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Flipped $SU(5)$ from $\mathbf{Z}_{12-\mathrm{I}}$ orbifold with Wilson line

Jihn E. Kim [a,*], Bumseok Kyae [b]

[a] *Department of Physics and Astronomy, and Center for Theoretical Physics, Seoul National University, Seoul 151-747, Republic of Korea*
[b] *School of Physics, Korea Institute for Advanced Study, 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-722, Republic of Korea*

Abstract

We construct a three family flipped $SU(5)$ model from the heterotic string theory compactified on the $\mathbf{Z}_{12-\mathrm{I}}$ orbifold with one Wilson line. The gauge group is $SU(5) \times U(1)_X \times U(1)^3 \times [SU(2) \times SO(10) \times U(1)^2]'$. This model does not derive any non-Abelian group except $SU(5)$ from E_8, which is possible only for two cases in case of one shift V, one in $\mathbf{Z}_{12-\mathrm{I}}$ and the other in $\mathbf{Z}_{12-\mathrm{II}}$. We present all possible Yukawa couplings. We place the third quark family in the twisted sectors and two light quark families in the untwisted sector. From the Yukawa couplings, the model provides the R-parity, the doublet–triplet splitting, and one pair of Higgs doublets. It is also shown that quark and lepton mixings are possible. So far we have not encountered a serious phenomenological problem. There exist vector-like flavor $SU(5)$ exotics (including $Q_{\mathrm{em}} = \pm \frac{1}{6}$ color exotics and $Q_{\mathrm{em}} = \pm \frac{1}{2}$ electromagnetic exotics) and $SU(5)$ vector-like singlet exotics with $Q_{\mathrm{em}} = \pm \frac{1}{2}$ which can be removed near the GUT scale. In this model, $\sin^2 \theta_W^0 = \frac{3}{8}$ at the full unification scale.
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1. Introduction

At present, it is of utmost importance to connect the high energy string theory with the low energy standard model, in particular with the minimal supersymmetric standard model (MSSM).

* Corresponding author.
  E-mail addresses: jekim@phyp.snu.ac.kr (J.E. Kim), bkyae@kias.re.kr (B. Kyae).

The initial attempt of the Calabi–Yau space compactification, which is geometrical, has been very attractive [1]. But the orbifold compactification, also being a geometrical device, got more interest due to its simplicity in model buildings [2,3]. Initially, the standard-like models were looked for [4], in an attempt to obtain minimal supersymmetric standard models (MSSMs), but it became clear that the standard-like models have a serious problem on $\sin^2 \theta_W$ to arrive at MSSMs [5]. All $\mathbf{Z}_N$ models without Wilson lines were tabulated a long time ago [6] and recently all $\mathbf{Z}_3$ models with Wilson lines are tabulated in a book [7].

The $\sin^2 \theta_W$ problem is that it is better for the bare value of $\sin^2 \theta_W^0$ at the unification or string scale to be close to $\frac{3}{8}$ [5] so that it reproduces the fact of the convergence of three gauge couplings at one point near the unification scale [8]. The so-called flipped $SU(5)$ does not fulfill this requirement automatically due to the leakage of $U(1)_Y$ beyond $SU(5)$.[1] Thus, the $\sin^2 \theta_W$ problem directs toward grand unified theories (GUTs) from superstring without the electroweak hypercharge Y leaking outside the GUT group. In this regard, one may consider simple GUT groups $SU(5)$ [12], $SO(10)$ [13], E_6 [14] and trinification $SU(3)^3$ [15]. The simplest orbifold without matter representations beyond the fundamentals require the Kac–Moody level $K = 1$. With $K = 1$, one cannot obtain adjoint representations [7]; thus among the above GUT groups the trinification group is the allowed one. Also, the Pati–Salam $SU(4) \times SU(2) \times SU(2)$ [16] can be broken to the standard model without an adjoint matter representation; above the GUT scale however three gauge couplings of the Pati–Salam model diverge rather than evolving in unison. Thus, trinification GUT seems to be the most attractive solution regarding the $\sin^2 \theta_W$ problem. The trinification is possible only in $\mathbf{Z}_3$ orbifolds [17].

