Chapter 3: Paradigm of particle physics: symmetries and their application to particle classification

The basic principles on continuous symmetries are first introduced, including the Poincare group, the internal symmetries, and Nöther current. Then, it is applied to $U(1)$ symmetry, the isospin space and the old $SU(3)_{\rm flavor}$ symmetry. A brief discussion on the $SU(6)$ symmetry which include $SU(3)_{\rm flavor} \times SU(2)_{\rm spin}$ is given.

I. INTRODUCTION

Symmetries make the study of physical laws concise and transparent. In late twenties the symmetries of various kinds are studied extensively, and it is understood that the quantum field theory(QFT), the basic tool of particle physics, is a theory completing the needed symmetries known in high energy physics.

II. SYMMETRIES IN QUANTUM MECHANICS

The symmetries in classical mechanics is elegantly studied from Hamilton's equation

$$\dot{\mathbf{q}}_i = \frac{\partial H}{\partial \mathbf{p}_i} \tag{1}$$

$$\dot{\mathbf{p}}_i = \frac{d\mathbf{p}_i}{dt} = -\frac{\partial H}{\partial \mathbf{q}_i} \tag{2}$$

where H is the classical Hamiltonian, and the coordinate q_i and momenta $\mathbf{p}_i = m(d\mathbf{q}_i/dt) = m\dot{\mathbf{q}}$ are the $6n$ degrees of freedom $(i = 1, 2, \cdot, n)$ for n particles. The above $\mathbf{q}_i$ and $\mathbf{p}_i$ are the conjugate variables. Making an infinitesimal transformation on $\mathbf{q}_i$,

$$\mathbf{q}_i \to \mathbf{q}_i + \delta\mathbf{q}_i, \tag{3}$$

the change of the Hamiltonian becomes,

$$\delta H = \sum_k \delta\mathbf{q}_k \cdot \frac{\partial H}{\partial \mathbf{q}_k} = -\sum_i \delta\mathbf{q}_i \cdot \dot{\mathbf{p}}_i \tag{4}$$

Let us choose, $\delta\mathbf{q}_i = \mathbf{q}$ so that the coordinate of every particle is translated by the same amount, i.e. the relative positions of the particles remain the same,

$$\delta H == -\delta\mathbf{q} \cdot \sum_i \dot{\mathbf{p}}_i \tag{5}$$

If the total momentum, $\sum_i \dot{\mathbf{p}}_i$, is the conserved quantity, it is a constant of motion and needs

$$\delta H = 0. \tag{6}$$

Therefore, conservation of the total momentum may equally be described by the invariance of H under the translation of the spatial coordinates. Even though we studied for an infinitesimal translation, the same conclusion can be drawn from a finite translation since a finite translation is obtained by compounding an infinite number of the infinitesimal translation.

The above symmetry argument on the classical Hamiltonian under the translation can be generalized in quantum mechanics also, since quantum mechanics is obtained by the correspondence principle. In the Schrödinger picture, the state vector $|\psi\rangle$ is time-dependent while operator O is time-independent. On the other hand, in the Heisenberg picture, the state vector is time-independent and the operator is time-dependent. Since $\mathbf{q}_i$ and $\mathbf{p}_i$ are operators in quantum mechanics, the symmetry principle can be easily copied in the Heisenberg picture. The Heisenberg equation of motion is

$$\frac{dO}{dt} = \frac{i}{\hbar}[H, O]. \tag{7}$$

Thus, if $[H, O] = 0$, then $\dot{O} = 0$. The expectation values of O, $\langle O \rangle$, are the same in the Schrödinger and Heisenberg pictures. Thus, if $\dot{O} = 0$, the measrement of O in the state ψ gives a constant of motion. Namely, the eigenvalues of O are conserved quantities. This is equivalent to the statement that O commutes with the Hamiltonian,

$$HO = OH, \quad \text{or} \quad OHO^{-1} = H, \tag{8}$$

where the second one is a similarity transformation for the infinitisimal transformation. In general, for a finite transformation in case for the continuous symmetry, we obtain also the similarity transformation which is not by O but by a finite transformation operator. The expression will be given explicitly later. Therefore, we have the following equivalent statements:

(i) Eigenvalues of O are conserved.

(ii) Hamiltonian is invariant under the similarity transformation.

(iii) The operator O, the transformation generator, commutes with the Hamiltonian.

Example 1. Parity

For a wave function $\psi(\mathbf{x})$, the parity operator changes $\psi(\mathbf{x})$ to

$$P\psi(\mathbf{x}) = \psi(-\mathbf{x}). \tag{9}$$

If $[P, H] = 0$, then

$$PH(\mathbf{x})\psi(\mathbf{x}) = H(-\mathbf{x})\psi(-\mathbf{x}) = H(\mathbf{x})P\psi(\mathbf{x}) = H(\mathbf{x})\psi(-\mathbf{x}). \tag{10}$$

Note that we are dealing with just the complex functions. Therefore, we obtain

$$H(-\mathbf{x}) = H(\mathbf{x}), \tag{11}$$

i.e. the Hamiltonian is invariant under the space inversion.

Example 2. Space displacement

The differential operator D on the wave function $\psi(\mathbf{x})$ is

$$D = 1 + \delta\mathbf{x} \cdot \frac{\partial}{\partial \mathbf{x}}, \tag{12}$$

i.e.

$$\psi(\mathbf{x}+\delta\mathbf{x}) = D\psi(\mathbf{x}). \tag{13}$$

Then,

$$DH(\mathbf{x})\psi(\mathbf{x}) = H(\mathbf{x}+\delta\mathbf{x})\psi(\mathbf{x}+\delta\mathbf{x}) = H(\mathbf{x}+\delta\mathbf{x})D\psi(\mathbf{x}). \tag{14}$$

If $[D,H]=0$, then the above equation is the same as $H(\mathbf{x})D\psi(\mathbf{x})$, implying

$$H(\mathbf{x}+\delta\mathbf{x}) = H(\mathbf{x}). \tag{15}$$

If the Hamiltonian is invariant under the displacement, the gradient operator(momentum),

$$-i\hbar\frac{\partial}{\partial\mathbf{x}}\psi = \mathbf{p}\psi, \tag{16}$$

must be conserved.

We can generalize the dynamical variables $\mathbf{x}_i$ to a generalized coordinates q_i.

Example 3. Orbital angular momentum in the z-direction

The rotation around the z-axis is generated by

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\frac{\partial}{\partial\phi} \tag{17}$$

where ϕ is the azimuthal angle, $\tan\phi = y/x$, or $\phi = \tan^{-1}(y/x)$. Thus,

$$L_z = -i\hbar\left(x\frac{1/x}{1+(y^2/x^2)} + y\frac{y/x^2}{1+(y^2/x^2)}\right) = -i\hbar\frac{\partial}{\partial\phi}. \tag{18}$$

The corresponding infinitesimal generator is defined as

$$O = 1 + \delta\phi\frac{\partial}{\partial\phi} \equiv 1 + \frac{i}{\hbar}\delta\phi L_z. \tag{19}$$

n successive infinitesimal(ϕ/n) rotation is $(1+i\frac{\phi/n}{\hbar}L_z)^n$ which becomes as $n\to\infty$,

$$e^{i\phi L_z/\hbar} \tag{20}$$

which is an operator for a finite rotation around the z-axis. In general, for a finite rotation, the transform operator is an exponentiated generator. The conservation of the angular momentum corresponds to the invariance of the Hamiltonian under the rotaion around the axis in consideration.

Example 4. Maxwell equations

The classical electromagnetic(we go back to a classical theory) is described by the vector potential $\mathbf{A}$ and the scalar potential ϕ. The Naxwell equations have the property that the scale of ϕ and $\mathbf{A}$ is arbitrary, which is called the scale invariance,

$$\begin{aligned}\mathbf{A}' &= \mathbf{A} + \vec{\nabla}\chi \\ \phi' &= \phi - \frac{1}{c}\frac{\partial\chi}{\partial t}.\end{aligned}$$

Table II. Some symmetries considered in physics

Transformation (symmetry of H)	conserved quantity (or eigenvalue)
space-displacement	momentum
time-displacement	energy
spatial-rotation	angular momentum
space inversion	parity ; ± 1
rotation in isospin space	isospin
charge conjugation	C ; ± 1
G conjugation	G parity ; ± 1
gauge transform $U(1)_{em}$	electric charge
gauge transform $SU(3)_c$	color
gauge transform $SU(2) \times U(1)$	weak charge
global transform $U(1)_B$	B number
global transform $U(1)_L$	L number

The above transformation do not change **E** and **B**, and so the physics is not changed by the above transformation. This invariance corresponds to the fact that the scale of the electrostatic potential is entirely arbitrary. Only the difference of the electrostatic potential matters. The invariance under the above gauge transformation leads to the electromagnetic charge conservation.

