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Packet transport along the shortest pathways in scale-free networks

C.-M. Ghim[a], E. Oh, K.-I. Goh, B. Kahng, and D. Kim

School of Physics, Seoul National University NS50, Seoul 151-747, Korea

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Abstract. We investigate a problem of data packet transport between a pair of vertices on scale-free networks without loops or with a small number of loops. By introducing load of a vertex as accumulated sum of a fraction of data packets traveling along the shortest pathways between every pair of vertices, it is found that the load distribution follows a power law with an exponent δ. It is found for the Barabási-Albert-type model that the exponent δ changes abruptly from $\delta = 2.0$ for tree structure to $\delta \simeq 2.2$ as the number of loops increases. The load exponent seems to be insensitive to different values of the degree exponent γ as long as $2 < \gamma < 3$.

PACS. 89.75.Fb Structures and organization in complex systems – 05.65.+b Self-organized systems – 02.10.Ox Combinatorics; graph theory

1 Introduction

Considerable efforts have been made to understand complex systems in terms of graphs consisting of vertices and edges which represent the constituents and their interactions, respectively [1–4]. In real systems they can be routers and optical wires in communication systems, individuals or *actors* and acquaintances in social networks, substrates and chemical reactions or enzymes catalyzing them in metabolic pathways, etc. One of the central measures characterizing a given graph is the degree, defined as the number of edges connecting to a certain vertex. Since its introduction about half a century ago the random graph theory [5] has successfully explained the *small-world* character of real-world networks. The degree distribution of the ER network, however, follows a Poissonian, while a majority of real world networks show the power-law behavior,

$$P_D(k) \sim k^{-\gamma}, \tag{1}$$

where k is degree and γ is the degree exponent. Such networks, called scale-free (SF), are ubiquitous in nature, examples of which include the world-wide web (WWW) [6–8], the Internet [9–11], the citation network [12], the collaboration network of scientific papers [13,14], and the metabolic networks in microbial organisms [15]. It was not until the advent of the so-called preferential attachment (PA) scheme that we understand the absence of typical scale in degree distribution thereof. To illustrate the mechanism of SF network formation, Barabási and Albert (BA) [16–18] introduced an evolving network model where the number of vertices N increases linearly with time rather than fixed, and a newly introduced vertex is connected to m already existing vertices with probability proportional linearly to the degree of the selected vertex, which lends the name PA. Then the degree distribution follows a power law with the exponent $\gamma = 3$.

Recently, another type of SF network model, called the static model, was introduced [19,20] in which the number of vertices N is fixed from the beginning, and vertices are indexed by an integer i ($i = 1, \cdots, N$) and assigned the weight or fitness $p_i = i^{-\alpha}$ each, where α is a control parameter in $[0, 1)$. We select two different vertices (i, j) with probabilities equal to the normalized weights, $p_i/\sum_k p_k$ and $p_j/\sum_k p_k$, respectively, and add an edge between them unless one exists already. This process is repeated until mN edges are made in the system leading to the mean degree $\langle k\rangle = 2m$. Since edges are connected to a vertex with frequency proportional to the weight of that vertex, the degree at that vertex is given by

$$k_i \simeq 2mN\frac{(1-\alpha)}{N^{1-\alpha}}\frac{1}{i^\alpha} \sim \left(\frac{N}{i}\right)^\alpha. \tag{2}$$

Then it follows that the degree distribution displays the power law, equation (1), where the degree exponent γ is given by

$$\gamma = 1 + 1/\alpha. \tag{3}$$

Thus, adjusting the parameter α in $[0, 1)$, we can obtain various values of the exponent γ in the range, $2 < \gamma < \infty$.

While the degree of a vertex represents one aspect of the centrality of the vertex, other measures should be used

[a] e-mail: `cmghim@phya.snu.ac.kr`

to quantify importance of a vertex in transport properties of the SF networks. In Section 2, we introduce a quantity, called load, which has been used in the study of data packet transport on networks. Interestingly, the load distribution displays a power law and the load exponent seems to be insensitive to the details of SF network structures. The load exponent, however, exhibits a distinct feature depending on the presence and absence of loop structures in scale-free networks. For all diverse spectra of degree distribution with different γ's, many real-world networks as well as stochastic and deterministic models mimicking them seem to fit a classification scheme [21] in terms of characteristic load exponent δ. Specific features of geodesics, mass-distance relation and resilience under attack can be attributed to the class identification [21]. In this vein, we introduce in Section 3, a modified BA model which interpolates the tree structure and the structure with a small number of loops and study how δ changes between the two structures.

