Kuramoto synchronization models with actively competing interaction

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Here we consider a Kuramoto model (KM) with actively competing interactions. Contrary to a passively competing interaction case generating two synchronized clusters, actively competing oscillators gather into a single synchronized cluster. Here we investigate the synchronization transition of the actively competing KM with several types of natural frequency distributions $g(\omega)$ using the mean-field approach. We find in general that the coupling constant K of the ordinary KM is replaced by the mean coupling constant $\langle K\rangle$ of the active competing system. However, when $g(\omega)$ is flat, the critical behavior of hybrid phase transition changes slightly in the subleading order.

I. INTRODUCTION

Synchronization is a collective phenomenon emerging in diverse complex systems in the real world. Examples include the flashing of fireflies, the chirp of the crickets, the pacemaker cells of the heart, and synchronous neural activity [1–6]. Globally coupled phase oscillators have been used to model such synchronizations. The conventional Kuramoto model (KM) is written as

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \tag{1}$$

where $\theta(t)$ is the phase of the i-th oscillator at time t, and ω_i is its natural frequency following the distribution $g(\omega)$, and K is a coupling constant. When K is small, each oscillator rotates almost independently with its own frequency ω_i; however, as K is increased, oscillators interact each other, and a synchronized cluster forms on a macroscopic scale at a transition point K_c.

Inspired by spin glass or neural networks with competing signs of interactions, generalizations of KM have been made [7–12]. One most precedent suggested by Daido [8] is written as

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N}K_{ij}\sin(\theta_j - \theta_i), \tag{2}$$

where K_{ij} is of Sherrington-Kirkpatrick type, which is distributed in a Gaussian form with zero mean. Under such competing interactions, the possibility of an oscillator glass state was questioned. A critical phenomenon was found regarding the motion of local fields in the complex plane. It was revealed that the frequency entrainment occurs but the phase-locking does not. The oscillator phases show a diffusive motion and therefore the initial coherence is always lost in the long time. Discovery of non-exponential relaxation of coherence in supercritical control parameter regime suggested the presence of potential glassy oscillators; yet it still remains as an inconclusive problem [8].

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The authors of Refs. [10, 11] considered simplified KMs with competing interactions based on nodes instead of edges. One can immediately notice that two generalizations are possible: passively (actively) coupled case with K_i (K_j) placed outside (inside) the summation.

The passively competing KM is written as

$$\dot{\theta}_i = \omega_i + \frac{K_i}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i). \tag{3}$$

K_i is placed outside the summation and is given as a mixture of different signs. In this case the synchronization order parameter is defined as that of the ordinary KM,

$$Z(t) \equiv R(t)e^{i\Psi(t)} = \frac{1}{N}\sum_{j} e^{i\theta_j}. \tag{4}$$

The oscillators are decoupled to each other but are coupled instead to the effective field of strength K_iR.

$$\dot{\theta}_i = \omega_i + K_i R\sin(\Psi - \theta_i). \tag{5}$$

Depending on the sign of K_i, the stability of an oscillator at the force-balancing position is reversed. Oscillators with positive coupling constants are attractive to the effective field, while the oscillators with negative coupling constants are repulsive. Recently, the KMs with passively competing interactions were extensively investigated; for the cases with $g(\omega)$ following the Lorentzian [10] and the uniform [12] distributions. In these cases the oscillators are clustered into two groups separated roughly by an angle π. They can be either static or traveling.

The actively competing KM is written as

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N}K_j\sin(\theta_j - \theta_i), \tag{6}$$

where K_j can have either a positive or negative value and follows the distribution $f(K)$. The natural frequency distribution $g(\omega)$ is assumed to be symmetric about zero. In this interaction form, each oscillator j actively attracts ($K_j > 0$) or repulses ($K_j < 0$) an oscillator i. In this case, a previous study [11] showed that the $\{K_j\}$ can be roughly replaced by the effective coupling constant $\langle K\rangle$ when $g(\omega)$ is unimodal.