In Table II, we list some interesting symmetries. Except **p**, E, **J**, Q_{em}, and color, any discrete or continuous symmetries are violated in the modern particle theory. **p**, E, and **J** should be conserved because we require the Poincare invariance. Any exact gauge symmetry is theoretically permissible, while global symmetries are in doubt, in particular if we consider the gravitational effects. In Table II, the baryon number conservation, the lepton number conservation are also shown, but it is believed that they are broken, and their discoveries are expected in the proton decay and in the neutrinoless double beta decay experiments.

III. SYMMETRIES IN QUANTUM FIELD THEORY

Relativistic quantum theory unifies the quantum idea with the relativity principle, and is based on the following two postulates:

1. For each state of a physical system, there corresponds a unique vector in a linear vector space (quantum theory), and

2. This linear vector space is a representation space of the Poincare group $L_+^\uparrow$, or in other words the vectors representing the physical states have well-defined transformation properties under the action in $L_+^\uparrow$ (relativity).

The above statement can be extended to quantum field theory. It is formulated in terms of the representation theory (due to Wigner): (1) Any symmetry group can be represented in the Hilbert space in terms of unitary operators except a very few antiunitary operators(e.g. time reversal), and (2) Generators of a symmetry group are constants of motion.

The relativity principle requires the relativistic field equations. QFT implies that we deal with the field operators. The bridge between the symmetries of a wave function in quantum mechanics and those of the quantum field operators is the interpretation of the matrix element of a quantum field operator between one particle state and the vacuum as the wave function.

Let us begin to see how the field operators, denoted as $\Phi(x)$, transform under the symmetry transformation, denoted as a unitary operator U(parameters for transformation), in consideration.

The operator equation remains the same under the transformation

$$AB = C \longrightarrow A'B' = C' \tag{21}$$

where $A' = UAU^{-1}$, etc. Then the equation of motion and the quantum commutators are invariant. The field operators transform as

$$\Phi(x) \to \Phi' = U\Phi(x)U^{\dagger} \tag{22}$$

where x is a label in the Hilber space which we, as stationary observers, do not want to change. A state vector $|\Psi\rangle$ also changes under the transformation,

$$|\Psi'\rangle = U|\Psi\rangle. \tag{23}$$

This representation is meaningful since the physically observable quantity $\langle\Psi|A|\Psi\rangle$ is invariant under the transformation,

$$\langle\Psi'|A'|\Psi'\rangle = \langle U\Psi|UAU^{\dagger}|U\Psi\rangle = \langle\Psi|U^{\dagger}UA|\Psi\rangle = \langle\Psi|A|\Psi\rangle. \tag{24}$$

How about that we are not stationary observers? Therefore, let us consider the coordinate change via

$$x' = \Lambda x + a \tag{25}$$

which corresponds to the Poincare transformation, with 4×4 matrix Λ and a 4×1 matrix four vector a. These transformation set is represented by (Λ, a).

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2)x' = \Lambda_2 x' + a_2 \tag{26}$$

$$x'' = \Lambda_1 x' + a_1 = \Lambda_1\Lambda_2 x + a_1 + \Lambda_1 a_2 U(0,\Lambda)U(a_1,\Lambda)U^{\dagger}(0,\Lambda) = U(\Lambda a_1, \Lambda\Lambda_1\Lambda^{-1}) \tag{27}$$

$$|(\Phi',\Psi')|^2 = |(\Phi,\Psi)|^2(\Phi',\Psi') = (\Phi,\Psi)^* = (\Psi,\Phi) \tag{28}$$

$$\phi_A'^{c-no}(x) = D_{AB}\phi_B^{c-no}(\Lambda^{-1}(x-a))(\Psi,\phi_A'(x)\Psi) = D_{AB}(\Psi,\phi_B(\Lambda^{-1}(x-a))\Psi) \tag{29}$$

$$U(a, \Lambda = I + \epsilon) = I + ia_\mu P^\mu - \frac{1}{2}\epsilon_{\mu\nu}M^{\mu\nu} \tag{30}$$

$$iM_{\mu\nu} = I_{\mu\nu}P^{\mu\dagger} = P^\mu, M^\dagger_{\mu\nu} = M_{\mu\nu} \tag{31}$$

$$U(0, \Lambda^{-1})U(a_1, \Lambda_1)U(0, \Lambda) = U(\Lambda^{-1}a, \Lambda^{-1}\Lambda_1\Lambda) \tag{32}$$

$$U(a, \Lambda^{-1})U(a, I)U(0, \Lambda) = U(\Lambda^{-1}a, I) = I + i(\Lambda^{-1}a_1)_\mu P^\mu = I + ia_\nu \Lambda^\nu{}_\mu P^\mu \tag{33}$$

$$U^\dagger(0, \Lambda)P^\mu U(0, \Lambda) = \Lambda^\mu{}_\nu P^\nu \tag{34}$$

$$U^\dagger(0, \Lambda)M_{\mu\nu}U(0, \Lambda) = \Lambda_{\mu\rho}\Lambda_{\nu\sigma}M^{\rho\sigma} \tag{35}$$

$$U^\dagger(0, I)P_\mu U(a, I) = P_\mu \tag{36}$$

$$[P^\mu, P_\nu] = 0 \tag{37}$$

$$\Lambda = I = \epsilon U^\dagger(0, \Lambda)P_\mu U(0, \Lambda) = \Lambda_\mu{}^\nu P_\nu \tag{38}$$

$$i[M_{\alpha\beta}, p_\mu] = -P_\alpha g_{\beta\mu} + P_\beta g_{\alpha\mu} \tag{39}$$

$$\Lambda = I = \epsilon U^\dagger(0, \Lambda)M_{\mu\nu}U(0, \Lambda) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma M_{\rho\sigma} \tag{40}$$

$$i[M_{\alpha\beta}, M_{\mu\nu}] = g_{\alpha\beta}M_{\beta\nu} - g_{\alpha\nu}M_{\beta\mu} - g_{\beta\mu}M_{\alpha\nu} + g_{\beta\nu}M_{\alpha\mu} \tag{41}$$

$$U^\dagger(0, \Lambda)P^\mu U(0, \Lambda) = \Lambda^\mu{}_\nu P^\nu \tag{42}$$

$$\left(I - ia_\mu P^\mu + i\frac{\epsilon_{\mu\nu}}{2}M^{\mu\nu}\right)\phi_A(x)\left(1 + ia_\rho P^\rho - i\frac{\epsilon_{\rho\sigma}}{2}M^{\rho\sigma}\right) \tag{43}$$

$$= D_{AB}\phi_B(\Lambda^{-1}(x - a)) = D_{AB}\phi_b(x - \epsilon x - a) \tag{44}$$

$$LHS = \phi'_A(x) = \phi(x) - ia_\mu[P_\mu, \phi_A(x)] + i\frac{\epsilon_{\mu\nu}}{2}[M^{\mu\nu}, \phi_A(x)] \tag{45}$$

$$RHS = D_{AB}\left\{\phi_B(x) - (\epsilon x)_\mu \frac{\partial \phi_B}{\partial x_\mu} - a_\mu \frac{\partial \phi_B}{\partial x_\mu}\right\} \tag{46}$$

$$D_{AB}\left\{\phi_B(x) - \frac{1}{2}\epsilon_{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\phi_B(x) - a_\mu\partial^\mu\phi_B(x)\right\} \tag{47}$$

$$= \left\{\phi_A(x) - \frac{1}{2}\epsilon_{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\phi_A(x) - a_\mu\partial^\mu\phi_B(x)\right\} \tag{48}$$

$$i[P_\mu, \phi_A(x)] = \partial_\mu \phi_A(x) \tag{49}$$

$$i[M_{\mu\nu}, \phi_A(x)] = (x_\mu \partial_\nu - x_\nu \partial_\mu)\phi_A(x) + (I_{\mu\nu})_{AB}\phi_B(x) \tag{50}$$

$$D_{AB} = \delta_{AB} + \frac{1}{2}\epsilon_{\rho\sigma}(I^{\rho\sigma})_{AB}\phi_B(x) \tag{51}$$

$$[P^0, P^\mu] = 0 = -i\dot{P}^\mu \tag{52}$$

$$i[M_{\alpha\beta}, P_0] = -P_\alpha g_{\beta 0} + P_\beta g_{\alpha 0} \tag{53}$$

$$U(a,\Lambda) = e^{ia_\mu P^\mu - \frac{i}{2}\epsilon_{\mu\nu}M^{\mu\nu}} \tag{54}$$

$$U(a,0) = e^{ia\cdot P} \tag{55}$$

$$\phi(x) \to \phi'(x) = U^\dagger \phi(x) U = \phi(x-a) \tag{56}$$

$$\phi(x+a) = U^\dagger \phi(x) U \tag{57}$$

$$\phi(a) = e^{-ia\cdot P}\phi(0)e^{ia\cdot P} \tag{58}$$

$$W_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}M^{\beta\gamma}P^\delta \tag{59}$$

