Geometric fractal growth model for scale-free networks

S. Jung,[1] S. Kim,[1] and B. Kahng[2]
[1]*Nonlinear and Complex Systems Laboratory, Department of Physics, Pohang University of Science and Technology, Pohang, Kyongbuk 790-784, Korea*
[2]*School of Physics and Center for Theoretical Physics, Seoul National University, Seoul 151-747, Korea*
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We introduce a deterministic model for scale-free networks, whose degree distribution follows a power law with the exponent γ. At each time step, each vertex generates its offspring, whose number is proportional to the degree of that vertex with proportionality constant $m-1$ $(m>1)$. We consider the two cases: First, each offspring is connected to its parent vertex only, forming a tree structure. Second, it is connected to both its parent and grandparent vertices, forming a loop structure. We find that both models exhibit power-law behaviors in their degree distributions with the exponent $\gamma=1+\ln(2m-1)/\ln m$. Thus, by tuning m, the degree exponent can be adjusted in the range, $2<\gamma<3$. We also solve analytically a mean shortest-path distance d between two vertices for the tree structure, showing the small-world behavior, that is, $d\sim\ln N/\ln \bar{k}$, where N is system size, and $\bar{k}$ is the mean degree. Finally, we consider the case that the number of offspring is the same for all vertices, and find that the degree distribution exhibits an exponential-decay behavior.

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I. INTRODUCTION

Recently, complex systems have received considerable attention as an interdisciplinary subject [1,2]. Complex systems consist of many constituents such as individuals, substrates, and companies in social, biological, and economic systems, respectively, showing cooperative phenomena between constituents through diverse interactions and adaptations to the pattern they create [3,4]. Recently, there have been a lot of efforts to understand such complex systems in terms of networks, composed of vertices and edges, where vertices (edges) represent constituents (their interactions). This approach was initiated by Erdös and Rényi (ER) [5]. In the ER model, the number of vertices is fixed, while edges connecting one vertex to another occur randomly with a certain probability. However, the ER model is too random to describe real complex systems. Recently, Barabási and Albert (BA) [6,7] introduced an evolving network where the number of vertices N increases linearly with time rather than fixed, and a newly born vertex is connected to already existing vertices, following the so-called preferential attachment (PA) rule; when the number of edges k incident upon a vertex is called the degree of the vertex, the PA rule means that the probability Π_i for the new vertex to connect to an already existing vertex i is proportional to the degree k_i of the selected vertex i, that is,

$$\Pi_i=\frac{k_i}{\sum_j k_j}. \tag{1}$$

The main difference between the ER and BA models appears in the degree distribution. For the ER network, the degree distribution follows the Poisson distribution, while for the BA network, it follows a power law, $P(k)\sim k^{-\gamma}$ with $\gamma=3$. The network whose degree distribution follows a power law is called the scale-free (SF) network [7]. SF networks are abundant in real world such as the world-wide web [8–11], the Internet [12–15], the citation network [16], the author collaboration network of scientific papers [17], and the metabolic networks in biological organisms [18].

While a lot of models have been introduced to describe SF networks in real world, most of them are stochastic models. However, a couple of models recently introduced by Barabási, Ravasz, and Vicsek (BRV) [19], and Dorogovtsev and Mendes (DM) [20] are deterministic. In general, deterministic model is useful for investigating analytically not only topological features of SF networks in detail, but also dynamical problems on the networks. Both the BRV and the DM models are meaningful as not only the first attempts for deterministic SF networks, but also as the ones constructed in a hierarchical way, so that analytic treatments can be made easily using recursive relations derived from the two structures in successive generations. In the BRV model, however, the mean shortest-path distance between two vertices averaged over all pairs, called the diameter, is independent of system size. Thus, the BRV model may be relevant to some specific systems such as the metabolic networks [18], where the diameter is independent of system size. In this paper, we introduce another type of the deterministic model for the SF network, which is also constructed in a hierarchical way. Our model is based on almost the same idea as that of the DM model. While the DM model starts from a triangle, our model does from a tree structure. This difference makes one easily modify the model into more general cases such as loopless or loop structures, and the ones with a various number of branches. Moreover, the simplicity of our model enables one to obtain the analytic solution for the degree distribution and the diameter. In particular, our model includes a control parameter, so that by tuning the parameter, we can obtain SF networks with a variety of degree exponents in the range, $2<\gamma<3$. Therefore our model should be useful to represent various SF networks in real world, in particular, of a tree shape. For example, the hierarchical tree structure is known as ''structural hole'' in sociology, a typical type of

social networks. Since the model of the structural hole is used for investigating the centrality in social networks [21], and moreover recent studies show that social systems also exhibit SF behaviors [22], our model could be used for studying social systems. Moreover, the internet structure in the autonomous level is effectively of a tree-type [23], so that the data packet transport on the internet can be understood analytically via our deterministic model.