In this paper, we extend the previous study to the cases of the bimodal and uniform distributions of $g(\omega)$. For these cases, the synchronization transitions are first-order and hybrid transitions [1, 12, 13], respectively. Here a hybrid phase transition means the transition in which properties of second-order and first-order transitions occur at the same transition point. The hybrid phase transition has been recently studied in k-core percolation models [19–22] and synchronization models [12–17]. We find that the actively competing KMs with the bimodal and uniform distributions of $g(\omega)$ also behave similarly to the corresponding classical KM but the role of the coupling constant K is replaced by the effective coupling constant $\langle K\rangle$. Interestingly, for the uniform distribution, we find an abnormal subleading hybrid scaling behavior, in addition to the result obtained by Pazó in Ref. [13].

This paper is organized as follows: In Sec. II, we introduce the order parameter of the actively competing KM. In Sec. III, the self-consisteny equations of the KMs with the bimodal and the uniform distributions are derived. The critical exponents associated with the order parameter in leading and subleading orders are obtained. The final section is devoted to the summary.

II. THE ACTIVELY COMPETING SYNCHRONIZATION

In the actively competing case, the order parameter is given as

$$W(t) \equiv S(t)e^{i\Phi(t)} \equiv \frac{1}{N}\sum_{j=1}^{N} K_j e^{i\theta_j(t)}. \tag{7}$$

Notice the difference with the conventional synchronization order parameter $Z(t)$ in that it is weighted by local coupling constants. In contrast to the passively competing model, where the K_i causes a sign difference, the actively competing oscillators are under the same mean field $S(t)$.

$$\dot{\theta}_i = \omega_i + S(t)\sin(\Phi(t) - \theta_i). \tag{8}$$

The system shows a synchronization transition from an incoherent to a coherent state as in the ordinary KM; group separation of π or traveling wave state does not occur.

III. THE SELF-CONSISTENCY EQUATION

The self-consistency equation of the actively competing model is written as

$$\begin{aligned} S &= \int dK f(K) \int d\omega g(\omega) K \sqrt{1-\left(\frac{\omega}{S}\right)^2} \\ &= \langle K\rangle \int_{-S}^{S} d\omega g(\omega)\sqrt{1-\left(\frac{\omega}{S}\right)^2}. \end{aligned} \tag{9}$$

Integrating the above equation after the series expansion of $g(\omega)$ at zero yields:

$$S = \frac{\pi}{2}g(0)\langle K\rangle S + \frac{\pi}{16}g^{(2)}(0)\langle K\rangle S^3 + \frac{\pi}{384}g^{(4)}(0)\langle K\rangle S^5 + \cdots, \tag{10}$$

where $g^{(2)}(0)$ and $g^{(4)}(0)$ denote the second and the fourth derivatives of $g(\omega)$ with respect to ω at $\omega = 0$.

A. Unimodal

For a unimodal natural frequency distribution, $g^{(2)}(0) < 0$. Thus,

$$S \sim \left(\langle K\rangle - \langle K\rangle_c\right)^{1/2}. \tag{11}$$

The critical exponent becomes $\beta = 1/2$ and the transition point is determined as

$$\langle K\rangle_c = \frac{2}{\pi g(0)}. \tag{12}$$

B. Bimodal

For a bimodal natural frequency distribution, $g^{(2)}(0) > 0$ and $g^{(4)}(0) < 0$. The transition is first-order and a hysteresis curve exists. There exist two transition points $\langle K\rangle_c^f$ and $\langle K\rangle_c^b$ given as

$$\langle K\rangle_c^f = \frac{2}{\pi g(0)} \tag{13}$$

$$\langle K\rangle_c^b = \frac{2}{\pi g(0)}\frac{1}{1+\frac{3g^{(2)}(0)^2}{4g(0)|g^{(4)}(0)|}} \tag{14}$$

where $\langle K\rangle_c^f > \langle K\rangle_c^b$. The jump sizes at $\langle K\rangle_c^b$ and $\langle K\rangle_c^f$ are determined as

$$S_c^b = \frac{S_c^f}{\sqrt{2}} = \sqrt{\frac{12 g^{(2)}(0)}{|g^{(4)}(0)|}}. \tag{15}$$