Another interesting GUT group, though not giving $\sin^2 \theta_W = \frac{3}{8}$ necessarily, is the flipped $SU(5)$ [9,10] where the exchanges $d^c \leftrightarrow u^c$ and $e^c \leftrightarrow$ (neutral singlet ν^c) in the representations of $SU(5)$ are adopted. The matter representation of the flipped $SU(5)$ is, under $SU(5) \times U(1)_X$,[2]

$$\mathbf{16}_{\text{flip}} \equiv \mathbf{10_1} + \bar{\mathbf{5}}_{-3} + \mathbf{1_5}. \tag{1}$$

The electroweak hypercharge is given by

$$Y = \frac{1}{5}(X + Y_5) \tag{2}$$

where $Y_5 = \text{diag}(\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \ -\frac{1}{2} \ -\frac{1}{2})$ and $X = \text{diag}(x \ x \ x \ x \ x)$. Then, the electroweak hypercharges of $\mathbf{1_5}$ and $\bar{\mathbf{5}}_{-3}$ are $+1$, $-\frac{2}{3}$, and $-\frac{1}{2}$, which are e^c, u^c, and electron doublet. There are some nice features of flipped $SU(5)$ [18].

From the string context, flipped $SU(5)$ was considered before in the fermionic construction scheme [11] and recently in orbifold construction also [19], Calabi–Yau compactification [20], and intersecting D-brane models [21]. Let us call flipped $SU(5)$ from string construction 'string flipped' $SU(5)$. In string flipped $SU(5)$, it does not necessarily predict $\sin^2 \theta_W = \frac{3}{8}$ at the unification scale [5]. However, if we introduce more parameters intrinsic in flipped $SU(5)$, we may fit parameters so that the gauge couplings meet at one point at the string scale M_s, the unification scale of $SU(5)$ and $U(1)_X$ couplings. These parameters include the symmetry breaking scale M_{GUT} for $SU(5) \times U(1)_X \to$ SM breaking and *intermediate scales of vector-like representations*. Above M_{GUT} the RG evolutions of $SU(5)$ and $U(1)_X$ couplings are different, and we do not expect a string scale around 0.7×10^{18} GeV [22] but can be determined by the unification

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[1] The terminology flipped $SU(5)$ was used as an $SU(5) \times U(1)_X$ subgroup of $SO(10)$ [9–11]. In this paper, we still use the same terminology if there appear $\mathbf{16}$s having the same quantum numbers as in the flipped $SU(5)$.

[2] The quantum number X of $U(1)_X$ in the flipped $SU(5)$ is highlighted.

ansatz at M_s and mass scales of vector-like representations [23]. In our string orbifold model, $\sin^2\theta_W$ turns out to be $\frac{3}{8}$ at the full unification scale due to the possibility of the electroweak hypercharge embedding in $SO(10)$. Thus such vector-like fields should be removed near the GUT scale.

String models in general include exotics. Electromagnetic exotics (E-exotics) are fractionally charged particles which are non-Abelian gauge group singlets. Color exotics (C-exotics) are quarks with non-standard charges, i.e. color triplet quarks not having $Q_{\rm em} = \frac{2}{3}$ or $-\frac{1}{3}$ and color anti-triplet quarks not having $Q_{\rm em} = -\frac{2}{3}$ or $\frac{1}{3}$. Flipped $SU(5)$ GUT exotics (G-exotics) are $SU(5)$ representations, not having the X charges of $\mathbf{10}_1, \overline{\mathbf{10}}_{-1}, \mathbf{5}_3, \bar{\mathbf{5}}_{-3}, \mathbf{5}_{-2}, \bar{\mathbf{5}}_2, \mathbf{1}_{\pm 5}, \mathbf{1}_0$. G-exotics contain C-exotics and fractionally charged leptons.

The fermionic construction of flipped $SU(5)$ has shown the existence of E-exotics with $Q_{\rm em} = \pm\frac{1}{2}$ and integer charged 'cryptons' where cryptons are defined to be the composites of the hidden sector confining group $SU(4)'$ [24]. Cosmological effect of cryptons was given in [25]. Because of the possibility that fractionally charged particles exist in most string vacua, discovery of fractionally charged particles may strongly hint the correctness of the idea of string compactification.

In this paper, we present the orbifold compactification with $\mathbf{Z}_{12-I}$ twist. This contains a detailed account of Ref. [19]. In addition, we present another orbifold model having a hidden sector $SU(4)'$. We succeeded in constructing a phenomenologically desirable flipped $SU(5)$ model from $\mathbf{Z}_{12-I}$.