This paper is organized as follows. In Sec. II, we will introduce deterministic models specifically for tree and loop structures, respectively. In Sec. III, analytic treatments will be performed for the deterministic models introduced in Sec. II. The final section will be devoted to conclusions and discussions.

II. DETERMINISTIC MODEL

It is known that the number of vertices in most of SF networks in real world increases with time exponentially. Thus, our deterministic model is constructed in this evolving way, that is, the number of vertices increases exponentially with time, where each already existing vertex generates its offspring, and connects them. Thus, vertices are ordered hierarchically. On the other hand, for real networks, it is known [24,25] that the PA probability Π_i' in Eq. (1) is modified into

$$\Pi_i' = \frac{k_i + \mu}{\sum_j k_j + \mu}, \tag{2}$$

where μ accounts for some randomness in connecting edges. To take into account of this modified PA behavior, we introduce two rules, called the addition and the multiplication rule, in the deterministic model, depending on how new vertices are generated from each old vertex. The details on both rules will be described below.

A. Tree structure

The network forms a tree structure when new vertices generated from an old vertex are connected to their parent only.

1. The addition rule

In the case of the addition rule, at each time step, a constant number of new vertices, say, m new vertices, are generated from each already existing vertex, and they are connected to their parent only. Then the degree $k_i(t)$ at vertex i at time t evolves as

$$k_i(t) = k_i(t-1) + m, \tag{3}$$

so that

$$k_i(t) = 1 + m(t - t_i), \tag{4}$$

for $t \geqslant t_i$, where t_i is its birth time. On the other hand, the number of vertices newly born at time t_i becomes

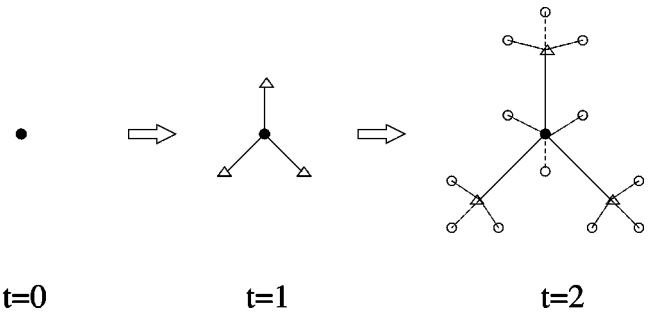

FIG. 1. Tree structures in the addition rule with $m=3$ at $t=0$, 1, and 2. The vertices born at $t=0$, 1 and 2 are denoted by ●, △ and ○, respectively.

$$L_A(t_i) = \begin{cases} 1 & \text{for } t_i = 0 \\ m(1+m)^{t_i - 1} & \text{for } t_i \geqslant 1, \end{cases} \tag{5}$$

where the subscript A stands for the addition rule. Thus, the total number of vertices $N_A(t)$ present at time t is

$$N_A(t) = \sum_{t_i=0}^{t} L_A(t_i) = (1+m)^t. \tag{6}$$

The definition of this model is illustrated schematically in Fig. 1.

2. The multiplication rule

In the case of the multiplication rule, the number of offspring generated from each old vertex is not the same, but it depends on the degree of each vertex. Let $k_i(t)$ be the degree of a given vertex i at time t. Under the multiplication rule, the number of offspring newly generated at time t from the vertex i is proportional to its degree at the previous time, i.e., $(m-1)k_i(t-1)$, where $m-1$ is a proportionality constant. Thus the degree of the vertex i increases by the factor m at each time step, that is,

$$k_i(t) = m k_i(t-1) \tag{7}$$

for $t \geqslant 2$. The degree of the vertex, born at $t=0$ $(t=1)$, at time $t=1$ is $k_0(1)=m$ $(k_1(1)=1)$. Then the degree of the vertex i at time t becomes

$$k_i(t) = m^{t - t_i}, \tag{8}$$

where t_i means the birth time of the vertex i. Next, let $L_M(t)$ be the total number of vertices newly born at time t, where the subscript M stands for the multiplication rule. Then we have the following relation:

$$L_M(t_i) = \sum_{t_j=0}^{t_i - 1} (m-1) m^{t_i - 1 - t_j} L_M(t_j) \tag{9}$$

for $t_i \geqslant 2$ with $L_M(0)=1$ and $L_M(1)=m$. Then we obtain $L_M(t_i)$ in a closed form to be