C. Uniform

Let us consider a uniform distribution of $g(\omega)$ ranging $[-\gamma, \gamma]$. In this case, the transition is hybrid. The self consistency equation can be solved exactly as follows:

$$\begin{aligned} S &= \frac{\langle K\rangle}{2\gamma}\int_{-\gamma}^{\gamma} d\omega \sqrt{1-\left(\frac{\omega}{S}\right)^2} \\ &= \frac{\langle K\rangle S}{2\gamma}\left[\arcsin\frac{\gamma}{S} + \frac{\gamma}{S}\sqrt{1-\left(\frac{\gamma}{S}\right)^2}\right]. \end{aligned} \tag{16}$$

The transition point occurs when $S_c = \gamma$ and is determined as

$$\langle K\rangle_c = \frac{4\gamma}{\pi} = \frac{2}{\pi g(0)}. \tag{17}$$

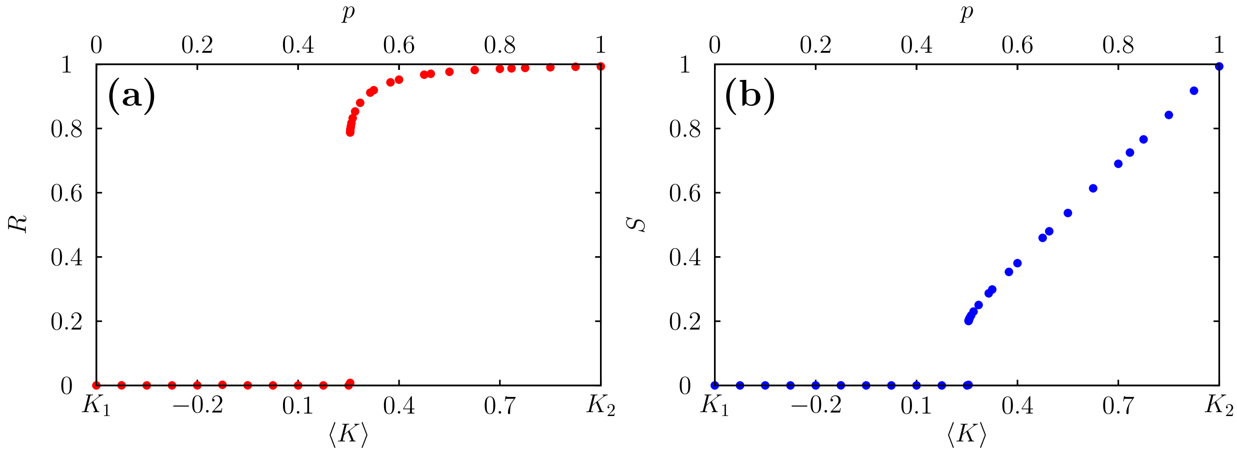

FIG. 1. Plots of the order parameter R of the ordinary KM (a) and the order parameter S of the KM with the actively competing coupling constant (b). Both cases take the uniform distribution of $g(\omega)$ in the range $[-\gamma, \gamma]$. Data points are obtained by simulations for $N = 25600, K_1 = -0.5, K_2 = 1$, and $\gamma = 0.2$, and where the distribution of coupling constants is set $f(K) = (1-p)\delta(K - K_1) + p\delta(K - K_2)$, for simplicity. The mixing fraction p of K_2 oscillators linearly interpolates the mean coupling $\langle K \rangle$ of the system between K_1 and K_2. The data points are obtained by taking average over the last ten percent of the total runtime $t = 10^5 s$. Both order parameters show a discontinuous jump at the transition point $\langle K \rangle_c = 4\gamma/\pi = 0.255$.