We need three families of $\mathbf{16}_{\rm flip}$, where

$$\mathbf{16}_{\rm flip} \equiv \mathbf{10}_1 + \bar{\mathbf{5}}_{-3} + \mathbf{1}_5 = \left(d^c, q, \nu^c\right) + \left(u^c, l\right) + e^c. \tag{3}$$

For spontaneous symmetry breaking, we need also the Higgs fields,

$$(\mathbf{10}_1 + \overline{\mathbf{10}}_{-1}) + (\mathbf{5}_{-2} + \bar{\mathbf{5}}_2). \tag{4}$$

Sometimes, it is useful to represent the components in terms of

$$\mathbf{10}_1: \quad \begin{pmatrix} d^c & q \\ q & \nu^c \end{pmatrix}, \qquad \bar{\mathbf{5}}_{-3}: \quad \begin{pmatrix} u^c \\ l \end{pmatrix}, \qquad \mathbf{1}_5: \quad e^c,$$

$$\mathbf{5}_{-2}: \quad \begin{pmatrix} D \\ h_d \end{pmatrix}, \qquad \bar{\mathbf{5}}_2: \quad \begin{pmatrix} \bar{D} \\ h_u \end{pmatrix}, \tag{5}$$

where q and l are lepton and quark doublets, D is $Q_{\rm em} = -\frac{1}{3}$ quark, and $h_{d,u}$ are Higgs doublets giving mass to d, u quarks. Spontaneous symmetry breaking of flipped $SU(5)$ proceeds via VEVs of $\mathbf{10}_1$ and $\overline{\mathbf{10}}_{-1}$ (components $\langle \nu^c \rangle, \langle \bar{\nu}^c \rangle$) and $\mathbf{5}_{-2}$ and $\bar{\mathbf{5}}_2$ (components in $\langle h_d \rangle, \langle h_u \rangle$).

In this model, there appear two light families from the untwisted sector and the third heavy family from twisted sectors. This is dictated from the Yukawa coupling structure. It also leads to (i) the doublet–triplet splitting, (ii) one pair of Higgs doublets, and (iii) the existence of R-parity.

Some standard models from fermionic construction are worthwhile to note since they may be free from some problems of GUTs [26]. But these models use $\mathbf{Z}_2 \times \mathbf{Z}_2$ which may have a very different phenomenology from the one we discuss here with $\mathbf{Z}_{12-I}$.

In Section 2, we present a review on orbifold construction with order $N = 12$. Here we include formulae with Wilson lines also. In Sections 3 and 4, the untwisted and twisted sector spectra are calculated in detail. In Section 5, we collect all observable sector fields. In Section 6, we present the Yukawa coupling structure and derive some phenomenological consequences. In Section 7 we show $\sin^2\theta_W^0 = \frac{3}{8}$. Section 8 is a conclusion. In Appendix A, we provide another model having $SU(4)'$.
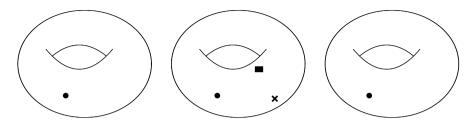
Fig. 1. Simplified showing of fixed points of the $\mathbf{Z}_{12-I}$ orbifold.

2. Orbifold method

An orbifold is constructed from a manifold by identifying points under a discrete symmetry group. The six internal space is orbifolded by a twist vector ϕ_s. With three complexified components, ϕ_s has three components ϕ_{s1}, ϕ_{s2} and ϕ_{s3}. The twist of $\mathbf{Z}_{12-I}$ orbifold is [7]

$$\phi_s = \left(\frac{5}{12}, \frac{4}{12}, \frac{1}{12} \right) \quad \text{with } \phi_s^2 = \frac{1}{12} \cdot \frac{7}{2}. \tag{6}$$

Torus corresponding to ϕ_{si} is twisted by ϕ_{si}. Hence the first and third tori have one fixed point while the second torus being modded by $\mathbf{Z}_3$ has three fixed points as schematically shown in Fig. 1. Multiplicity due to fixed points are three. These can be distinguished by Wilson lines.