$$W^2 = \frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 - M_{\mu\nu}M^{\rho\nu}P^\mu P_\rho \tag{60}$$

$$W_\alpha P^\alpha = 0 \tag{61}$$

A A global $U(1)$ symmetry

Now let us restrict for the case with a stationary observer. For a c-number wave function $\phi(x)_{\mathrm{c-no}}$, the phase symmetry is given for a constant infinitesimal number ϵ as,

$$\phi_{\mathrm{c-no}}(x) \to \phi'_{\mathrm{c-no}}(x) = e^{i\epsilon}\phi_{\mathrm{c-no}}(x) = (1+i\epsilon)\phi_{\mathrm{c-no}}(x), \tag{62}$$

or

$$\delta\phi_{\mathrm{c-no}} = i\epsilon\phi_{\mathrm{c-no}}(x). \tag{63}$$

On the other hand, the q-number $\phi_{\mathrm{q-no}}(x) \equiv \phi(x)$, the quantum field operator, transforms under a $U(1)$ transformation as($U = e^{i\epsilon Q}$)

$$\phi'(x) = U\phi(x)U^\dagger = (1+i\epsilon Q)\phi(x)(1-i\epsilon Q) = \phi(x) - i\epsilon[\phi(x), Q] \tag{64}$$

where Q is the generator of the transformation. Taking an expectation value of Eq. (64) between the vacuum and one particle state, one must produce (62) which is an expression for the c-number wave function $\phi_{\rm c-no} = \langle 0|\phi|p\rangle$. It is satisfied for the quantum field satisfying commutater

$$[Q, \phi] = \phi. \tag{65}$$

Acting this commutater on the vaccum state,

$$[Q, \phi(x)]|0\rangle = \phi(x)|0\rangle, \tag{66}$$

we obtain

$$Q|\phi\rangle = |\phi\rangle, \tag{67}$$

since the vacuum has the vanishing Q eigenvalue. Namely, the state $|\phi\rangle$ has the one unit eigenvalue of Q, such as the electromagnetic charge if Q is the electromagnetic charge operator.

We observed that if $[Q, H] = 0$, then the charge Q is conserved. Namely, the difference(=the total charge Q) between the number of particles and the number of antiparticles is conserved.

B The Gell-Mann-Levy equation

Let us generalize the above global $U(1)$ case, in the Lagrangian field theory. The action is expressed in terms of a Lagrangian $\mathcal{L}(\phi(x), \dot{\phi}(x))$ which is a function of the quantum field ϕ and its time derivative $\dot{\phi}$,

$$I = \int d^4x \mathcal{L}(x) \tag{68}$$

where the space-time dependence comes from the field operators.

In general, there can be many parameters, $\alpha_s(s = 1, 2, \cdots)$, for the transformation, instead of just one(ϵ in the $U(1)$ case). The generator(combination of field operators) for the s-direction transformation is denoted as T^s, i.e. we suppose

$$\begin{aligned} \delta\phi(x) &= \delta\alpha_s(x) T^s \phi(x) \\ \delta\phi^*(x) &= -\delta\alpha_s(x) T^s \phi^*(x). \end{aligned} \tag{69}$$

Without considering the surface term, if the Lagrangian is invariant under the above transformation, then the invariance of the action is implied, The variation of $\mathcal{L}$ under the above transformation is

$$\mathcal{L}[\phi] \Rightarrow \mathcal{L}[\phi + \delta\phi, \cdots] = \mathcal{L}[\phi, \cdots] \to \delta I = 0. \tag{70}$$

The variation of $\mathcal{L}$ under the above transformation is

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi). \tag{71}$$

Note that

$$\delta\partial_\mu\phi = \partial_\mu(\phi(x) + \delta\phi(x)) - \partial_\mu\phi(x) = (\partial_\mu\delta\alpha_s)T^s\phi + \delta\alpha_s T^s\partial_\mu\phi, \tag{72}$$

or we have

$$\delta I = \int d^4x \left[\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\right)\right]\delta\alpha_s T^s\phi \tag{73}$$

where the surface term is neglected. Therefore, $\delta I = 0$ implies,

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \right) = \frac{\delta \mathcal{L}}{\delta \phi} \tag{74}$$

which is the so-called Gell-Mann-Levy equation. It is useful to study the continuous internal(not including the space-time transformation) symmetries in quantum field theory. The Gell-Mann-Levy equation can be expressed as conserved 4-divergence

$$\partial^\mu j^s_\mu = 0, \tag{75}$$

where

$$j^s_\mu(x) = \frac{\delta \mathcal{L}}{\delta \partial^\mu \alpha_s}. \tag{76}$$

Because of the conserved 4-divergence, the charge Q^s

$$Q^s = \int d^3x j^{s0}(x) \tag{77}$$

is a constant of motion, and we obtain the conservation laws for Q_{em}, B, L, etc. given in Table 3.1.

C Isospin: the internal $SU(2)$ symmetry

In Eq. (69), we considered many parameters for the continuous transformation. If these parameters are not related, we will have the continuous group $U(1)^n$ for n parameters of transformation. However, some parameters of the transformation can be related. This relation is the relation between the generators, T^s.

The first internal symmetry example along this line was suggested by Heisenberg in 1932. He noted that the masses of proton and neutron are very similar, and conjectured that it arises from a deeper symmetry principle so that proton and neutron are just a different way of looking the same thing. So, there must be an internal symmetry, realizing a two component state. Since we know that the spinor representation of rotation group $SO(3)$ allows a doublet state, he conjectures a similar group structure for this internal space and called it isotopic spin. Nowadays, we call it *isospin*. Namely, the nucleon state vector $|N\rangle$ has two components

$$N = \begin{pmatrix} p \\ n \end{pmatrix}. \tag{78}$$

The rotation in the isospin space is done by an operator U. For the state (78), U must be a 2×2 unitary matrix which is generated by 4 independent 2×2 matrices. If we factor out one symmetry which describes the conservation of the nucleon number,

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}, \tag{79}$$

then U has three independent parameters or it is generated by traceless 3 independent Hermitian matrices, since for an infinitesimal rotation,

$$e^{i\alpha_i T_i} e^{i\alpha_i T_i^\dagger} = I = (I + i\alpha_i T_i + \cdots)(I - i\alpha_i T_i^\dagger + \cdots) \tag{80}$$

implies $T_i = T_i^\dagger$. These Hermitian matrices are the Pauli matrices σ_i $(i = 1, 2, 3)$, satisfying

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}. \tag{81}$$

Threfore, the previous T_s are not independent, but they are intimately related by the commutators as given by (81). Then, an infinitesimal rotation is done by an operator

$$e^{i\alpha_i T_i} \simeq I + i\alpha_i T_i \tag{82}$$

where $T_i = \sigma_i/2$.

The isospin symmetry is called an $SU(2)$ symmetry, the group action isomorphic to the multiplication of 2×2 special(determinant $= 1$) unitary($U^{-1} = U^\dagger$) matrices. The identity

$$\det e^A = e^{\mathrm{Tr}A} = 1 \tag{83}$$

implies that special unitarity is generated by traceless Hermitian matrices.

A state vector in the isospin space has the components with similar masses, but the electromagnetic charges can be different by one units as the an angular momentum representation has components differinf J_3 eigenvalues by one units. The state vector N given above has this property. The generators for the isospin space rotation for the doublet is

$$T_1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{84}$$

Another example is the pion system

$$\pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \tag{85}$$

In other words,

$$\pi^+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \pi^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \pi^- = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{86}$$

The K meson system(τ or θ of Chapter 2) is also a state vector in the isospin space,

$$K^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{87}$$

Also their antiparticles form an isospin doublet,

$$\bar{K}^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{88}$$

The group action on the isospin space is identical to the familiar rotation in the three dimensional space.

G-parity

We noted in Chapter 2 that π^0 is an eigenstate of the charge conjugation operation C, while $\pi^\pm$ are not eigenstates of C. But as we argued above, in the isospin space formalism it is better to have a unfied quantum number for an isospin multiplet. Note that the isospin multiplet has components with different charges, which was the definition of the isospin space. Namely, the electromagnetic charge operator Q_{em} is T_3. For this purpose, we define *G-parity* so that the members of an isospin multiplet have the same quantum number for G. It must involve C, of course, and can involve a rotation in the isospin space. The isospin space rotation of 180^o around the y(or the 2nd) axis fulfils our requirement,

$$G = Ce^{i\pi T_y}. \tag{89}$$

To show it explicitly for the pion multiplet, for which the algebra is a bit more complicated than for the doublet, we note that for the triplet

$$T_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 - i & 0 \\ i \;\; 0 & -i \\ 0 \;\; i & 0 \end{pmatrix}. \tag{90}$$

Note that

$$M \equiv iT_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad M^2 = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}, \tag{91}$$

and the rest are simply calculated, noting

$$M^3 = -M, \quad M^4 = -M^2, \quad \text{etc.}$$

Therefore, $e^{i\theta T_y}$ becomes

$$\begin{aligned} e^{\theta M} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots \right) + M \left(\theta - \frac{\theta^3}{3!} + \cdots \right) \\ &= \begin{pmatrix} \frac{1+\cos\theta}{2} & 0 & \frac{1-\cos\theta}{2} \\ 0 & \cos\theta & 0 \\ \frac{1-\cos\theta}{2} & 0 & \frac{1+\cos\theta}{2} \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sin\theta & 0 \\ -\sin\theta & 0 & \sin\theta \\ 0 & -\sin\theta & 0 \end{pmatrix}. \end{aligned} \tag{92}$$

Rotation with 180^0 is generated by the matrix

$$e^{i\pi T_y} = e^{\pi M} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{93}$$

The above isospin rotation gives,

$$e^{i\pi T_y} \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} = \begin{pmatrix} \pi^- \\ -\pi^0 \\ \pi^+ \end{pmatrix}. \tag{94}$$

Acting C, we obtain

$$G \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} = - \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}. \tag{95}$$

Thus, we have the same G-parity quantum number for the pion multiplet. It can be interpreted as a generalization of C to the isospin multiplet.