At a postcritical $\langle K \rangle = \langle K \rangle_c + \delta\langle K \rangle$, let $S = \frac{\langle K \rangle S}{2\gamma} \int_{-\theta_m}^{\theta_m} \frac{1+\cos 2\theta}{2} d\theta$ where $\theta_m \equiv \arcsin \frac{\gamma}{S} \equiv \frac{\pi}{2} - \delta\theta$. The self consistency equation yields

$$\begin{aligned} 1 &= \frac{\langle K \rangle_c + \delta\langle K \rangle}{2\gamma} \left[\frac{\pi}{2} - \delta\theta + \frac{1}{2}\sin(\pi - 2\delta\theta) \right] \\ &= \left(\langle K \rangle_c + \delta\langle K \rangle \right) \left[\frac{1}{\langle K \rangle_c} - \frac{\delta\theta}{2\gamma} + \frac{1}{4\gamma} \left(2\delta\theta - \frac{(2\delta\theta)^3}{3!} \right) \right] \\ &= \left(\langle K \rangle_c + \delta\langle K \rangle \right) \left[\frac{1}{\langle K \rangle_c} - \frac{(\delta\theta)^3}{3\gamma} \right], \end{aligned} \tag{18}$$

up to the lowest order in $\delta\theta$ and thus

$$\delta\langle K \rangle = \frac{\langle K \rangle_c^2}{3\gamma} (\delta\theta)^3. \tag{19}$$

From $\gamma = S \sin\theta_m$, we have

$$\begin{aligned} \gamma &= (S_c + \delta S) \sin\left(\frac{\pi}{2} - \delta\theta \right) \\ &= (\gamma + \delta S) \left(1 - \frac{(\delta\theta)^2}{2} \right) \end{aligned} \tag{20}$$

and therefore,

$$\begin{aligned} \delta S &= \frac{\gamma}{2} (\delta\theta)^2 \\ &= \frac{\gamma}{2} \left(\frac{3\pi^2}{16\gamma} \right)^{2/3} \delta\langle K \rangle^{2/3}. \end{aligned} \tag{21}$$

Note that the leading order calculation gives the critical exponent $\beta = 2/3$. We stress that β is a non-integer. Therefore the synchronization transition of the actively competing KM with uniform frequency distribution falls into the category of hybrid phase transition.

Up to the next order we find,

$$\delta\langle K \rangle = \frac{\langle K \rangle_c^2}{3\gamma} (\delta\theta)^3 - \frac{\langle K \rangle_c^2}{15\gamma} (\delta\theta)^5 + O(\delta\theta)^6 \tag{22}$$

and reversing the above series gives

$$\delta\theta = \left(\frac{3\gamma}{\langle K \rangle_c^2} \delta\langle K \rangle \right)^{1/3} + \frac{\gamma}{5\langle K \rangle_c^2} \delta\langle K \rangle. \tag{23}$$

Therefore,

$$\begin{aligned} \delta S &= \frac{\gamma(1 - \cos\delta\theta)}{\cos\delta\theta} = \frac{\gamma}{2}(\delta\theta)^2 + \frac{5\gamma}{24}(\delta\theta)^4 \\ &= \left(\frac{9\pi^4\gamma}{2048} \right)^{1/3} \delta\langle K \rangle^{2/3} + \frac{289\pi^2}{5760} \left(\frac{3\pi^2}{16\gamma} \right)^{1/3} \delta\langle K \rangle^{4/3}. \end{aligned} \tag{24}$$

Thus the subleading term gives a different exponent $\beta' = 4/3$ for the order parameter S.