In ten-dimensional (10D) heterotic string, left and right movers are treated as gauge group degrees and $\mathcal{N} = 1$ supersymmetry, respectively. The embedding possibility is in the gauge group space of left movers, NS sector of left and right movers, and R sector of right movers. So we focus on the embedding in the group space for left movers and in the R sector for right movers. In 10D, the R sector embedding is given by ϕ_s. The group space embedding is given by sixteen numbers, $V^I (I = 1, 2, \ldots, 16) \equiv \{v(I = 1, \ldots, 8), v'(I = 9, \ldots, 16)\}$. Factoring out $\frac{1}{12}$ by defining $\phi_{sa} = \frac{1}{12} \phi_a$, $v_a = \frac{1}{12} w_a$, $v'_a = \frac{1}{12} w'_a$, we must satisfy for a $\mathbf{Z}_{12-I}$ orbifold [27]

$$\sum_{a=1}^{3} \phi_a^2 - \sum_{a=1}^{8} w_a^2 - \sum_{a=1}^{8} w_a'^2 = 0 \mod 24,$$
$$3a_3 = 3a_4 = 0, \qquad a_1 = a_2 = a_5 = a_6 = 0. \tag{7}$$

2.1. Dynkin diagram technique for finding gauge group

Just for finding out a gauge group structure, the Dynkin diagram technique is extremely useful [28]. In the Dynkin diagram, each simple root is endowed with a Coxeter label. A Dynkin diagram technique of obtaining maximal subgroups is to strike out a simple root from the extended Dynkin diagram. In orbifold, this is generalized to strike out some roots where sum of the eliminated Coxeter labels add up to order N of $\mathbf{Z}_N$. To have $SU(5)$ only without a Wilson line, there must remain four linearly connected simple roots. So, for $N \leqslant 8$ orbifold, it is impossible.[3] For $\mathbf{Z}_{12}$, there is only possibility. Suppose $\sum_i c_i = N$ where c_i is the Coxeter label of simple root α_i. There is only one possibility which is $c_0 + c_1 + c_2 + c_6 + c_7 = 12$. See Fig. 2. Thus, from orbifold compactification, there are only two possibilities for constructing flipped $SU(5)$

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[3] By a two step process using Wilson lines, it is possible to obtain $SU(5)$ in other $\mathbf{Z}_N$ orbifolds.
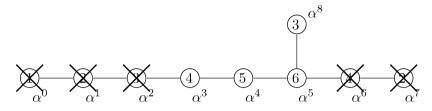
Fig. 2. The $SU(5)$ subgroup of E_8. The Coxeter labels are shown inside circles.

models, one in $\mathbf{Z}_{12-I}$ and another in $\mathbf{Z}_{12-II}$. For $N = \sum_i c_i = N$, the shift vector V is given by $V = \sum_i \Lambda_i$ where Λ_i are fundamental weights [27]. Thus, for $\mathbf{Z}_{12}$ orbifolds V is given by $V_1 = \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_6 + \Lambda_7$ with $\Lambda_0 = 0$. Thus, the shift vector for flipped $SU(5)$ is

$$V_1 = \frac{1}{12}\left(\frac{17}{2} \; \frac{5}{2} \; \frac{3}{2} \; \frac{1}{2} \; \frac{1}{2} \; \frac{1}{2} \; \frac{1}{2} \; -\frac{1}{2}\right)(\cdots).$$

Now we shift the origin by $-\frac{5}{24}$, to obtain

$$V_2 = \frac{1}{12}(11 \; 5 \; 4 \; 3 \; 3 \; 3 \; 3 \; 2)(\cdots). \tag{8}$$

Both V_1 and V_2 give an unbroken $SU(5)$. But the entries of V_1 and V_2 do not have five common entries. So, we try to add an integer times $\mathbf{Z}_{12-I}$ shift ϕ_s so that the resulting entries manifestly show five common entries. In this way, of course the $SU(5)$ non-Abelian group is kept. Usually, if one adds ϕ_s to three entries of E_8, some non-Abelian groups are broken. So our strategy is to add a multiple of ϕ_s such that an $SU(5)$ survives. For this, we add $(\frac{4}{12} \; 0 \; \frac{8}{12} \; 0 \; 0 \; 0 \; 0 \; \frac{4}{12})$ to obtain $\frac{1}{12}(15 \; 5 \; 12 \; 3 \; 3 \; 3 \; 3 \; 6)$. Subtracting torus lattice and rearranging entries, we obtain

$$V_3 = \frac{1}{12}(3 \; 3 \; 3 \; 3 \; 3 \; 5 \; 6 \; 0) \tag{9}$$

which has five common entries. This form is perfectly simple enough in obtaining $SU(5)$ weights since there are five common entries. Otherwise, i.e. with V_1 or V_2, it is cumbersome to work out all the $SU(5)$ weights as tried out in [7].