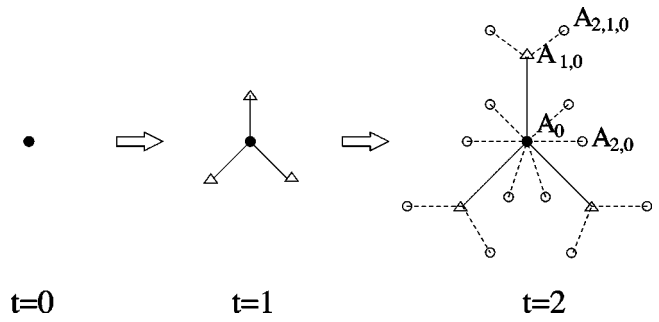

FIG. 2. Tree structures in the multiplication rule with $m=3$ at $t=0$, 1, and 2. A_0 (●) stands for the vertex at center born at $t=0$. $A_{1,0}$ (△) is the offspring of A_0 born at $t=1$. $A_{2,0}$ and $A_{2,1,0}$ (○) are the offspring of A_0 and $A_{1,0}$, respectively, born at $t=2$.

$$L_M(t_i)=\begin{cases}1 & \text{if } t_i=0\\ m & \text{if } t_i=1\\ 2m(m-1)(2m-1)^{t_i-2} & \text{if } t_i\geqslant 2.\end{cases} \tag{10}$$

The total number of vertices $N_M(t)$ at time t is given by

$$N_M(t)=\sum_{t_i=0}^{t} L_M(t_i)=\begin{cases}1 & \text{if } t=0\\ 1+m(2m-1)^{t-1} & \text{if } t\geqslant 1.\end{cases} \tag{11}$$

The definition of this model is illustrated schematically in Fig. 2.

B. Loop structure

A loop structure can be generated when each newly born vertex is connected to both its parent and grandparent vertices as illustrated in Fig. 3. In particular, if the parent of a certain vertex is the vertex born at $t=0$, then the grandparent is regarded as one of the vertices born at $t=1$. This rule is valid for both cases of the addition and the multiplication rule.

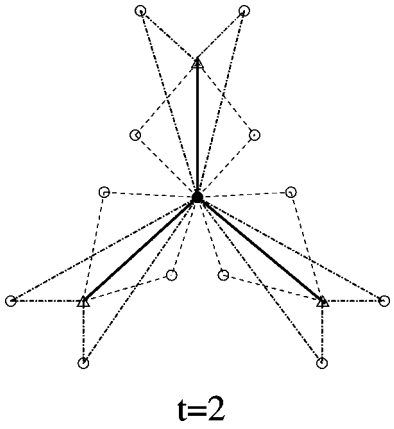

FIG. 3. Loop structure in the multiplication rule with $m=3$ at $t=2$. The vertices with the symbols ●, △, and ○ are born at $t=0$, 1, and 2, respectively.

III. ANALYTIC SOLUTION

A. The degree distribution for the tree structure

Since the degree of each vertex has been obtained explicitly as in Eqs. (4) and (8) and the degree of each vertex is ordered with time, one can obtain the degree distribution via its cumulative distribution, i.e.,

$$P(k)=P(k_i>k-1)-P(k_i>k), \tag{12}$$

or

$$P(k)=\frac{\partial[1-P(k_i>k)]}{\partial k}. \tag{13}$$

The details of the analytic treatments are given as follows.