But for nucleons, it is not meaningful to have the G quantum number since any of the nucleon multiplet is not an eigenstate of C. Nevertheless, it is important to see the action of G on the nucleon multiplet. When we generalize the charge conjugation operation to an isospin multiplet, it must be G. For N, the G-conjugated nucleon is called $\bar{N}$,

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \quad \bar{N} = \begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}, \tag{96}$$

where the second equation is obtained from

$$\bar{N} = Ce^{i\pi T_2} \begin{pmatrix} p \\ n \end{pmatrix}. \tag{97}$$

Note

$$e^{i\theta\sigma_2/2} = \cos(\theta/2) + i\sigma_2 \sin(\theta/2). \tag{98}$$

Thus,

$$e^{i\pi T_2} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{99}$$

And we obtain,

$$\bar{N} = C \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}. \tag{100}$$

With the above notation, let us calculate an $SU(2)$ invariant,

$$\frac{1}{\sqrt{2}}\epsilon^{ij} N_i \bar{N}_j = \frac{p\bar{p} + n\bar{n}}{\sqrt{2}}, \tag{101}$$

where the $SU(2)$ Levi-Cicvita tensor is

$$\epsilon^{12} = -1, \quad \epsilon^{21} = 1. \tag{102}$$

The group multiplication conserves the total number of states, denoted as, for the multiplication of the above doublet and doublet, we have $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$. Therefore, for the triplet we have

$$\begin{pmatrix} p\bar{n} \\ \frac{1}{\sqrt{2}}(p\bar{p} - n\bar{n}) \\ n\bar{p} \end{pmatrix}. \tag{103}$$

Sometimes, it is easy to construct the algebra, using the creation and annihilation operators[1] Since the triplet state (103) has zero nucleon number it can be interpreted as a triplet meson multiplet. Indeed, we have such ones, the spin-0 π-meson and spin-1 ρ-meson. Therefore, consider for example Eq. (103) indeed describes the pion system. Let us represent its components by creation operators of particle(a_i^*) and antiparticle(b_i^*) with $a_1 = p, a_2 = n, b_1 = \bar{p}, b_2 = \bar{n}$. In view of (103), we represent

$$|\pi^+\rangle = |p\bar{n}\rangle = a_1^* b_2^* |0\rangle \equiv T_1^2 |0\rangle, \tag{104}$$

$$|\pi^0\rangle = |\frac{p\bar{p} - n\bar{n}}{2}\rangle = \frac{1}{2}(a_1^* b_1^* - a_2^* b_2^*)|0\rangle = \frac{1}{2}(T_1^1 - T_2^2)|0\rangle, \tag{105}$$

Table 3.2. Spin-0 mesons

	(π^+, π^0, π^-)	η	(K^+, K^0)	$(\bar{K}^0, K^-)$
I	1	0	1/2	1/2
Y	0	0	1	-1
S	0	0	1	-1
Mass(MeV)	140	550	495	495

Table 3.3. Spin-$\frac{1}{2}$ baryons

	$(\Sigma^+, \Sigma^0, \Sigma^-)$	Λ	(p, n)	(Ξ^0, Ξ^-)
I	1	0	1/2	1/2
Y	0	0	1	-1
S	-1	-1	0	-2
Mass(MeV)	1195	1115	939	1318

$$|\pi^-\rangle = |n\bar{p}\rangle = a_2^* b_1^* |0\rangle = T_2^1 |0\rangle. \tag{106}$$

In general, the traceless matrices can be constructed as

$$T_\nu^\mu = a_\nu^* b_\mu^* - \frac{1}{2}\delta_\nu^\mu a_\lambda^* b_\lambda^*. \tag{107}$$

The above T_ν^μ have one to one correspondence with the generators T_i, and represent the isospin $SU(2)_I$.

IV. FLAVOR $SU(3)$ AND INTRODUCTION OF COLOR

The identification of the pion system with nucleon system is very intreaguing. It renders a proposal that pions are the nucleon and antinucleon composite particles. Indeed, this proposal was put forward by Fermi and Yang and called the Fermi-Yang model.

In 1950's plenty of hadrons were discovered. Classifying them in the vicinity of similar masses for the same spin, we have particles as shown in Tables 3.2, 3.3, and 3.4.

In these Tables, the number of particles come in a multiplet **8**. One should find an internal space which allows a reasonable **8**. However, first note that the isospin space has a connection with Q_{em}. For the pion, $T_3 = Q_{em}$. On the other hand, for the nucleon, $T_3 = \text{diag.}(\frac{1}{2}, -\frac{1}{2})$ does not describe the electromagnetic charge of proton and neutron. But notice that it is

Table 3.4. Spin-1 mesons

	(ρ^+,ρ^0,ρ^-)	ω,ϕ	(K^{*+},K^{*0})	$(\bar{K}^{*0},K^{*-})$
I	1	0,0	1/2	1/2
Y	0	0,0	1	-1
S	0	0	1	-1
Mass(MeV)	750	740,1020	890	890

shifted by a $\frac{1}{2}$ unit. Since the nucleon carries the baryon number(B), we can add half of B to T_3 to interpret Q_{em}. Note that pions do not carry B and previous interpretaion for pions is identical to this new definition. In other words, Q_{em} is a combination $Q_{em} = T_3 + \frac{1}{2}B$. Note, however, the K meson system is also a doublet, but does not carry B. But notice again that K and $\bar{K}$ mesons carry a new quantum number which we introduce to obtain the correct electromagnetic charge. Since it is very strange from the baryon number,[2] let us call it the *strangeness* S. The S quantum numbers of K and $\bar{K}$ are defined to be –1 and +1, respectively. Then, for the K meson systems, $Q_{em} = T_3 + \frac{1}{2}S$ correctly produces the electromagnetic charges of the K mesons. In these examples, we note that T_3 is not all for the electromagnetic charge. A generalization of the above two facts is achieved by introducing a new quantum number, *hypercharge* Y,

$$Q_{em} = T_3 + \frac{1}{2}Y \tag{108}$$

where

$$Y = B + S. \tag{109}$$

Eq. (108) is the so-called Gell-Mann-Nishijima formula. In Tables 3.2, 3.3, and 3.4, the Y quantum numbers are also shown.

Since a new quantum number is introduced, the proton and neutron creation operators cannot fulfil the needed algebra. The two creation operators were B and T_3 in the isospin formalism with $SU(2)$. With one more quantum number, we need one more creation operator. This new entity must carry S. In Table 3.3, the particle Λ appearing together with p and n is recommendable since it is neutral, but its baryon number is 1. Thus, it must carry $S = -1$ to satisfy the Gell-Mann-Nishijima formula. This was strange, and the quantum number strangeness was introduced. Therefore, let us try creation operators of particles and antiparticles of p, n, and Λ. It is called the Sakata model.

