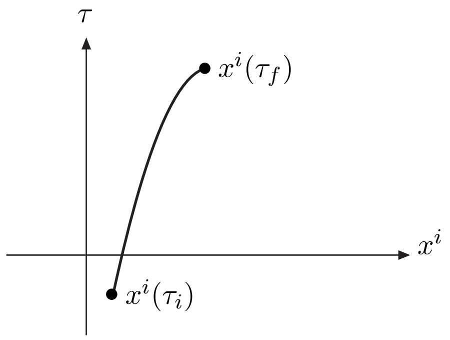

Figure 2.6: A world-line trajectory of a point particle.

particle, $\mathbf{x}_{\rm op}$, at proper time τ. The nonrelativistic counterpart of the action (2.148) reduces to the quantum Hamiltonian

$$H = \frac{\mathbf{p}_{\rm op}^2}{2m}.$$

How does time come in the quantization? The quantization condition has the root in our postulation on measurements. The simultaneous measurement is the concept we have introduced. So the quantization condition must be on the dynamical variables at an equal time. Thus, the condition (2.146) is called 'equal time commutator'(ETC). Since the action(2.148) includes a 'time' as the integration variable, the quantization condition must be of the ETC form

$$[x_i(t_1), p_j(t_2)]_{t_1=t_2} = i\hbar.$$

The commutators of operators at different times, must be obtained by applying time translation transformations on operators.

2.8.2 The uncertainty relation

We note that for a proper definition of simultaneous measurement, i.e. for the irrelevance of the time ordering of the measurement('simultaneous' implies this), of A and B we must have $AB|\psi\rangle = BA|\psi\rangle$, i.e. the commutator vanishes, $[A, B] = 0$. This observation implies that if the commutator does not vanish, then A and B cannot be measured simultaneously with an arbitrary accuracy, i.e. there must be an uncertainty in the measurement of A and B. The uncertainty of the measurement A can be mathematically defined as a dispersion ΔA,

$$(\Delta A)^2 \equiv \langle\psi|(A - \overline{A})^2|\psi\rangle \tag{2.149}$$

in terms of the expectation value

$$\overline{A} = \langle\psi|A|\psi\rangle. \tag{2.150}$$

The uncertainty $(\Delta B)^2$ can be similarly given. Then, $(\Delta A)^2(\Delta B)^2$ on state $|\psi\rangle$ can be given as

$$\begin{aligned}(\Delta A)^2(\Delta B)^2 =& \langle\psi|(A-\overline{A})^2|\psi\rangle\langle\psi|(B-\overline{B})^2|\psi\rangle \\ &\geq |\langle\psi|(A-\overline{A})(B-\overline{B})|\psi\rangle|^2 \end{aligned} \tag{2.151}$$

where we used the Schwartz inequality. Since the LHS of (2.151) is written in terms of scalar numbers $(\Delta A)^2$ and $(\Delta B)^2$, it is equivalent to $(\Delta B)^2(\Delta A)^2$. So it can be expressed in terms of a real parameter α,

$$\begin{aligned}(\Delta A)^2(\Delta B)^2 &= \alpha(\Delta A)^2(\Delta B)^2 - (\alpha-1)(\Delta B)^2(\Delta A)^2 \\ &\geq \alpha|\langle\psi|(A-\overline{A})(B-\overline{B})|\psi\rangle|^2 \\ &\quad - (\alpha-1)|\langle\psi|(B-\overline{B})(A-\overline{A})|\psi\rangle|^2 \end{aligned} \tag{2.152}$$

Since any operator can be written as a sum of the symmetric and antisymmetric combinations, the operators in (2.151) can be written as

$$\begin{aligned}(A-\overline{A})(B-\overline{B}) &= \tfrac{1}{2}[(A-\overline{A})(B-\overline{B}) + (B-\overline{B})(A-\overline{A})] \\ &\quad + \tfrac{1}{2}[(A-\overline{A})(B-\overline{B}) - (B-\overline{B})(A-\overline{A})],\end{aligned}$$

or

$$(A-\overline{A})(B-\overline{B}) \equiv S_{AB} + \tfrac{1}{2}[A,B]. \tag{2.153}$$

Defining the following complex numbers

$$S \equiv \langle\psi|S_{AB}|\psi\rangle, \quad T \equiv \langle\psi|[A,B]|\psi\rangle, \tag{2.154}$$

Eq. (2.152) becomes

$$\begin{aligned}(\Delta A)^2(\Delta B)^2 &\geq \alpha\left[|S|^2 + \tfrac{1}{2}(S^*T + T^*S) + \tfrac{1}{4}|T|^2\right] \\ &\quad - (\alpha-1)[|S|^2 - \tfrac{1}{2}(S^*T + T^*S) + \tfrac{1}{4}|T|^2] \\ &= \left|S + \tfrac{2\alpha-1}{2}T\right|^2 + (\alpha-\alpha^2)|T|^2 \end{aligned} \tag{2.155}$$