1. The addition rule

Using the fact, $P_{A,T}(k_i>k)=P_{A,T}(t_i<\tau=t-[k-1]/m)$, where the subscript T means the tree structure, we obtain that

$$\begin{aligned}P_{A,T}(k_i>k)&=\sum_{t_i=0}^{\tau-1}\frac{L_A(t_i)}{N_A(t)}\\ &=\frac{1}{(1+m)^t}+\frac{m}{(1+m)^{t+1}}\sum_{t_i=1}^{\tau-1}(1+m)^{t_i}\\ &=(1+m)^{-(k-1)/m}.\end{aligned} \tag{14}$$

Using the relation that $P_{A,T}(k)=P_{A,T}(k_i>k-1)-P_{A,T}(k_i>k)$, we obtain

$$P_{A,T}(k)=(1+m)^{-(k-1)/m}[(1+m)^{1/m}-1]. \tag{15}$$

So, the degree distribution $P_{A,T}(k)$ decays exponentially with k under the addition rule.

2. The multiplication rule

Since the degree k_i has been obtained explicitly as a function of time in Eq. (8), $P_{M,T}(k_i>k)$ is replaced by $P_{M,T}(t_i<\tau)$, where $\tau=t-\ln k/\ln m$. Thus,

$$\begin{aligned}P_{M,T}(k_i>k)&=\sum_{t_i=0}^{\tau-1}\frac{L_M(t_i)}{N_M(t)}\\ &=\frac{1+m}{1+m(2m-1)^{t-1}}\\ &\quad+\sum_{t_i=2}^{\tau-1}\frac{2(m-1)(2m-1)^{t_i-2}}{1+m(2m-1)^{t-1}}\\ &\propto k^{-\ln(2m-1)/\ln m}.\end{aligned} \tag{16}$$

Thus the degree distribution is obtained to be

$$P_{M,T}(k)=\frac{\partial[1-P_{M,T}(k_i>k)]}{\partial k}\propto k^{-\gamma(m)}, \tag{17}$$

where

$$\gamma(m)=1+\ln(2m-1)/\ln m. \tag{18}$$

In the limit of $m\to 1$, one can get $\gamma(1)=3$, while as m goes to infinity, one gets $\gamma(\infty)=2$. Thus by tuning the parameter m, one can get a variety of SF networks with different exponents in the range, $2<\gamma<3$.

B. The degree distribution for the loop structure

1. The addition rule

Let $n_i(t)$ be the degree of vertex i, born at time t_i, at time t for the loop structure. As a new vertex is connected to its parent and grandparent, each old vertex is connected to its m children and m^2 grandchildren. So, Eq. (3) is modified as

$$n_i(t)=n_i(t-1)+(m+m^2). \tag{19}$$

Taking the same steps as for the loopless case, we obtain the degree distribution following the exponential decay,

$$P_{A,L}(n)\propto(1+m+m^2)^{-n/(m+m^2)}, \tag{20}$$

where the subscript L means the loop structure.

2. The multiplication rule

Let $n_i(t)$ be the degree of vertex i at time t for the loop structure in the multiplicative rule. The degree of vertex i can be obtained via the relation,

$$n_i(t)=n_i(t-1)+(m-1)k_i(t-1)+(m-1)^2k_i(t-2), \tag{21}$$

where the second term on the right hand side of the above equation results from the children of the vertex i, and the third term from the grandchildren of the vertex i. Thus, the degree at the vertex i becomes

$$n_i(t)\approx\left(\frac{2m-1}{m}\right)m^{t-t_i}, \tag{22}$$

for $t\gg t_i$. Since the degree $n_i(t)$ depends on time t similarly to Eq. (8), we can apply Eq. (16) even to the loop case, except that τ is replaced by $\tau=t+\ln(2m-1)/\ln m-1-\ln n/\ln m$. This replacement, however, does not affect the degree exponent at all. Thus, even for the loop structure, the degree exponent is reduced to the same value, $\gamma=1+\ln(2m-1)/\ln m$ as the one in Eq. (18).

C. The diameter for the tree structure

The diameter $d(t)$ is defined as a chemical distance between two distinct vertices along the shortest path averaged over all pairs of vertices at time t, that is,

$$d(t)=\frac{1}{N(t)(N(t)-1)}\sum_{i\neq j}d_{i,j}(t), \tag{23}$$

where $d_{i,j}(t)$ is the chemical distance between vertex i to j. For simplicity, let $\mathcal{D}(t)$ denote the sum of the chemical distances between two vertices over all pairs, that is,

$$\mathcal{D}(t)=\sum_{i\neq j}d_{i,j}(t). \tag{24}$$

It is not easy to derive a closed formula for $\mathcal{D}(t)$ for both the tree and the loop structures, however, we list $\mathcal{D}(t)$ for the tree structure at a few early times in the Appendix. We trace the formula for the tree structure in two limiting cases, $m\to 0$ and $m\to\infty$, as follows.