Sakata model

Now the nucleon triplet is

$$N = \begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}. \tag{110}$$

Note that a_1^* creates p, b_1^* creates $\bar{p}, \cdots,$, b_3^* creates $\bar{\Lambda}$. Because the state(complex state) (110) is a 3×1 matrix, the rotation in this internal space is generated by 3×3 traceless Hermitian matrices. This forms indeed a group called $SU(3)$. The generalization of Pauli's

σ's is Gell-Mann's λ's,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{111}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{112}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{113}$$

The λ's staisfy

$$\mathrm{Tr}\lambda_i = 0, \quad \mathrm{Tr}\lambda_i\lambda_j = 2\delta_{ij}. \tag{114}$$

The $SU(3)$ group generators are

$$F_i = \frac{\lambda_i}{2}, \tag{115}$$

which satisfies

$$\mathrm{Tr}F_iF_j = \frac{1}{2}\delta_{ij}. \tag{116}$$

In general, for the funtamental representation $\mathbf{N}$ of $SU(N)$, the F_i matrices are normalized as in Eq. (116). As in the $SU(2)$ case with $T_i = \frac{\sigma_i}{2}$, and $[T_i, T_j] = i\epsilon_{ijk}T_k$, we have for $SU(3)$

$$[F_i, F_j] = if_{ijk}F_k, \tag{117}$$

where completely antisymmetric f_{ijk} is called the structure constants of the group $SU(3)$. Note that the anticommutator, unlike in the case of $SU(2)$, is nonvanishing.

$$\{F_i, F_j\} = d_{ijk}F_k + \frac{1}{3}\delta_{ij}, \tag{118}$$

where completely symmetric d_{ijk} depends on the representation of $SU(3)$. Eqs. (117) and (118) are useful in reducing the number of F's. For example,

$$\mathrm{Tr}F_l[F_i, F_j] = if_{ijk}\frac{1}{2}\delta_{lk} = \mathrm{Tr}[F_j, F_l]F_i = \frac{i}{2}f_{jli}. \tag{119}$$

To simplify the anticommutator, we define

$$F_0 = \frac{1}{2}\sqrt{\frac{2}{3}}I, \quad f_{ij0} = 0, \quad d_{ij0} = \sqrt{\frac{2}{3}}\delta_{ij}. \tag{120}$$

Then, the anticommutator is supplied with

$$\{F_0, F_j\} = \sqrt{\frac{2}{3}}F_j, \quad \{F_0, F_0\} = \sqrt{\frac{2}{3}}F_0. \tag{121}$$

Eq. (118) becomes

$$\{F_i, F_j\} = d_{ij\mu}F_\mu, \quad (\mu = 0, 1, \cdots, 8). \tag{122}$$

Thus, we summarize

$$[F_i, F_j] = if_{ijk}F_k, \quad \{F_i, F_j\} = d_{ijk}F_k, \quad i,j,k = (0,1,\cdots,8). \tag{123}$$

As in the case of the Fermi-Yang model, Sakata tried to introduce one more entity as a fundamental one. For this, he added Λ. Thus, the nucleon triplet is introduced,

$$N = \begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}. \tag{124}$$

The isospin and strange quantum numbers of p, n, and Λ are shown in Table 3.3. In particular, the eigenvalues of I_3 are

$$I_3|p\rangle = \frac{1}{2}|p\rangle, \quad I_3|n\rangle = -\frac{1}{2}|n\rangle, \quad I_3|\Lambda\rangle = 0|\Lambda\rangle. \tag{125}$$

The triplet **3** given in Eq. (124) is interpreted as the fundamental representation of $SU(3)$. The antitriplet $\bar{\mathbf{3}}$ is the G-conjugated N,

$$\bar{N} = \begin{pmatrix} \bar{n} \\ -\bar{p} \\ \bar{\Lambda} \end{pmatrix}. \tag{126}$$

Thus, $\mathbf{3} \times \bar{\mathbf{3}}$ contains a singlet $\mathbf{1}= p\bar{p} + n\bar{n} + \Lambda\bar{\Lambda}$, and the rest is an octet **8** of $SU(3)$. The **8** contains,

$$\begin{aligned} &\pi^+ = p\bar{n}, \quad \pi^- = n\bar{p}, \quad \pi^0 = \frac{1}{\sqrt{2}}(p\bar{p} - n\bar{n}), \quad \eta = \frac{1}{\sqrt{6}}(p\bar{p} + n\bar{n} - 2\Lambda\bar{\Lambda}), \\ &K^+ = p\bar{\Lambda}, \quad K^0 = n\bar{\Lambda}, \quad \bar{K}^+ = K^- = \Lambda\bar{p}, \quad \bar{K}^0 = \Lambda\bar{n}. \end{aligned} \tag{127}$$

Now we can generalize the $SU(2)$ tensor notation of (107) to $SU(3)$. Thus,

$$\begin{gathered} B_1^2 = F_1 + iF_2, \quad B_2^1 = F_1 - iF_2, \quad B_1^3 = F_4 + iF_5, \quad B_3^1 = F_4 - iF_5, \quad B_2^3 = F_6 + iF_7, \\ B_3^2 = F_6 - iF_7, \quad \tfrac{1}{2}(B_1^1 - B_2^2) = F_3, \quad B_3^3 = -\tfrac{1}{2}(B_1^1 + B_2^2) = -\tfrac{2}{\sqrt{3}}F_8, \end{gathered}$$

which can be summarized as

$$(B_\rho^\sigma)_\mu^\nu = \delta_\rho^\nu \delta_\mu^\sigma - \frac{1}{3}\delta_\rho^\sigma \delta_\mu^\nu. \tag{128}$$

Then, the group algebra becomes,

$$[B_\alpha^\beta, B_\mu^\lambda] = \delta_\mu^\beta B_\alpha^\lambda - \delta_\alpha^\lambda B_\mu^\beta. \tag{129}$$

The components are denoted as P_ν^μ. Then,

$$\begin{gathered} |\pi^+\rangle = P_1^2|0\rangle, \quad |\pi^-\rangle = P_2^1|0\rangle, \quad |\pi^0\rangle = \frac{1}{\sqrt{2}}(P_1^1 - P_2^2)|0\rangle, \\ |\eta\rangle = \frac{1}{\sqrt{6}}(P_1^2 + P_2^2 - 2P_3^2)|0\rangle \\ |K^+\rangle = P_1^3|0\rangle, \quad |K^0\rangle = P_2^3|0\rangle, \quad |K^-\rangle = P_3^1|0\rangle, \quad |\bar{K}^0\rangle = P_3^2|0\rangle. \end{gathered} \tag{130}$$

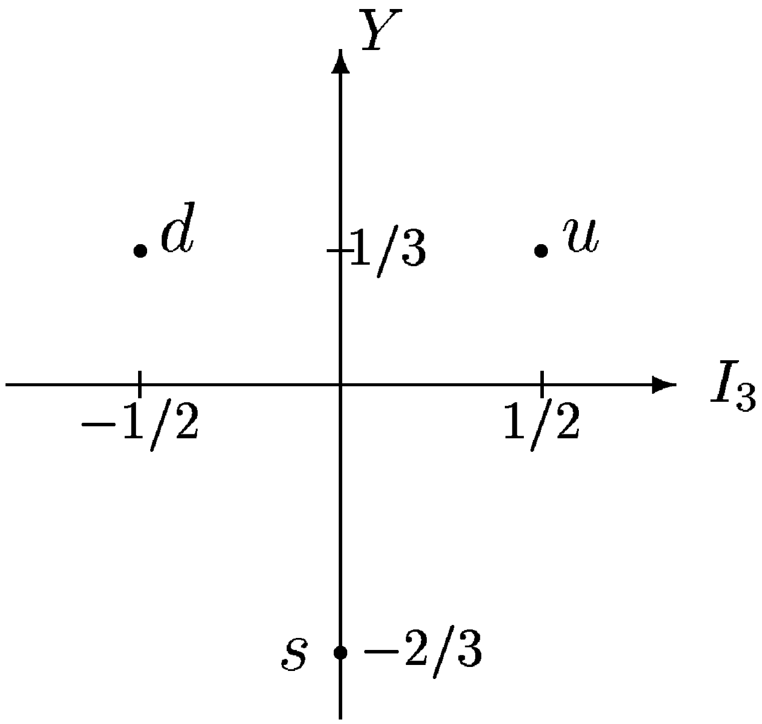

FIG. 1: The quark triplet.

Therefore, we can successfully interpret the meson octet given in Table 3.2 in terms of the triplet of $SU(3)$. This model, (124) with $SU(3)$, is called the Sakata model.

The quark model

Even though the Sakata model interprets the mesons($S = 0, 1$) successfully, it fails in interpreting the nucleon octet given in Table 3.3. It is because the Sakata model treats p, n, Λ of Table 3.2 differently from the rest members of the Table.