2 Load or betweenness centrality

Let us suppose that a data packet is sent from a vertex to another on a scale free network. It is transmitted along the shortest pathway between them. If there exist more than one shortest pathways, the data packet would encounter one or more branching points, where the data packet is presumed to take one of them with equal probability, and is substantially divided evenly by the number of branches at each branching point as it travels. Then the load ℓ_i at a vertex i is defined as the total amount of data packets passing through that vertex (i) when all pairs of vertices send and receive one unit of data packet between them. Note that the contribution from the pathway with the source/target pair (s, t), denoted by $\ell_i^{(s\to t)}$, may be different from that of (t, s) even for undirected networks. Hence we define the load ℓ_i of a vertex i as the sum over all pairs of vertices, $\ell_i \equiv \sum_{s,t} \ell_i^{(s\to t)}$. The definition of the load is illustrated in Figure 1a. Here, we do not take into account the time delay of data transfer at each vertex or edge, so that all data are delivered in a unit time, regardless of the distance between any two vertices. In fact, if we assume data travel with constant speed so that the time delay of data transfer is proportional to the distance between two vertices, the time delay effect does not change the load distribution. The reason for this result is that when the time delay is accounted, load at each vertex is reduced roughly by a factor $\log N$ or $\log\log N$, in turn, proportional to the diameter, which is negligible compared with the load without the time delay [19]. Because of this small-world property, the universal behavior remains unchanged under the time delay of data transmission. On the other hand, since the packets are conserved, the total load contributed by a pair is simply related to the shortest pathway length d_{st} between them, leading to $\sum_i \ell_i^{(s\to t)} = d_{st} + 1$. Thus we have the sum rule for ℓ_i:

$$\sum_i \ell_i = \sum_{s,t} (d_{st} + 1) = N(N-1)(d+1) \sim N^2 d. \quad (4)$$

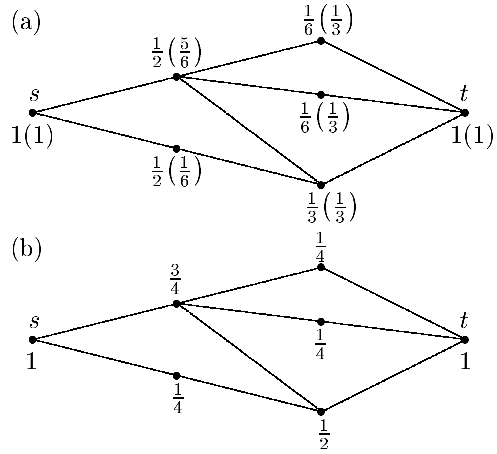

Fig. 1. (a) The load at each vertex due to a unit packet transfer from the vertex s to the vertex t, $\ell_i^{(s\to t)}$. In this diagram, only the vertices along the shortest paths between (s, t) are shown. The quantity in parentheses is the corresponding value of the load due to the packet from t to s, $\ell_i^{(t\to s)}$. (b) The BC at each vertex due to a packet transfer between the vertices s and t, $b_i^{(s\to t)}$.

The dynamic quantity, load, is closely related to its static counterpart used in sociology called "betweenness centrality" (BC) to quantify how much power is centralized to people in social networks [22,23], defined as follows. Let us consider a communication between two persons represented by a pair of vertices (s, t). The communication is supposed to travel along the shortest pathway between the two. When there are more than one geodesics between them, each one of them is taken with equal probability. The BC for a certain vertex is defined as the accumulated fraction of total number of the shortest pathways passing on that vertex over all pairs. That is, the betweenness centrality at a vertex i is

$$b_i = \sum_{s\neq t} b_i^{(s\to t)} = \sum_{s\neq t} \frac{G_i(s,t)}{G(s,t)}, \quad (5)$$

where $G(s, t)$ is the total number of geodesics connecting the vertices s and t, and $G_i(s, t)$ the number of those passing through the vertex i among them. The definition of BC is also illustrated in Figure 1b. Slightly different in the definitions, the two quantities, load or BC, behave closely and their distributions are indistinguishable within our numerical resolution. Hence we shall not distinguish them throughout this paper, unless otherwise noted explicitly.

