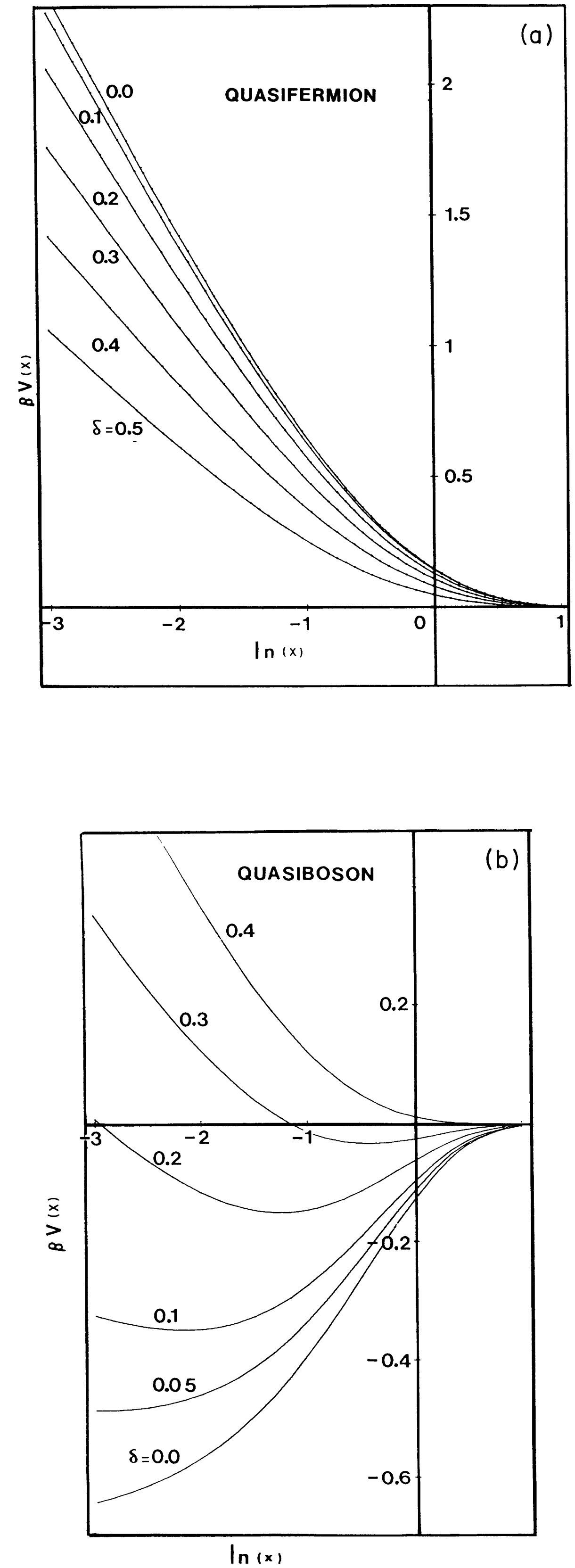

FIG. 1. Statistical potential $\beta v(x)$ for (a) quasifermions and (b) quasibosons vs $\ln x$ with $x=mr^2/2\beta\hbar^2$.

$$e^{-\beta v(x)}=1-e^{-2x} . \tag{12}$$

For small $|\delta|$ we expand $I_{2n+1\pm\delta}(x)$ in powers of δ and obtain the relation

$$e^{-\beta v(x)}=4e^{-x}\sum_{n=0}^{\infty}\left[I_{2n+1}(x)+\frac{\delta^2}{2}\left.\frac{\partial^2 I_\nu(x)}{\partial\nu^2}\right|_{\nu=2n+1}\right] . \tag{13}$$

We can see that the terms of first order in δ are canceled out, as in the case of the second virial coefficient.[3] We note that the two terms of Eq. (13) have the same signs, while they have opposite signs for the quasibosons below.

For quasibosons we take $\alpha=2j+\delta$, again with $|\delta|<1$, leading to $|\delta|$, $2\pm\delta$, $4\pm\delta$, etc. In contrast to the quasifermion case, however, the statistical potential for quasibosons behaves very differently:

$$e^{-\beta v(x)}=2e^{-x}\left[I_{|\delta|}(x)+\sum_{n=1}^{\infty}I_{2n+\delta}(x)+I_{2n-\delta}(x)\right] . \tag{14}$$

The statistical potential is not an analytic function of δ, as in the case of the second virial coefficient.[3] The nonanalytic term in δ may cause anomalous behavior of the statistical potential for quasibosons. We plot the statistical potential $\beta v(x)$ obtained from Eq. (14) versus $\ln x$ in Fig. 1(b). As shown in the figure, the statistical potential is no longer a monotonous function, but displays a minimum for $\delta<0.4$, which implies that there exists a stable interparticle distance between the two bosonic anyons for this range of δ. Note that the stable position r depends on δ and the temperature, through the relation $x=\pi r^2/\lambda_T^2$. On the other hand, it is also easy to check that, for $\delta=0$, Eq. (14) reduces to the relation for ordinary bosons:

$$e^{-\beta v(x)}=1+e^{-2x} . \tag{15}$$

In order to understand how the minimum occurs, we expand $I_{2n\pm\delta}(x)$ in powers of δ and obtain the relation

$$e^{-\beta v(x)}=2e^{-x}\left[2\left[\sum_{n=1}^{\infty}I_{2n}(x)+\frac{\delta^2}{2}\left.\frac{\partial^2 I_\nu}{\partial\nu^2}\right|_{\nu=2n}\right]+I_0(x)+|\delta|\left.\frac{\partial I_\nu}{\partial\nu}\right|_{\nu=0}\right] . \tag{16}$$