Since the five entries are common, the simple roots for $SU(5)$ take the following standard form,

$$\alpha_1 = (1 \; -1 \; 0 \; 0 \; 0; \; 0 \; 0 \; 0),$$
$$\alpha_2 = (0 \; 1 \; -1 \; 0 \; 0; \; 0 \; 0 \; 0),$$
$$\alpha_3 = (0 \; 0 \; 1 \; -1 \; 0; \; 0 \; 0 \; 0),$$
$$\alpha_4 = (0 \; 0 \; 0 \; 1 \; -1; \; 0 \; 0 \; 0). \tag{10}$$

Then, the highest weights of some representations we use are

$$\mathbf{5}: \quad \begin{cases} (1 \; 0 \; 0 \; 0 \; 0; \; 0 \; 0 \; 0), \\ (+\frac{1}{2} \; -\frac{1}{2} \; -\frac{1}{2} \; -\frac{1}{2} \; -\frac{1}{2}; \; 0 \; 0 \; 0), \end{cases} \tag{11}$$

$$\bar{\mathbf{5}}: \quad \begin{cases} (-1 \; 0 \; 0 \; 0 \; 0; \; 0 \; 0 \; 0), \\ (-\frac{1}{2} \; +\frac{1}{2} \; +\frac{1}{2} \; +\frac{1}{2} \; +\frac{1}{2}; \; 0 \; 0 \; 0), \end{cases} \tag{12}$$

$$\mathbf{10}: \quad \begin{cases} (1 \; 1 \; 0 \; 0 \; 0; \; 0 \; 0 \; 0), \\ (+\frac{1}{2} \; +\frac{1}{2} \; -\frac{1}{2} \; -\frac{1}{2} \; -\frac{1}{2}; \; 0 \; 0 \; 0), \end{cases} \tag{13}$$

$$\overline{\mathbf{10}}: \quad \begin{cases} (-1 \ -1 \ 0 \ 0 \ 0; \ 0 \ 0 \ 0), \\ (-\frac{1}{2} \ -\frac{1}{2} \ +\frac{1}{2} \ +\frac{1}{2} \ +\frac{1}{2}; \ 0 \ 0 \ 0). \end{cases} \tag{14}$$

For the E_8', we find roots and weights in a similar way.

2.2. Choosing shift vector and Wilson lines

Embedding of the orbifold action can be found by satisfying the modular invariance conditions (7). If one tries to have a specific gauge group, the Dynkin diagram technique is very helpful as we have discussed in Section 2.1. Or one can study the components of $(\cdots)(\cdots)'$ to guess the gauge group, but this method of finding gauge group is completed only after one obtains all nonzero roots of the gauge multiplet. For example, if one tries V_2 of (8) and V_3 of (9) then in both cases he will obtain $SU(5)$. But guessing $SU(5)$ from (8) is not straightforward. In this sense, the Dynkin diagram technique is superior. On the other hand, a computer search of gauge groups will cover all these cases. In the computer search, the identical shift vector is given by several different forms as done in V_2 and V_3. Usually, it is very difficult to identify all the same shift vectors [7].

Choosing the shift vector (V) and Wilson lines $(a_1, \ldots, a_6)$ fixes the embedding of the orbifold action in the group space. So gauge groups and representations are fixed by the shift vector and Wilson lines, consistently with the modular invariance condition (7).

In summary, the compactification is specified by twisting represented by a shift form in the six internal space ϕ_s, shift V, and Wilson lines $a_1, \ldots, a_6$,

$$\begin{aligned} \text{internal space:} \quad & \phi_s = (\phi_{s1}, \phi_{s2}, \phi_{s3}), \\ \text{group space:} \quad & V, \ a_i \ (i = 1, 2, \ldots, 6). \end{aligned} \tag{15}$$

Here, s_0 determines the chirality and $\tilde{s}$ encodes the orbifolding information of the R sector.

2.3. Massless modes

Finding out all the massless modes below the compactification scale is the key problem in the compactification process. The left movers and right movers have different relations for the Einstein mass-shell condition, even though the form has a similarity. In the symmetric orbifold [2], let us bosonize the Ramond sector of right movers, which is represented by four half integers in s. The orbifold action is modding out the 6D torus, and s contains the orbifold information of right movers under translation in the internal space. For left movers, momenta P corresponding to translation in the group space have the orbifold information. All these satisfy the level matching condition, $M_L^2 = M_R^2$. Thus, left moving and right moving states on torus have the following vanishing vacuum energy for massless states,