2.1 The load distribution

Once a SF network is generated artificially or adopted from the real world, we select an ordered pair of vertices (i, j) on the network, and identify the shortest pathway(s) between them and measure the load on each vertex along the shortest pathway using the modified version of the breath-first search algorithm introduced by Newman [23] and independently by Brandes [24]. We have measured load ℓ_i of each vertex i for SF networks with various γ. It is found numerically that the load distribution $P_L(\ell)$ follows the formula,

$$P_L(\ell) \sim \ell^{-\delta}. \tag{6}$$

When the index of the vertices are ordered according to the rank of the load, we have $\ell_1 \geq \cdots \geq \ell_N$. Then, the power-law behavior of the load distribution implies that

$$\frac{\ell_i}{\sum_j \ell_j} \sim \frac{1}{N^{1-\beta}} \frac{1}{i^\beta} \ . \tag{7}$$

with $\delta = 1 + 1/\beta$. The relation, equation (7), is valid in the region, $\ell_{\min} < \ell < \ell_{\max}$, where

$$\ell_{\min} \sim \ell_{\max}/N^\beta \sim \begin{cases} Nd & \text{for } \beta < 1 \\ Nd/\ln N & \text{for } \beta = 1 \\ N^{2-\beta}d & \text{for } \beta > 1. \end{cases} \tag{8}$$

Note that equations (1) and (6) combined may give a scaling relation between load and degree of a certain vertex as

$$\ell \sim k^{(\gamma-1)/(\delta-1)}, \tag{9}$$

which holds when degree and load are correlated in the way that the vertex with higher degree has larger value of load. On the other hand, to assess the degree-degree correlation of a network, the Pearson correlation coefficient r between the degrees of the two end vertices of an edge averaged over all edges has been introduced [25]. In terms of the value r, networks are categorized as assortative ($r > 0$), neutral ($r = 0$), and dissortative ($r < 0$). equation (9) holds for the neutrally and dissortatively mixed networks but breaks down for the assortatively mixed ones [26]. However, where analytic results are available, the trees satisfy the relation,

$$\ell \sim k^{\gamma-1}, \tag{10}$$

since $\delta = 2$ for all γ.

2.2 Classification of scale-free networks

Based on numerical measurements of load exponents for a variety of SF networks, it is likely that load exponent is apparently robust, insensitive to the details of network structure such as the degree exponent γ in the range, $2 < \gamma \leq 3$, and mean degree, directionality of edge, etc. Empirical measurements also allude us to the classification of SF networks into two classes, say, Class I and II, which are shown in Figure 3. For Class I, the load exponent is $\delta \simeq 2.2(1)$ [1] and for Class II, it is $\delta \simeq 2.0(1)$.

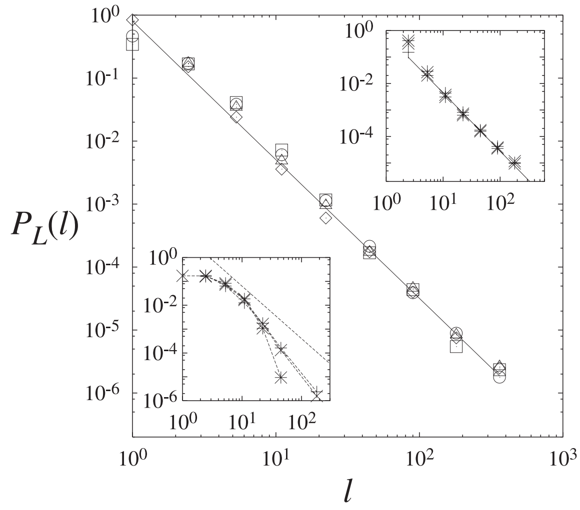

Fig. 2. Plot of the load distribution $P_L(\ell)$ versus ℓ for various $\gamma = 2.25$ (◇), 2.5 (△), 2.75 (○) and 3.0 (□) in double logarithmic scales. The data are obtained from the static model. The linear fit (solid line) has a slope -2.2. Simulations are performed for $N = 10000$ and $m = 2$ and all data points are averaged over 10 configurations. Lower Inset: Same plot for $\gamma = 4$ (+), 5 (×), and ∞ (*). The line having a slope -2.2 is drawn to compare the data with the case for $2 < \gamma \leq 3$. Upper Inset: Plot of $P_L(\ell)$ versus ℓ for different $m = 2$, 4 and 6, but for the same $\gamma = 2.5$.