Let us first consider the case of $m\to 1$. For this case, we denote $m=1+\epsilon$ and $\epsilon\ll 1$. The total number of nodes $N_M(t)$ at time t is given by

$$N_M(t)=1+(1+\epsilon)(1+2\epsilon)^{t-1}\approx 2+(2t-1)\epsilon+O(\epsilon^2), \tag{25}$$

for large t. So, $\ln(N_M-1)\approx(2t-1)\epsilon$ within the first order of ϵ. Moreover, the sum of all chemical distances $\mathcal{D}(t)$ becomes

$$\mathcal{D}(t)\approx 2+4(2t-1)\epsilon+O(\epsilon^2). \tag{26}$$

Using the relation in Eq. (23), we can obtain the average distance to be

$$\begin{aligned} d&=\frac{2+4(2t-1)\epsilon}{2+3(2t-1)\epsilon}+O(\epsilon^2),\\ &\approx\frac{2+4\ln(N_M-1)}{2+3\ln(N_M-1)}+O(\epsilon^2), \end{aligned} \tag{27}$$

Therefore, the diameter converges to 4/3 in the limit of $N_M\to\infty$.

Next, we consider the case of $m\to\infty$. In this case, the term in the highest order of m could be dominant, so that we trace the coefficient of the term in the highest order of m as a function of time.

$$\begin{aligned} \mathcal{D}(0)&=0,\\ \mathcal{D}(1)&=2m^2+\text{lower order terms},\\ \mathcal{D}(2)&=[(2+3)+(3+4)]m^4+\text{lower order terms},\\ \mathcal{D}(3)&=[(2+2\times 3+4)+2\times(3+2\times 4+5)+(4+2\times 5\\ &\quad+6)]m^6+\text{lower order terms},\\ \mathcal{D}(4)&=[(2+3\times 3+3\times 4+5)+3(3+3\times 4+3\times 5+6)\\ &\quad+3(4+3\times 5+3\times 6+7)+(5+3\times 6+3\times 7\\ &\quad+8)]m^8+\text{lower order terms},\\ &\vdots\\ \mathcal{D}(t+1)&=\sum_{p=0}^{t}\binom{t}{p}\sum_{k=0}^{t}\binom{t}{k}(k+p+2)m^{2(t+1)}\\ &\quad+\text{lower order terms}. \end{aligned} \tag{28}$$

Using the formula [26],

$$\sum_{k=0}^{t} k\binom{t}{k} = t2^{t-1}, \tag{29}$$

$\mathcal{D}(t)$ in the highest order of m is obtained explicitly to be

$$\begin{aligned}\mathcal{D}(t) &\approx \sum_{p=0}^{t-1}\binom{t-1}{p}\sum_{k=0}^{t-1}\binom{t-1}{k}(k+p+2)m^{2t} \\ &= \sum_{p=0}^{t-1}\binom{t-1}{p}m^{2t}[t-1+2(2+p)]2^{t-2} \\ &= (t+1)2^{2t-2}m^{2t}. \end{aligned} \tag{30}$$

On the other hand, using Eq. (11),

$$N_M(t)[N_M(t)-1] \approx 2^{2t-2}m^{2t}. \tag{31}$$

Therefore, the diameter $d(t)$ at time t becomes simply

$$d(t) \approx t+1 \approx \frac{\ln(N_M-1)}{\ln(2m-1)} + 1. \tag{32}$$

Thus, for large N_M, the above equation is rewritten simply as

$$d(N_M) \sim \ln N_M / \ln \overline{k} \tag{33}$$

with the mean degree $\overline{k} \approx 2m$, which confirms the small-world behavior.