Thus, Gell-Mann and Zweig introduced more fundamental entities than the nucleons. Because of the success of the Sakata model in interpreting the spin-0 and spin-1 mesons in terms of spin-$\frac{1}{2}$ nucleons with the $SU(3)$ symmetry, it is necessary to keep the $SU(3)$ and the spin-$\frac{1}{2}$ property of the fundamental entiry. The new entity is called *quarks*. A quark and an antiquark make mesons. Because these quarks are more fundamental than the nucleons, one expects that the nucleons are also made of quarks. Because of the spin-$\frac{1}{2}$ property of the nucleons, it is necessary to have an odd number of quarks to form a nucleon. The simplest case is with three quarks. Here, we have to check whether the octet gicen in Table 3.3 is obtained in this scheme. The $SU(3)$ group theory allows the following tensor products

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}, \tag{131}$$

and

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8}' \oplus \mathbf{10}. \tag{132}$$

Thus, three spin-$\frac{1}{2}$ entities can make an octet. Therefore, let us introduce the quark triplet

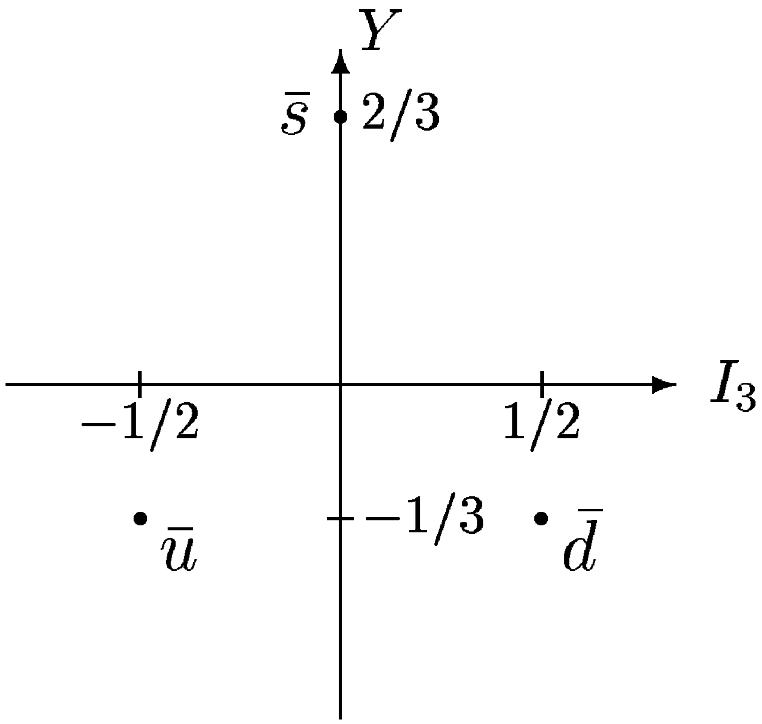

FIG. 2: The anti-triplet.

Table 3.5. The Gell-Mann-Zweig three quarks.

Quarks	I_3	Y	Q_{em}	B	S
u	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0
d	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	0
s	0	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	-1

in $SU(3)$,

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{133}$$

whose quntum numbers are shown in Table 3.5. In terms of the quarks, the pseudo-scalar mesons are reproduced as shown in Table 3.6.

The three quark states contain octets, one of which is interpreted as the baryon octet. The quantum numbers of $q_iq_jq_k$ are shown in Table 3.7 and we can match the baryon octet successfully.

Thus, the group $SU(3)$ was accepted as a symmetry of particle physics, and some believed the physical nature of quarks but others were in doubt in 1960's. This model is the so-called quark model. A more fundamental concept is $SU(3)$, and the quark model is an explicit realization in terms of the fundamental entity quarks: up-type quark u, down-type quark d, and strange quark s. The $SU(3)$ considered here is called the flavor $SU(3)$ to distinguish it from the *color* $SU(3)$ in the modern theory of strong interactions.

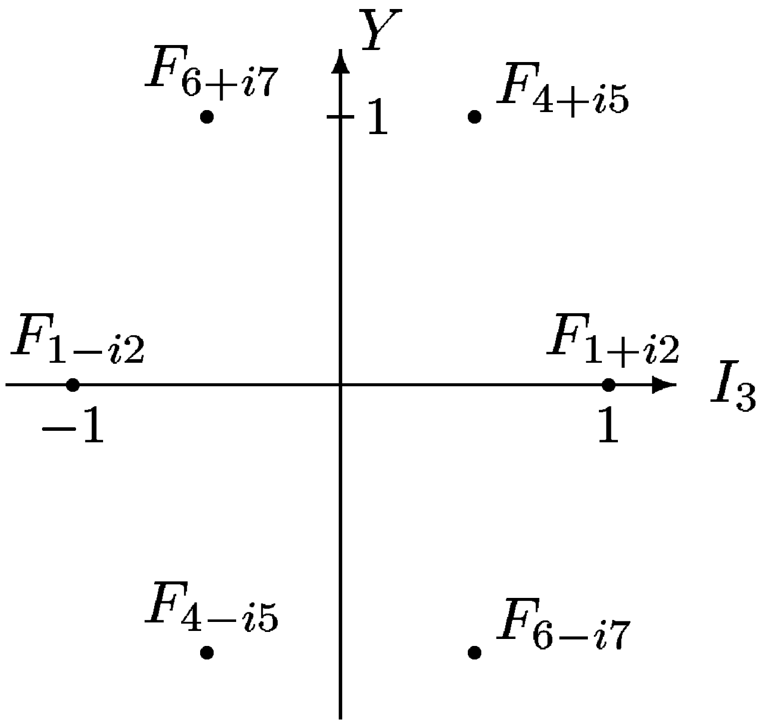

FIG. 3: Generators of $SU(3)$.

If there exists the $SU(3)$ symmetry in strong interactions, one can predict a relation between the mass of the octet multiplet. Suppose that the hadrons respect the $SU(3)$. Then the masses of the octet are the same. However, the observed masses are almost same, but not exactly. So, the $SU(3)$ symmetry must be broken, not strongly but weakly. Suppose that the $SU(3)$ breaking Hamiltonian transforms like the 8^{th} member of the eight generators,

$$\Delta H \propto F_8. \tag{134}$$

Of course, the dominant piece is the $SU(3)$ invariant one H_0. The isospin multiplet has the same mass since ΔH does not break the isospin symmetry. Now, let us consider the masses of the baryon octet. It can be shown that there exists a relation between the masses of the baryon octet,

$$\frac{M_N + M_\Xi}{2} = \frac{M_\Sigma + 3M_\Lambda}{4} \tag{135}$$

which is called the Gell-Mann-Okubo mass formula. The left-hand side and the right-hand-side of Eq. (135) are, 1129 MeV and 1135 MeV, respectively. Thus, we can see that the $SU(3)$ symmetry description of the nucleon masses is very successful. Similar conclusion can be drawn for the pseudoscalar and vector meson octets, but here one must include the singlet effect also, in terms of the mixing of the $SU(3)$ singlet with the $I = 0, Y = 0$ member of the octet.

The group $SU(3)$ is similar to $SU(2)$, but is fundamentally different from it. The rank of $SU(N)$ group is $(N-1)$, meaning that there are $N-1$ independent Casimir operators, or equivalently there are $(N-1)$ mutually commuting generators. Among N^2-1 generators, therefore we usually diagonalize $N-1$ generators. In $SU(3)$, we diagonalize F_3 and F_8. Therefore, we can assign the quantum numbers of F_3 and F_8 for each member in the

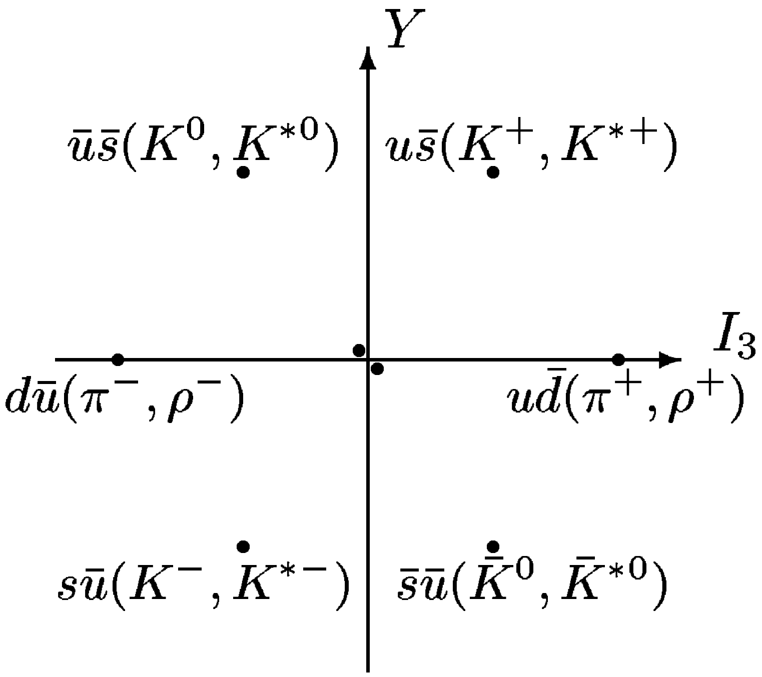

FIG. 4: The pseudo-scalar and vector meson octets in terms of quarks.

multiplet. It is easier to view by representing them on the plane since we need just two axes to represent the quantum numbers of $F_3 \equiv I_3$ and $F_8 \equiv \frac{\sqrt{3}}{2}Y$. The representation of the members on this $F_3 - F_8$ plane is called the weight diagram. For the quarks and antiquarks, they are shown in Figs. IV and IV, respectively.

The quark contents of the pseudoscalar meson octet and the nucleon octet are shown in Figs. IV and IV, respectively.