We conjecture the load exponent for Class II to be exactly $\delta = 2$ since it can be derived analytically for simple models: (i) Szabó and Kertész [28] derived $\delta = 2.0$ for BA tree by using mean field approach. (ii) Noh [29] did the same for the hierarchical model proposed by Ravasz and Barabási [30], where the load exponent invariably is $\delta = 2.0$ as the degree exponent is varied as a function of the increasing rate of the number of edges at a hub in each hierarchical level. (iii) Goh et al. [21] also did for the generalized BA tree structure including the PA proportional to $k + a$, but they obtained the edge-load exponent to be $\delta = 2.0$, independent of the degree exponent γ. Moreover, we derive the load exponent explicitly for the deterministic tree structure proposed by Jung et al. [31] in Appendix. In the deterministic tree model of SF network, the load exponent $\delta = 2.0$ is kept intact while the degree exponent can be controlled by adjusting a parameter μ for $2 < \gamma < 3$. The real-world as well as the model networks that are found to belong to each class are listed in Table 1. On the other hand, for $\gamma > 3$, δ depends on γ in a way that it increases as γ increases. Eventually, the load distribution decays exponentially for $\gamma = \infty$ as shown in the lower inset of Figure 2. Thus, the transport properties of the SF networks with $\gamma > 3$ are fundamentally different from those with $2 < \gamma \leq 3$.

The different behaviors of the load distribution in the Class I and II may originate from different generic topological features of networks. For the BA model, when $m > 2$, the network contains loop structure, so that $\delta \simeq 2.2$ while, for $m = 1$, the network is of tree leading to $\delta = 2.0$. Since the two values are too close, we plot the load

[1] Although δ for real networks in Class I is measured to be $\sim 2.2(1)$, detailed measurements on the static model show considerable variation of δ on γ between 2.0 and 2.2 as noted by Barthélemy [27].

Table 1. Networks investigated and classified.

Class	System	Node	Link	Ref.
I	Coauthorship networks of neuroscience 1991–1998	Scientists	Coauthorship	[14]
	Metabolic networks of 5 species of eukaryotes and 32 of bacteria	Metabolites	Chemical reaction	[15]
	BA model with $m \geq 2$			[16–18]
	Protein interaction network of the yeast *S. cerevisiae*	Proteins	Physical binding	[32,33]
	Geometrical growth model by Huberman & Adamic			[34]
	Copying model by Kumar et al.			[35]
	Accelerated growth model by Dorogovtsev and Mendes.			[36]
	Fitness model by Bianconi and Barabási			[37]
	Protein interaction network model by Solé et al.			[38]
II	WWW within `www.nd.edu` domain	Webpages	Hyperlinks	[6]
	Metabolic networks of 6 species of Archaea	Metabolites	Chemical reaction	[15]
	BA model with $m = 1$			[16–18]
	Hierarchical model by Ravasz and Barabási			[30]
	Deterministic tree model by Jung et al.			[31]
	Internet at the autonomous systems (AS level)	ASes	Hardwire connection	[39]

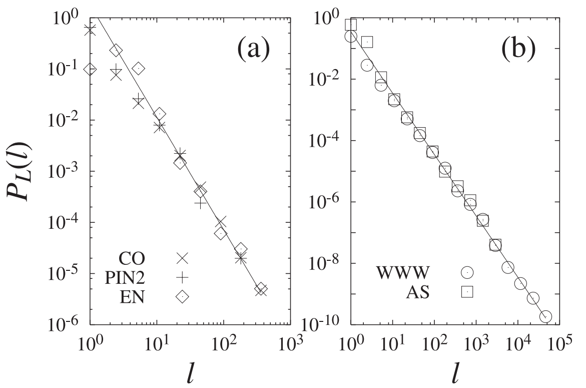

Fig. 3. The load distribution for (a) the coauthorship network (CO), the protein interaction network by Ito et al. (PIN2), and the metabolic network of an eukaryotic organism *Emericella nidulans*; (b) the Internet at the autonomous system level (AS) and the WWW within `nd.edu` domain (WWW). The solid lines, drawn for guide to the eyes, have slopes -2.2 (a) and -2.0 (b), respectively.