Choosing α such that the $|T|^2$ term in the RHS of (2.155) becomes maximum, we obtain the minimum uncertainty relation

$$(\Delta A)^2(\Delta B)^2 \geq \tfrac{1}{4}|T|^2 \tag{2.156}$$

Reference books

In this course, it has been assumed that students have taken wave mechanics course. Some introductory quantum physics books covering wave mechanics are

1. F. Mandl, *Quantum mechanics* (Butterworths Scientific Publications, London, 1957).
2. R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures in Physics*, Vol. 3, Addison-Wesley Publishing Co., Reading, MA, 1965.
2. S. Gasiorowicz, *Quantum Physics*, John Wiley and Sons, Inc., 1974.
3. H. S. Song, *Quantum Mechanics* (in Korean), Gyo-Hak-Yun-Gu-Sa, Seoul, 1984.
4. R. L. Liboff, *Introductory Quantum Mechanics*, Addison-Wesley Publishing Co., Reading, Massachusetts, 1989.
5. A. Goswami, *Quantum Mechanics*, Wm. C. Brown Publishers, Boston, 1997.
6. C. Lee, *Essential Quantum Physics*, Chungbum Publishing Co., Seoul, 2000.
7. J. S. Kang, *Quantum Mechanics* (in Korean), Sorim Press, Seoul, 2003.
8. D. J. Griffiths, *Introduction to Quantum Mechanics*, Pearson Prentice Hall, 2005.

This book covers a 32 week lecture course after one of the above books, in which however all the materals and problems cannot be dealt with in full depth. The course was aimed just for preparing basics needed in the research fields using quantum mechanics. More complete treatments can be found in the following books

9. J. J. Sakurai, *Advanced Quantum Mechanics*, John Wiley and Sons, Inc., 1967.
10. L. I. Schiff, *Quantum Mechanics* (third edition), Mc-Graw-Hill Book Co., 1968.
11. A. Messiah, *Quantum Mechanics*, Vol. 1 and Vol. 2, John Wiley and Sons, Inc., 1968.
12. E. Merzbacher, *Quantum mechanics* (second edition), John Wiley and Sons, Inc., 1970.
13. J. J. Sakurai, *Modern Quantum Mechanics* (revised edition), Addison-Wesley Publishing Co., Reading, Massachusetts, 1994.
14. K. Gottfried and T.-M. Yan, *Quantum Mechanics: Fundamentals* (the second edition), Springer, Berlin, 2003.

symbol	name, equation	value
$\hbar$	Planck constant = 1	$1.054\ 571\ 68(18)\times 10^{-34}$ J s
c	speed of light = 1	299 792 458 m/s
	GeV·s	1.519255×10^{24}
	GeV·cm	0.5067689×10^{14}
$\alpha_{\rm em}$	fine structure constant	1/137.035 999 11(46)
G_F	Fermi coupling constant	$1.166\ 37 \times 10^{-5}$ GeV^{-2}
M_P	Planck mass = $(8\pi G_N)^{-1/2}$	2.436×10^{18} GeV
m_e	electron mass	0.510 998 918(44) MeV
r_e	classical electron radius $e^2/4\pi m_e c^2$	$2.817\ 940\ 325(28) \times 10^{-13}$ cm
$\bar{\lambda}_e$	Compton wave length $\hbar/m_e c$	$3.861\ 592\ 678(26) \times 10^{-11}$ cm
m_p	proton mass	938.272 029(80) MeV
m_d	deuteron mass	1875.612 82(16) MeV
a_0	Bohr radius $r_e/\alpha_{\rm em}^2$	$0.529\ 177\ 2108(18) \times 10^{-8}$ cm
Ry	Rydberg energy $m_e c^2 \alpha_{\rm em}^2/2$	13.605 6923(12) eV
μ_B	Bohr magneton $e\hbar/2m_e$	$5.788\ 381\ 804(39) \times 10^{-11}$ MeV/T
μ_N	nuclear Bohr magneton $e\hbar/2m_p$	$3.152\ 451\ 259(21) \times 10^{-14}$ MeV/T
$(g-2)_e$	of electron	$\alpha_{\rm em}/\pi \simeq 0.00232$
$(g-2)_p$	of proton	$\simeq 3.59$
$(g)_n$	of neutron	$\simeq -3.83$
k	Boltzmann constant	0.862×10^{-4} eV/K
N_0	Avogadro number	6.02205×10^{23}/ mol