The success of the flavor $SU(3)$ was the prediction of Ω^- particle. The product $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$ contains $\mathbf{10}$. The weight diagram of $\mathbf{10}$ is shown in Fig. **??**, together with the quark contents. In this figure, the isospin quartet $Y = 1$ member is the spin-$\frac{3}{2}$ Δ particles, Δ^{++}, Δ^+, Δ^0, and Δ^-. The $Y = 0$ isospin triplet is the spin-$\frac{3}{2}$ Y_1^* particles, Y_1^{*+}, Y_1^{*0}, and Y_1^{*-}. The $Y = -1$ isospin doublet is Ξ^{*0} and Ξ^{*-}. These were the observed particle when the flavor $SU(3)$ quark model was proposed. But the 10^{th} spin-$\frac{3}{2}$ member was not observed until then. The flavor $SU(3)$ predicts the $Y = -2$(or $S = -3$) isospin singlet Ω^-. Indeed, this prediction was confirmed by the discovery [3].

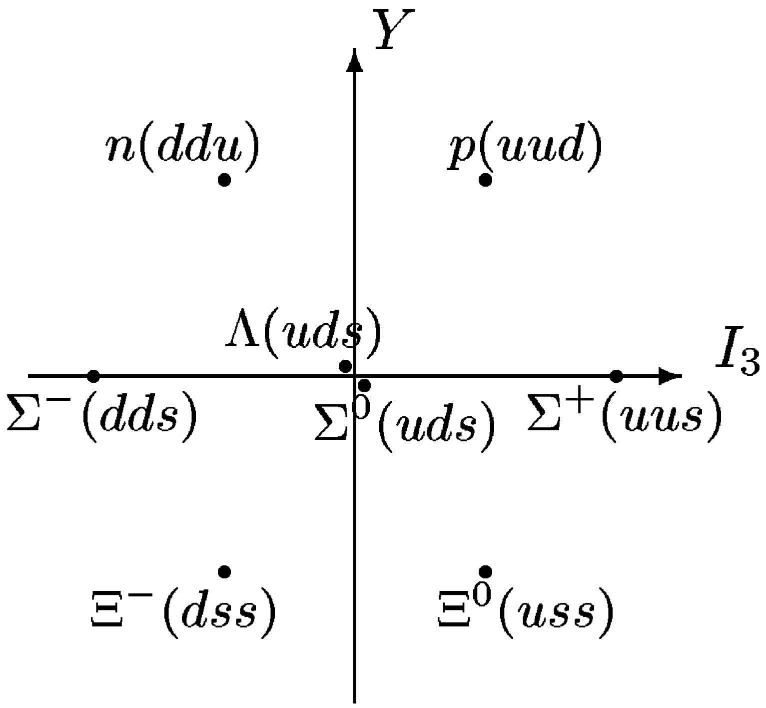

FIG. 5: The baryon octet.

V. $SU(6)$, NUCLEON MAGNETIC MOMENTS, AND THE NEED FOR INTRODUCTION OF COLOR

Let us look into the three quark system more closely. A two quark system has a symmetric and antisymmetric combinations, viz $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} + \oplus \bar{\mathbf{3}}$

$$\mathbf{6} = \begin{pmatrix} q_1 q_1 \\ q_2 q_2 \\ q_3 q_3 \\ \frac{1}{\sqrt{2}}(q_1 q_2 + q_2 q_1) \\ \frac{1}{\sqrt{2}}(q_1 q_3 + q_3 q_1) \\ \frac{1}{\sqrt{2}}(q_2 q_3 + q_3 q_2) \end{pmatrix}$$

$$\bar{\mathbf{3}} = \begin{pmatrix} \frac{1}{\sqrt{2}}(q_1 q_2 - q_2 q_1) \\ \frac{1}{\sqrt{2}}(q_1 q_3 - q_3 q_1) \\ \frac{1}{\sqrt{2}}(q_2 q_3 - q_3 q_2) \end{pmatrix}.$$

Let us add the third quark, so that we consider $\mathbf{6} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8}$ and $\bar{\mathbf{3}} \otimes 3 = \mathbf{8} + \mathbf{1}$, where $\mathbf{10}$ is a completely symmetric state and $\mathbf{1}$ is a completely antisymmetric state. The baryon octet belongs to a mixed state. If we try to interpret the three quark system as describing the spin-$\frac{1}{2}$ baryon octet and spin-$\frac{3}{2}$ baryon ducaplet, we must take into account the spin of quarks since the octet and the ducaplet have different spins. Therefore, we consider the spin-up u quark, $u\uparrow$, the spin-down u, $u\downarrow$, the spin-up d, $d\uparrow, \cdots,$. If we consider the spin, the u, d, s endowed with the up and down spins, the fundamental representation is $\mathbf{6}$ and

Table 3.6. Mesons in terms of quarks.

Mesons	Quark contents	Q_{em}	Y	S	B
π^+	$u\bar{d}$	+1	0	0	0
π^-	$d\bar{u}$	−1	0	0	0
π^0	$u\bar{u} - d\bar{d}$	0	0	0	0
η	$u\bar{u} + d\bar{d} - 2s\bar{s}$	0	0	0	0
K^+	$u\bar{s}$	+1	+1	+1	0
K^0	$d\bar{s}$	0	+1	+1	0
K^-	$s\bar{u}$	−1	−1	−1	0
$\bar{K}^0$	$s\bar{d}$	0	−1	−1	0

Table 3.7. Baryons in terms of quarks.

Baryons	Quark contents	Q_{em}	Y	S	B
p	uud	+1	+1	0	1
n	udd	0	+1	0	1
Σ^+	suu	+1	0	−1	1
Σ^0	sud	0	0	−1	1
Σ^-	sdd	−1	0	−1	1
Λ	sud	0	0	−1	1
Ξ^0	ssu	0	−1	−2	1
Ξ^-	ssd	−1	−1	−2	1

we have to consider $SU(6)$. But the spin is the doublet representation of $SU(2)$. The three spins of the three quark system involves

$$\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = (\mathbf{3}(S)\oplus\mathbf{1}(A))\otimes\mathbf{2} = \mathbf{4}(S)\oplus\mathbf{2}(M_S)\oplus\mathbf{2}(M_A)$$

where the S, M_S, and M_A denote how the combinations are obtained. For example, $S_z = \frac{1}{2}$ components have the following spin combinations

$$\chi(S) = \sqrt{\frac{1}{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$

$$\chi(M_S) = \sqrt{\frac{1}{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) \tag{136}$$

$$\chi(M_A) = \sqrt{\frac{1}{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \tag{137}$$

Thus, possible symmetry properties of the $SU(3)\times SU(2)$ representations are, the completely symmetric S, one kind of mixed symmetry M_S, another kind of mixed symmetry M_A, and the completely antisymmetric A. In the mixed symmetry case, the symmetry properties of the first two quark combinations are shown as subscripts. Thus, possible combinations are

$$\begin{aligned} S &: (10,4)+(8,2)\\ M_S &: (10,2)+(8,4)+(8,2)+(1,2)\\ M_A &: (10,2)+(8,4)+(8,2)+(1,2)\\ A &: (1,4)+(8,2). \end{aligned} \tag{138}$$

If we want to construct a totally symmetric combination for an octet, we can consider

$$\sqrt{\frac{1}{2}}[(8(M_S),2(M_S))+(8(M_A),2(M_A))]. \tag{139}$$

The above consideration is a bit complicated. Since we are familiar with $SU(N)$ groups by now, we can directly consider the $SU(6)$. In $SU(6)$, the tensor product of three fundamental representations is

$$\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}=(\mathbf{21}(S)\oplus\mathbf{15}(A))\otimes\mathbf{6}=\mathbf{56}(S)\oplus\mathbf{70}(M_S)\oplus\mathbf{70}(M_A)\oplus\mathbf{20}(A). \tag{140}$$

To assign the known baryon spin state, consider the spin-$\frac{3}{2}$ state made of three $u^\uparrow$ quarks. The state $S_z=+\frac{3}{2}$ is a physical state realized and it is completely symmetric under the exchange of any two $u^\uparrow$. Thus, it must belong to **56** which is the only completely symmetric state. Thus, physically realized sates must belong to this multiplet, in particular the spin-$\frac{3}{2}$ ducaplet **10** which counts $4\times 10=40$ states in **56**. The other members must be also realized in nature with the spin-$\frac{1}{2}$ baryon octet **8** whose symmetry property is shown in (139). Thus, the $SU(6)$ classifies both **8** and **10** baryon multiplets successfully within a single representation in $SU(6)$.

This is again verified in the meson multiplets made of a quark and an antiquark,

$$\mathbf{6}\otimes\bar{\mathbf{6}}=\mathbf{1}+\mathbf{35}. \tag{141}$$

The representation **35** can house the vector meson octet($3\times 8=24$ states), the vector meson singlet(3 states), and the pseudoscalar meson octet. The singlet **1** of $SU(6)$ must be a pseodoscalar, presumably η'. The meson states are realized fully. But the baryon states are not realized in full because of the symmetry restriction.