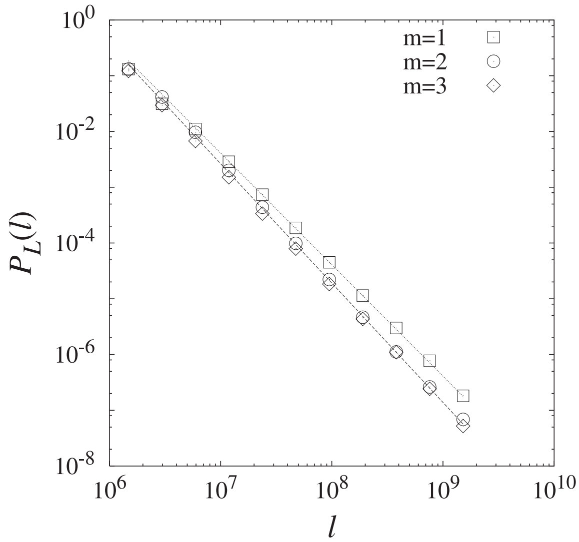

Fig. 4. The load distribution for the BA model ($\gamma = 3$) with $m = 1, 2$, and 3, and $N = 3 \times 10^5$. The dotted (dashed) line has a slope -2.0 (-2.2).

distributions for the BA model with $m = 1, 2$ and 3 in Figure 4, obtained from large system size, $N = 3 \times 10^5$. We can see clearly different behaviors between the two load distributions for the cases of $m = 1$ (Class II) and of $m = 2$ and 3 (Class I). One may wonder why the hierarchical model by Ravasz and Barabási belongs to the Class II, even though it contains loop structures. Here we emphasize that the load exponent is determined by generic topological features of the *shortest pathways* of networks. In Class I, the presence of the long-range loops in multiple shortest pathways between a given pair of vertices leads to the load-sharing between the hubs and a compact localized blob of the hubs is formed. On the other hand, in Class II, such a blob is absent, and the shortest pathways are tree-like. For the Internet at the autonomous system level, and the metabolic networks of Archaea, even though the shortest pathways are composed of loops, they form in a trivial manner, and the blob structure of the shortest pathways is absent, leading to the result that the number of vertices residing on the shortest pathways with a given length depends on that length linearly, like the case of tree structure. Thus, $\delta \approx 2.0$ [21]. To illustrate the reliance of the load exponent on topological feature, in the next section, we investigate the crossover behavior between the two behaviors of the load distribution as the fraction of loops in the BA networks changes.

3 Crossover in load distributions between tree and loop scale-free networks

We investigate how the value of the exponent $\delta = 2.0$ changes as the number of loops increases. To this end, we modify the BA model in such a way that a new vertex attaches one or two edges to existing network with probability $1 - p$ or p, respectively. Then the mean number of edges emanating from a new vertex is given by $\langle m \rangle = 1+p$.