Finally we consider the conventional order parameter $R = \left| \frac{1}{N} \sum_j e^{i\theta_j} \right|$ in the actively competing KM. R is obtained as

$$R = \int d\omega g(\omega) \sqrt{1 - \left(\frac{\omega}{S} \right)^2}. \tag{25}$$

The two order parameters are related to each other by

$$S = \langle K \rangle R \tag{26}$$

as long as the coupling constant K and the natural frequency ω are uncorrelated. Above the transition point,

$$\delta S = \delta\langle K \rangle R_c + \langle K \rangle_c \delta R + \text{higher order}. \tag{27}$$

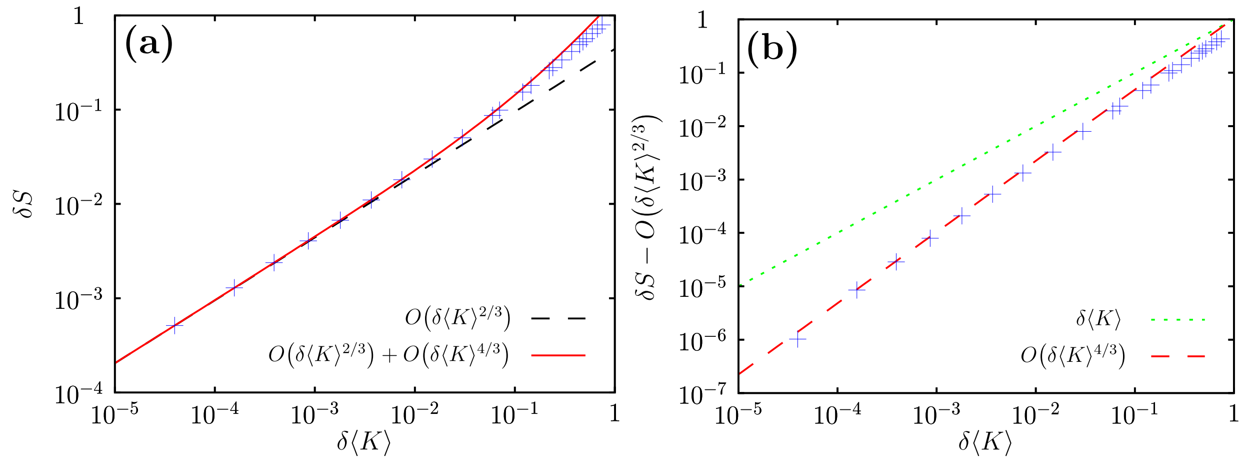

FIG. 2. (a) Plot of δS versus $\delta\langle K\rangle$. The black dashed line denotes the leading order δS of Eq. (24) and the critical exponent $\beta = 2/3$ is clearly noticed. The red solid line counts up to the next leading order of Eq. (24). Data points are obtained by simulations with $N = 25600, K_1 = -0.5, K_2 = 1$, and $\gamma = 0.2$. We used time averaged values during the last ten percent of the total runtime $t = 10^5 s$. For larger values of $\langle K\rangle$ beyond the critical point, a small deviation is noticed. (b) Plot of the subleading correction values versus $\delta\langle K\rangle$ to check the exponent of the subleading order $\beta' = 4/3$. Red dashed line denotes the subleading correction for the actively competing model (24). The dotted green line with slope one is drawn for comparison, which represents the subleading correction for the Pazo model.