IV. CONCLUSIONS AND DISCUSSIONS

We have introduced a deterministic model for the scale-free network, which is constructed in a hierarchical way. At each time step, each already existing vertex produces its offspring, whose number is proportional to the degree of the vertex. Depending on whether each new offspring is connected to only one or more than one old vertices, the network forms either a tree structure or a loop structure, respectively. We have obtained the analytic solution for the degree distribution and the diameter explicitly for the deterministic model. By tuning a control parameter in the model, we can adjust the degree exponent in the range, $2<\gamma<3$. Thus this model can represent a variety of SF networks in real world. Moreover, we obtained the diameter of the deterministic model analytically to be $d \sim \ln N/\ln \overline{k}$, where N is the system size and $\overline{k}$ is the mean degree. Since the network is generated in a hierarchical way, it is expected that a variety of physical problems can be solved through this deterministic model by constructing recursive relations derived from two structures in successive generations. On the other hand, the deterministic model has a shortcoming that it does not include any long-ranged edge, connecting two vertices belonging to different branches separated at $t=0$. Thus, this model can be used only for the model for a tree structure. Despite this shortcoming, we think that our deterministic model could offer a guide toward generating more realistic deterministic model for SF networks.

ACKNOWLEDGMENTS

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APPENDIX

The closed formula for the sum of the chemical distance between two vertices $\mathcal{D}(t)$ is shown for $0 \leqslant t \leqslant 3$.

$$\mathcal{D}(t=0)=0,$$

$$\mathcal{D}(t=1)=\mathcal{N}_0 m+\mathcal{N}_{1,0}[1+2(m-1)],$$

$$\begin{aligned}\mathcal{D}(t=2)=&\mathcal{N}_0[m^2+2m(m-1)]+\mathcal{N}_{1,0}[m+2m(m-1)+2(m-1)+3(m-1)(m-1)]+\mathcal{N}_{1,0}[1+2m^2-2(m+1)+2m \\ &+3m(m-1)]+\mathcal{N}_{2,1,0}[1+2(m-1)+3(m^2-1)+4(m-1)^2],\end{aligned}$$

$$\begin{aligned}\mathcal{D}(t=3)=&\mathcal{N}_0[m^3+2(m^2-m)(m-1)+2m(m-1)+2m(m^2-m)+3m(m-1)(m-1)]+\mathcal{N}_{1,0}[1m^2+2m^3-2+2(m-1) \\ &\times(m-1)+3\{(m-1)^2+(m^2-m)(m-1)+(m-1)(m^2-m)\}+4(m-1)^3]+\mathcal{N}_{2,0}(m+2(m^3-1) \\ &+3\{[m(m-1)-1](m-1)+m(m^2-m)+m(m-1)\}+4m(m-1)^2)+\mathcal{N}_{2,1,0}[m+2(m^2-1)+3\{(m^3-1) \\ &+(m-1)(m-2)\}+4\{(m^2-m)(m-1)+(m-1)^2+(m-1)(m^2-m)\}+5(m-1)^3]+\mathcal{N}_{3,0}[1+2(m^3-1) \\ &+3\{m^2(m-1)+m(m^2-m)\}+4m(m-1)^2]+\mathcal{N}_{3,2,0}(1+2(m-1)+3(m^3-1)+4\{[m(m-1)-1](m-1) \\ &+m(m^2-1)\}+5m(m-1)^2)+\mathcal{N}_{3,1,0}[1+2(m^2-1)+3\{(m^3-1)+(m-1)(m-1)\}+4\{(m^2-1)(m-1)+(m-1) \\ &\times(m^2-m)\}+5(m-1)^3]+\mathcal{N}_{3,2,1,0}[1+2(m-1)+3(m^2-1)+4\{(m^3-1)+(m-1)(m-1-1)\} \\ &+5\{(m^2-1)(m-1)+(m-1)(m^2-1)\}+6(m-1)^3],\end{aligned}$$

$$\vdots$$

where

$$\mathcal{N}_0 = 1,$$

$$\mathcal{N}_{1,0} = m,$$

$$\mathcal{N}_{2,0} = m(m-1),$$

$$\mathcal{N}_{2,1,0} = m(m-1),$$

$$\mathcal{N}_{3,0} = m^2(m-1),$$

$$\mathcal{N}_{3,1,0} = m^2(m-1),$$

$$\mathcal{N}_{3,2,0} = m(m-1)^2,$$

and

$$\mathcal{N}_{3,2,1,0} = m(m-1)^2.$$

$\mathcal{N}_{i,\{j\}}$ means the number of the vertices denoted by $A_{i,\{j\}}$ in Fig. 2, where the first index i stands for its birth time and the rest indices $\{j\}$ are its parent vertex.

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