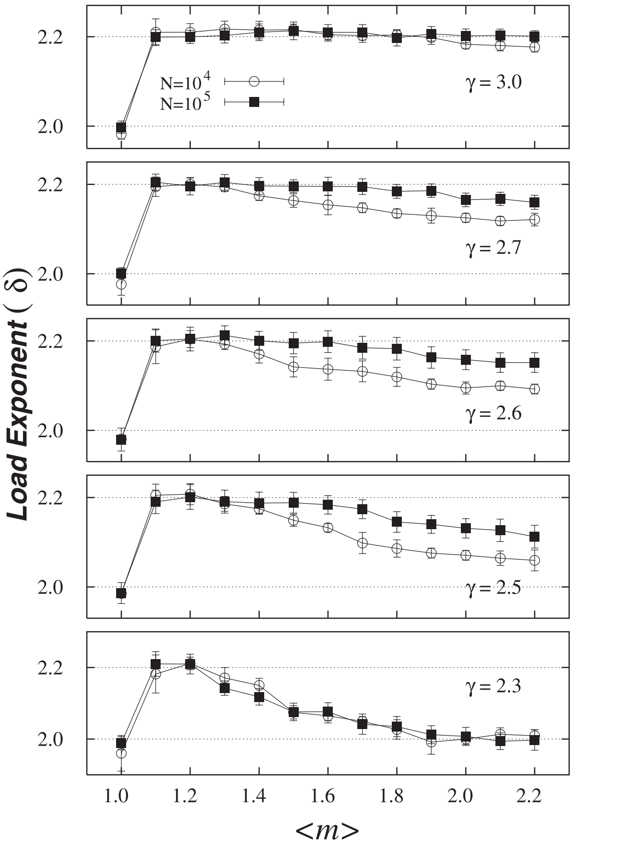

Fig. 5. The load exponent as a function of the mean number of edges $\langle m\rangle$ emanating from a new vertex for various degree exponents γ in the BA model and different system sizes, $N = 10^4$ (◯) and $N = 10^5$ (■).

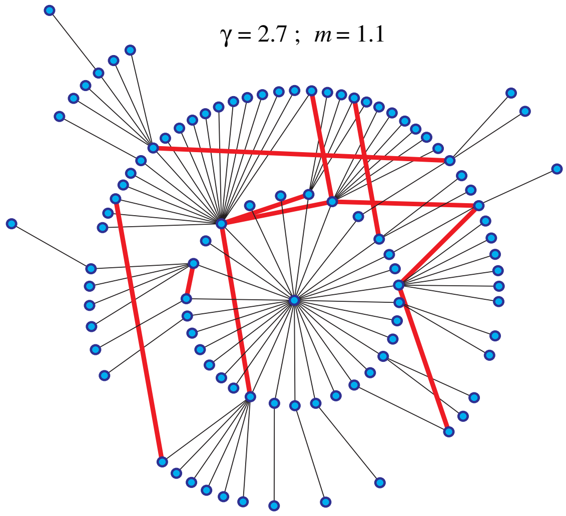

Fig. 6. (Color online). Snapshot of a network with $\gamma = 2.7$ and $m = 1.1$. Thick red lines represent edges without which the structure is tree.

A target vertex j is selected according to the preferential attachment scheme, that is, chosen with the probability linearly proportional to $k_j + m_j(a-1)$, where k_j is the degree of the vertex j at the time when the connection is attempted and m_j is the one or two assigned upon its birth, and a is a control parameter. The degree exponent of the BA model network generated in this way is $\gamma = 2+a$ and its mean degree is $2(1+p)$. For fixed γ (or a), we adjust the fraction of edges forming loops by controlling p. Once a SF network is generated in this manner, we measure load exponent δ for $2 < \gamma < 3$ and $\langle m\rangle$ by applying the algorithm proposed by Newman and Brandes [23,24].

When $p = 0$, the network is a tree, and the load exponent is confirmed to be $\delta = 2.0$. In addition, we find that the load exponent increases to $\delta \simeq 2.2$ by increasing $\langle m\rangle$ to $\langle m\rangle \simeq 1.1$ at which the edges connecting different branches of the tree structure form sparse loops in a nontrivial manner. As shown in Figure 5, the value $\delta \simeq 2.2$ at $\langle m\rangle \simeq 1.1$ turns out to be robust, largely independent of the degree exponent γ for $2 < \gamma < 3$. Such a behavior persists as long as $\langle m\rangle$ is smaller than a γ-dependent characteristic value, $\langle m_c\rangle$, beyond which δ depends on γ. Moreover, we find that the plateau region of $\delta \simeq 2.2$ is extended as the system size N increases as shown in Figure 5. Note that the degree distribution of the BA model contains hump in the tail region generically, which becomes broader as $\gamma \to 2$. Due to this property, the power-law region in the load distribution becomes narrower as $\gamma \to 2$, resulting in larger error bars for δ as γ approaches 2.0. The saturated value decreases with decreasing γ. Therefore, for fixed $\langle m\rangle = 2$, δ is likely to decrease from $\simeq 2.2$ for $\gamma = 3$ to $\delta \simeq 2.0$ as γ decreases. The decreasing rate seems to be larger as γ approaches 2. Based on the result for $\langle m\rangle \geq 2$, Barthélemy argued [27,40] that the universal behavior of the load exponent in γ breaks down. The robustness of the load exponent, however, appears apparently when $\langle m\rangle < \langle m_c\rangle(\gamma)$, where the graph is sparse. Moreover, we study the size-dependent behavior of the load exponent by comparing the behavior of δ for different sizes, $N = 10^4$ and $N = 10^5$. We find that the overall behavior of the load exponent is likely to remain the same but the characteristic value $\langle m_c\rangle(\gamma)$ becomes larger for larger size N, implying that the plateau region of the load exponent being $\delta \simeq 2.2$ is extended as N increases. Note that the abrupt transition of the load exponent from $\delta = 2.0$ to $\simeq 2.2$ by a small amount increment of $\langle m\rangle$ is reminiscent of the crossover behavior from the Heisenberg model class to the Ising model class by imposing a small mount of anisotropy on the coupling constant in one direction [41].