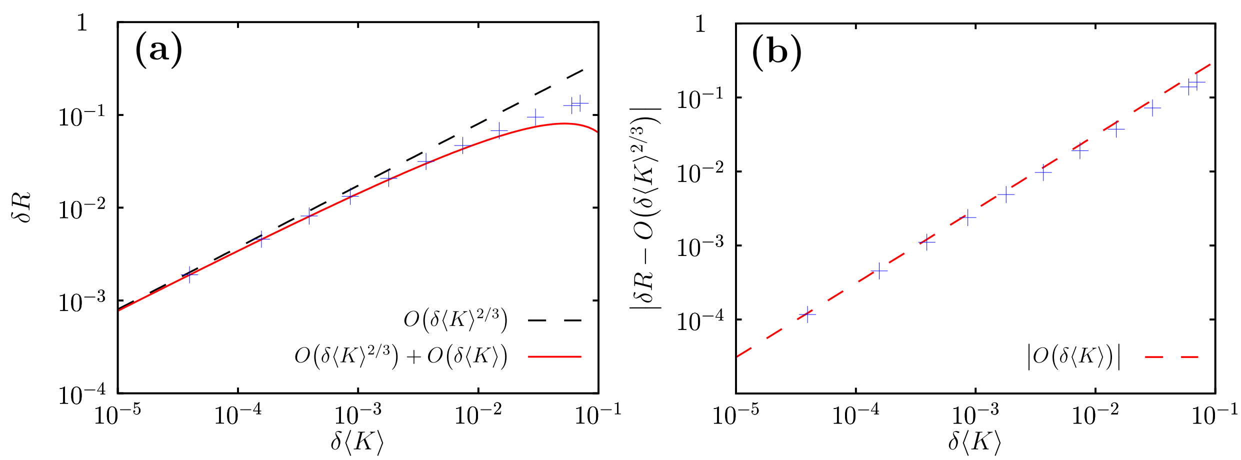

FIG. 3. (a) Plot of δR versus $\delta\langle K\rangle$. Up to the sublinear leading order, scaling of δR is governed by the same hybrid critical exponent $\beta = 2/3$. The black dashed line represents the leading order of Eq. (28), while the red solid line counts up to the next leading order of Eq. (28). (b) Plot of the magnitudes of the subleading correction values versus $\delta\langle K\rangle$ to check the exponent of the subleading order $\beta' = 1$. The dashed red line denotes the absolute value of the subleading correction of Eq. (28), which is linear in $\delta\langle K\rangle$. The data points are obtained by the simulation.

Using Eqs. (24) and (27), we find that

$$\begin{aligned}\delta R &= \frac{\gamma}{2\langle K\rangle_c}\left(\frac{\langle K\rangle_c^2}{3\gamma}\right)^{2/3}\delta\langle K\rangle^{2/3} - \frac{R_c}{\langle K\rangle_c}\delta\langle K\rangle \\ &= \frac{\pi}{8}\left(\frac{3\pi^2}{16\gamma}\right)^{2/3}\delta\langle K\rangle^{2/3} - \frac{\pi^2}{16\gamma}\delta\langle K\rangle, \qquad (28)\end{aligned}$$

where we used $R_c = \pi/4 = S_c/\langle K\rangle_c$. This scaling of δR is the same as the one obtained by Pazó [13], except that the coupling constant K has been replaced by the mean coupling $\langle K\rangle$. We remark that from Eq. (27), we observe that the scalings of S and R are of the same leading order. However, in contrast to our result of the critical exponent $\beta' = 4/3$ for the order parameter S, the subleading term had given a noncritical exponent $\beta' = 1$ for the order parameter R in Ref. [13].

IV. SUMMARY

We investigated the synchronization transition of the Kuramoto model (KM) with actively competing interactions, and compared the result with that of the ordinary KM. A common feature between the two results is that the transition point $\langle K\rangle_c$ of the mean coupling constant of the competing mixture plays the same role of K_c of the ordinary KM. We verified that this is generally valid for the unimodal, bimodal and uniform natural frequency distributions. Furthermore, the critical exponents of continuous transition $\beta = 1/2$ and hybrid transition $\beta = 2/3$ remains unchanged up to the leading order. Unexpectedly, however, we have found another hybrid critical exponent $\beta' = 4/3$ for δS in the subleading order for the actively competing KM with uniform frequency distribution, different from $\beta' = 1$ for δR. This suggests that further interesting features are expected from the hybrid phase transition of the actively competing generalization of the KM.

ACKNOWLEDGMENTS

This work was supported by the National Research Foundation of Korea by Grant No. NRF-2014R1A3A2069005.

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