We find that the zeroth-order terms of δ have an opposite sign to the first-order term for all x, while the second-order term changes its sign near the minimum position. Accordingly, not only does the presence of the first-order term cause the anomalous behavior, but also the second-order term plays an important role in forming the minimum. We plot the statistical potential $\beta v(x)$ as a function of $\ln x$ and δ in Fig. 2 to see overall how the statistical potential for anyons interpolates from a purely attractive one to a purely repulsive one. From the figures the statistical potential shows purely repulsive behavior for $\delta\geq 0.4$. Hence we can say that the statistical potential for semions, which corresponds to $\delta=0.5$, is repulsive.

So far, we have considered the statistical potential for anyons, which possesses stable interparticle separation for two bosonic anyons and is purely repulsive for fermionic anyons. Moreover, numerical data in Fig. 1 suggest that the statistical potential diverges to infinity in the limit of $r\rightarrow 0$. Therefore we may conclude that two-anyon particles are not at the same space point. This result supports the nature of anyons that the configuration space is multiply connected, so that two particles cannot be at the same position.[7–9] Very recently, Loss and Fu imposed a hard-core-type potential to free anyons, in order to understand the cause of the cusp behavior of the second virial coefficient of a noninteracting anyon gas in the limit of bosonic gas, and they concluded that the cause of the cusp behavior lies in the lack of a hard-core interaction.[10] On the other hand, since the statistical potential diverges as $r\rightarrow 0$ for bosonic and fermionic anyons—in other words, the statistical potential is at-

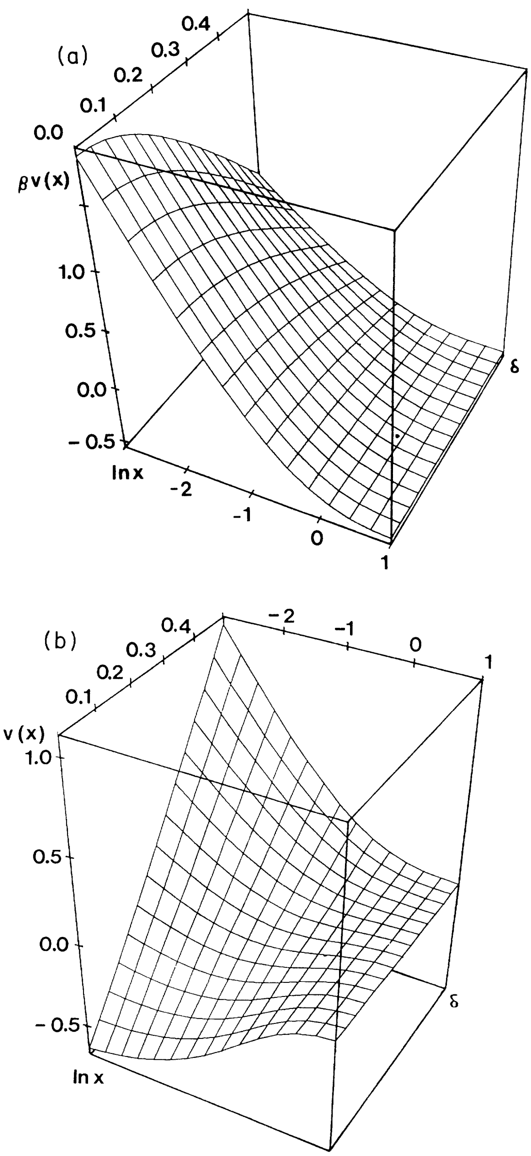

FIG. 2. Statistical potential $\beta v(x)$ for (a) quasifermions and (b) quasibosons vs $\ln x$, with $x=mr^2/2\beta\hbar^2$, and vs δ.

tractive as $r \rightarrow 0$ only for bosons, and the cusp-behavior occurs only for bosons—we may say that our result supports their arguments. On the other hand, it would be interesting to investigate the behavior of the statistical potential for the case that the Hamiltonian includes a nonstatistical hard-core interaction or other types of nonstatistical potentials to check if the minimum behavior in the statistical potential does not occur. This subject is currently under consideration.

In conclusion, we have considered the statistical potential for anyons, carrying both electric charges and magnetic-flux tubes in two dimensions. The statistical potential for quasifermions exhibits behavior similar to that of ordinary fermions. For quasibosons, in contrast, it displays a striking difference: The statistical potential possesses a minimum at a finite interparticle distance, which implies that there exists a stable interparticle distance between two bosonic anyon particles. The stable interparticle distance depends on the temperature and statistics determining the parameter δ.

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