We investigate the qualitative mechanism of the crossover from the Class I to II and vice versa around $\langle m\rangle \simeq 1.1$. When $\langle m\rangle = 1$, the number of branches at the hub, located at the center in Figure 6, is $N^{1/(\gamma-1)}$ on average. As $\langle m\rangle$ increases, the edges connecting different branches form as denoted by thick (red: color online) lines in Figure 6, playing a role of "weak ties". Such edges provide alternative shortest pathways between a pair of vertices belonging to different branches that are otherwise connected only by detouring through the hub. Due to the presence of such edges, the average distance between two vertices can be shorter and vertices become multiply connected in their shortest pathways. As a result, the load of the hub is reduced, while those of the vertices attaching to the weak ties become larger, leading to the increase of the load exponent δ. It is not manifest yet, however, why the numerical value of the load exponent remains robust as $\delta \simeq 2.2$ for $1.1 < \langle m \rangle < \langle m_c \rangle$ regardless of the degree exponent γ as long as $2 < \gamma < 3$. To clear it, analytic solution is needed.

4 Conclusions and discussion

We have introduced a quantity called *load* to study the transport phenomena of data packets on SF network, finding that the load distribution follows a power law with exponent δ. Interestingly, the measured values of the load exponent δ for real world networks and correlated artificial networks seem to fall into two classes with $\delta \approx 2.2(1)$ and 2.0, although δ changes continuously in some uncorrelated models such as the static model. The classification is mainly rooted in generic topological features of the shortest pathways between a pair of vertices. For the former, most shortest pathways are made of multiple pathways and have a blob structure, while for the latter, it is effectively tree. In addition, we have studied the crossover behavior between the two classes by controlling the number of edges emanating from a newly born vertex in the BA model, from which is drawn that the topological changes induced by the nucleation of loops yield a robust and generic crossover. The contents of this review article is mainly based on our works published in [19,21,26,40,42].

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Appendix: Derivation of $\delta = 2.0$ for the deterministic tree

Let us introduce a modified version of the geometric fractal growth model for SF networks proposed by Jung et al. [31]. In this model, we start with $v_0 = 2$ vertices, each having degree 1 at $t = 0$ (see Fig. 7). At each time step, a new generation of vertices are added in such a way that the degree of each node is multiplied by a multiplication factor μ; that is, the degree of the vertex i is evolving via $k_t^{(i)} = \mu k_{t-1}^{(i)}$, where $k_t^{(i)}$ is the degree of the vertex i at time t. Then the number of vertices newly born at time $t \geq 1$ is expressed as

$$v_t = (\mu - 1)\Big(v_{t-1} + \mu v_{t-2} + \cdots + \mu^{t-1} v_0\Big), \qquad \text{(A.1)}$$

or by subtracting μv_{t-1} from this, as $v_t = (2\mu - 1)v_{t-1}$. The number of newborn vertices and the total number of vertices at time t are,

$$v_t = 2(\mu - 1)(2\mu - 1)^{t-1}; \quad v_0 = 2, \qquad \text{(A.2a)}$$
$$N_t = (2\mu - 1)^t + 1\ , \qquad \text{(A.2b)}$$

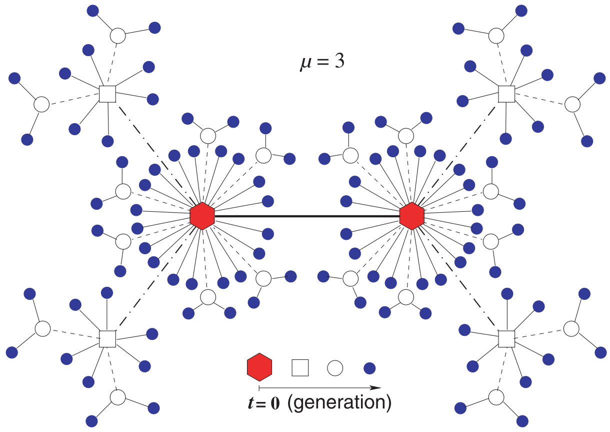

Fig. 7. (Color online) Early stage ($t = 3$) of network growth in the deterministic tree with $\mu = 3$. Initially there are two vertices connected by an edge.