Even though the symmetry property gather together the known states in a single $SU(6)$ multiplet, it was a mystery why only the **56** is realized. There must be a *deep principle*. Indeed, people found a rationale to introduce a new quantum number, the *color*.

We introduced the quarks as spin-$\frac{1}{2}$ entities. This implies that the baryon wave function made of three quarks must be antisymmetric under the exchange of two quarks since they are fermions. But the physically observed baryon wave function **56** is symmetric under the exchange of any two quarks. This is in gross contradiction to the spin-statistics theorem.

In 1964, people worried this spin-statistics theorem. One resolution wa that the quarks do not obey the Bose-Einstein statistics, nor the Fermi-Dirac statistics, but obey another kind of statistics, the so-called parastatistics of rank 3. It must be equivalent to introducing

Table 3.9. The Han-Nambu integer-charged quark model.

Quarks	p_1	n_1	λ_1	p_2	n_2	λ_2	p_3	n_3	λ_3
Q_{em}	1	0	0	1	0	0	0	-1	-1
I_3	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
Y	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$
Y'	1	1	1	1	1	1	-2	-2	-2
B	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

a new quantum number which let us call 'color', so that the baryon wave function can be written as

$$\Psi = \Phi(orbital) \times \chi(spin) \times I(internal\ symmetry) \times \psi(color) \tag{142}$$

where $\chi \times I$ has the $SU(6)$ symmetry. Since it is a wisdom that the orbital motion gives the s-wave(symmetric under any exchange of the components) state as the ground state, the s-wave **56** is the ground state. Thus, $\psi(color)$ must be completely antisymmetric under exchange of its components so that the spin-statistics theorem is satisfied. Without introducing this new quantum number, we cannot satisfy the spin-statistics theorem. Suppose, this new quantum number belongs to a fundamental representation of a new $SU(N)_c$. In $SU(N)$, there exists a completely antisymmetric Levi-Civita tensor, $\epsilon_{i_1 i_2 \cdots i_N}$. Thus, the baryon wave function can be made completely antisymmetric if $N = 3$ since the baryons are made of three quarks $|q_1 q_2 q_3\rangle$. We must choose the combination in the $SU(3)_c$ space as

$$\epsilon_{\alpha\beta\gamma} q^\alpha q^\beta q^\gamma \tag{143}$$

where α, β, γ counts the color. Sometimes, the color is labeled as *red, green* and *blue*, but the natural color does not have any relevance here. It is just a way of labeling the indices. We must distinuish the flavor $SU(3)$ (the u, d, s) and the color $SU(3)$ (the *red, green, blue*).

This color concept was introduced first by Han and Nambu, which is a big leap forward to the modern particle theory. But their model is not the current color quark model. In 1964, the fractionally charged quarks were not observed, and hence Han and Nambu introduced integer-charged quarks. To have the correct Gell-Mann-Nishjima formula for three quarks, the integer-charged quarks must average to the fractionally charged quarks of 2/3, and –1/3. Thus, the Han-Nambu model is Table. 3.9.
We must have a new quantum number which we denoted as Y' so that the Gell-Mann-Nishijima type equation is satisfied

$$Q_{em}^{HN} = I_3 + \frac{Y}{2} + \frac{Y'}{3}. \tag{144}$$

Nowadays, the nonobservation of the fractionally charged quarks is attributed to the realization of only color singlet state in nature, notably by Gell-Mann. The modern quark model is the *fractionally charged quark model* in that the *fractionally charged quarks repeat three times* exactly in the same way but with different colors

$$\text{Modern colored quarks}: u^r, u^g, u^b, d^r, d^g, d^b, s^r, \cdots \tag{145}$$

Since we know how to write the proton and neutron wave function, including the color factor, we can proceed to calculate the static properties of the baryons. Let us itry to calculate the nucleon magnetic moment μ_p and μ_n. The proton spin-up state is a completely symmetric state in $SU(6)$, viz. (139),

$$|p^\uparrow\rangle = \frac{1}{\sqrt{2}}|\Big\{\frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}(udu+duu-2uud)(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow -2\uparrow\uparrow\downarrow)+\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(udu-duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)\Big\}\rangle$$

$$\begin{aligned} &= \frac{1}{6\sqrt{2}}\,|\Big\{\ 4u\uparrow d\downarrow u\uparrow -2u\downarrow d\uparrow u\uparrow -2u\uparrow d\uparrow u\downarrow \\ &\qquad - 2d\uparrow u\downarrow u\uparrow +4d\downarrow u\uparrow u\uparrow -2d\uparrow u\uparrow u\downarrow \\ &\qquad - 2u\uparrow u\downarrow d\uparrow -2u\downarrow u\uparrow d\uparrow +4u\uparrow u\uparrow d\downarrow \Big\}\rangle. \end{aligned} \tag{146}$$

Let us calculate the magnetic moment in the limit of large quark mass(below the chiral symmetry breaking scale). Assume that the proton magnetic moment is the sum of the magnetic moments of the three constituent quarks,

$$\mu_p = \sum_i \mu_{q_i}, \tag{147}$$

i.e.

$$\begin{aligned} \langle p^\uparrow|\mu_z|p^\uparrow\rangle &= \frac{1}{2}\sum_i \mu_i\langle p^\uparrow|\sigma_{z_i}|p^\uparrow\rangle \\ &= \frac{1}{2}\,\tfrac{1}{72}\,\langle 4u^\uparrow d^\downarrow u^\uparrow - 2u^\downarrow d^\uparrow u^\uparrow - 2u^\uparrow d^\uparrow u^\downarrow - 2d^\uparrow u^\downarrow u^\uparrow \\ &\quad +\ 4\ d^\downarrow u^\uparrow u^\uparrow - 2d^\uparrow u^\uparrow u^\downarrow - 2u^\uparrow u^\downarrow d^\uparrow - 2u^\downarrow u^\uparrow d^\uparrow + 4u^\uparrow u^\uparrow d^\downarrow| \\ &\quad |(\ 2\ \mu_u - \mu_d)4u^\uparrow d^\downarrow u^\uparrow - \mu_d 2u^\downarrow d^\uparrow u^\uparrow - \mu_d 2u^\uparrow d^\uparrow u^\downarrow - \mu_d 2d^\uparrow u^\downarrow u^\uparrow \\ &\quad +\ 4\ (2\mu_u - \mu_d)d^\downarrow u^\uparrow u^\uparrow - \mu_d 2d^\uparrow u^\uparrow u^\downarrow - \mu_d 2u^\uparrow u^\downarrow d^\uparrow \\ &\quad - \mu_d\, 2u^\downarrow u^\uparrow d^\uparrow + 4(2\mu_u - \mu_d)u^\uparrow u^\uparrow d^\downarrow\rangle \\ &= \tfrac{1}{144}\Big[16(2\mu_u - \mu_d) + 4\mu_d + 4\mu_d + 4\mu_d + 16(2\mu_u - \mu_d) \\ &\quad +\ 4\ \mu_d + 4\mu_d + 4\mu_d + 16(2\mu_u - \mu_d)\Big]. \end{aligned} \tag{148}$$

Thus, we obtain

$$\begin{aligned} \mu_p &= \tfrac{1}{6}(4\mu_u - \mu_d) \\ \mu_n &= \tfrac{1}{6}(4\mu_d - \mu_u) \end{aligned} \tag{149}$$

where we used the fact that the exchange of $u \leftrightarrow d$ in the proton state gives the neutron state. If $m_u = m_d$ below the chiral symmetry breaking scale and the strong interaction is assumed to preserve the isospin symmetry, we expect $\mu_d = -\frac{1}{2}\mu_u$ since they must be proportional to their charges. Thus, the $SU(6)$ predicts

$$\frac{\mu_n}{\mu_p} = -\frac{2}{3}. \tag{150}$$

Experimental number is $(0-1.91)/(1+1.79) = -0.684$ which is close to the $SU(6)$ prediction (150). This shows how powerful the symmetry is in predicting the ratios physical properties.

Consideration of color does not change the success of the above prediction since the color factor just takes the average over the differently colored quarks.

Homework

1. Does the strong interaction allow the process

$$\pi + M \to \pi + \pi + M, \tag{151}$$

where M is a self-adjoint multiplet(the meson containing its own antiparticles).

2. Obtain the Gell-Mann-Okubo mass formula for the baryon octet. For the pseudoscalar and vector meson octets, what is your conclusion on the Gell-Mann-Okubo mass formula and a possible modification thereof?

3. What is the magnetic moments of Σ^0 and Λ^0 in units of the proton magnetic moment?

REFERENCES

[1] Note that the operators in quantum field theory involves particle creation and annihilation operators.

[2] Originally, the need for the new quantum number was known from the associated production, $\pi^+ + n \to K^+ + \Lambda$, while it does not occur in $\pi^- + n \to K^- + \Lambda$. This was strange, and the quantum number strangeness was introduced.

[3] Omega