respectively. This, in turn, implies that the total number of vertices with degree $k = \mu^\tau$ or those of $(t-\tau)$th generation can be written as

$$V_t(\tau) = \begin{cases} 2(\mu-1)(2\mu-1)^{t-\tau-1} & \text{for } \tau \leq t-1 \\ 2 & \text{for } \tau = t. \end{cases} \qquad \text{(A.3)}$$

Thus, the degree exponent can be written as $\gamma = 1 + \ln(2\mu - 1)/\ln\mu$ and the mean degree of the tree is

$$\langle k \rangle = \frac{1}{N_t}\sum_{\tau=0}^{t} \mu^\tau V_t(\tau) = 2 + \mathcal{O}\left(\frac{\mu}{2\mu-1}\right)^t. \qquad \text{(A.4)}$$

Similarly, the total number of offsprings of a vertex with degree μ^τ, $\Omega_t(\tau)$, is related to that of its descendants via

$$\Omega_t(\tau) = \sum_{j=1}^{\tau} \mu^{j-1}(\mu-1)\Big\{\Omega_t(\tau-j)+1\Big\} \qquad \text{(A.5)}$$

to yield

$$\Omega_t(\tau) = \frac{(2\mu-1)^\tau - 1}{2}. \qquad \text{(A.6)}$$

In the tree, the shortest pathway being unique, the load of a vertex is simply given by the number of distinct routes passing through the vertex on their way to each other. Since the number of vertices with degree μ^j that are direct daughters of a vertex with degree μ^τ is $\mu^{\tau-j-1}(\mu-1)$, the load of a τ-generation-old vertex denoted as $\ell_t(\tau)$ becomes

$$\ell = \ell_t(\tau) = \Big\{N_t - \big[\Omega_t(\tau)+1\big]\Big\}\Omega_t(\tau) + \binom{\Omega_t(\tau)}{2} - \sum_{j=1}^{\tau-1}\mu^{\tau-j-1}(\mu-1)\binom{\Omega_t(j)+1}{2}. \qquad \text{(A.7)}$$

In the grown stage of the network($t \gg 1$), from one-to-one correspondence between ℓ and τ, the load distribution is given by the relation,

$$P_L(\ell) = \frac{V_t(\tau)}{N_t} \simeq \begin{cases} \dfrac{2(\mu-1)}{(2\mu-1)^{\tau+1}} & \text{for } \tau \leq t-1 \\ \dfrac{2}{(2\mu-1)^\tau + 1} & \text{for } \tau = t\ . \end{cases} \qquad \text{(A.8)}$$

and $\ell = \ell_t(\tau) \simeq (1/2)(2\mu-1)^{t+\tau}$. Since the deterministic tree can only have discrete values of ℓ, it is more natural to consider the accumulated distribution $P_L^{\rm acc}(\ell) = {\rm Prob}(\ell \geq \ell_t(\tau))$;

$$\begin{aligned} P_L^{\rm acc}(\ell)) &= {\rm Prob}(\ell \geq \ell_t(\tau)) = \sum_{k=\tau}^{t} P_L(\ell_t(k)) \\ &\simeq \frac{2(\mu-1)}{(2\mu-1)^{\tau+1}} \frac{1-(2\mu-1)^{-t+\tau}}{1-(2\mu-1)^{-1}} + 2(2\mu-1)^{-t} \\ &= (2\mu-1)^{-t} + (2\mu-1)^{-\tau} \sim \ell_t(\tau)^{-1}\ , \end{aligned} \quad \text{(A.9)}$$

implying

$$P_L(\ell) = -\frac{\rm d}{{\rm d}\ell} P_L^{\rm acc}(\ell) \sim \ell^{-2}\ . \quad \text{(A.10)}$$

Thus the load exponent $\delta = 2.0$, independent of μ and γ.

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