$$\begin{aligned} \text{left movers:} \quad & \frac{(P + kV)^2}{2} + \sum_j N_j^L \tilde{\phi}_j - \tilde{c}_k = 0, \\ \text{right movers:} \quad & \frac{(s + k\phi_s)^2}{2} + \sum_j N_j^R \tilde{\phi}_j - c_k = 0, \end{aligned} \tag{16}$$

where j runs over $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$. Here $\tilde{\phi}_i \equiv k\phi_i \mod Z$ such that $0 < \tilde{\phi}_i \leqslant 1$, $\tilde{\phi}_{\bar{i}} \equiv -k\phi_i \mod Z$ such that $0 < \tilde{\phi}_{\bar{i}} \leqslant 1$. (If $k\phi_i$ is an integer, $\tilde{\phi}_j = 1$ [29,30].) For $k = 0$, conditions for massless

left and right movers are given by

$$P^2 = 2 - 2\sum_j N_j^L \tilde{\phi}_j,$$

$$s^2 = 1 - 2\sum_j N_j^R \tilde{\phi}_j, \tag{17}$$

where $\tilde{c}_0 = c_0 + \frac{1}{2}$.

The massless modes include graviton $g_{\mu\nu}$, antisymmetric tensor field $B_{\mu\nu}$, dilaton, gravitino, gauge bosons, gauginos, and chiral matter. So the matter states (P, s) must satisfy

$$(P + kV)^2 = 2\tilde{c}_k - 2\sum_j N_j^L \tilde{\phi}_j,$$

$$(s + k\phi_s)^2 = 2c_k - 2\sum_j N_j^R \tilde{\phi}_j. \tag{18}$$

Among these massless modes, we are interested in a resulting $\mathcal{N} = 1$ SUSY gauge theory. The SUSY condition for orbifold compactification is given by right movers, which are given by four component s, including three component $\tilde{s}$. When we compactify six dimensions, ten-dimensional supersymmetry generators can be decomposed into $Q_{(10)} = Q_{(4)} \otimes Q_{(6)}$. The six-dimensional internal space part $Q_{(6)}$ transforms as **4** of $SO(6) \sim SU(4)$. Because the remaining part $Q_{(4)}$ becomes the four-dimensional generator, the dimension **4** counts the number of super-symmetries. Its spinorial representation is given by $|\tilde{s}\rangle = |s_1\ s_2\ s_3\rangle = |\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle$ with even number of minus signs. Under point group, it transforms as

$$Q_{(6)} \to \exp(2\pi i s \cdot \phi) Q_{(6)}.$$

The invariant component corresponds to the unbroken supersymmetry generator. For $\mathcal{N} \geqslant 1$ supersymmetry, we need at least one solution, say if we choose $s = (+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, the argument of exponent vanishes for

$$\phi_1 - \phi_2 - \phi_3 = 0 \bmod Z.$$

The number of solutions, $\mathcal{N}$, counts the number of unbroken supersymmetry generators from orbifold compactification. Note that for our choice $\phi_s = (\frac{5}{12}, \frac{4}{12}, \frac{1}{12})$ the above condition is satisfied only for $\tilde{s} = (+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. We can introduce another set, $\tilde{r}$, of half integers so that entries of $\tilde{s} + \tilde{r}$ become integers,

$$\tilde{r} = (r_1, r_2, r_3). \tag{19}$$

Namely, the SUSY condition for orbifolded spectrum is $\phi_s \cdot \tilde{r} = 0$ [7].

2.3.1. Gauge multiplet

Gauge boson multiplets appear in the untwisted sector U, satisfying

$$\text{gauge group:} \quad P^2 = 2, \quad P \cdot V = 0, \quad P \cdot a_i = 0 \quad \text{for all } i, \tag{20}$$

where the first one is the masslessness condition and the second and the third ones are the orbifold conditions. The corresponding right movers, satisfying the mass-shell and orbifold conditions chooses two s's with $s^2 = 1$, which are always $\mathcal{CPT}$ conjugates of each other. In this way, we obtain the gauge multiplet. As expected, the multiplicity ($\mathcal{P}$) of gauge bosons is 1.

2.3.2. Matter multiplets

Other massless states can appear in U also, for

$$\text{untwisted matter:} \quad P^2 = 2, \quad P \cdot V = \frac{k}{N} \quad \text{for } k \neq 0 \mod N, \qquad P \cdot a_i = 0 \quad \text{for all } i. \tag{21}$$

We combine $\tilde{s}$ and $\tilde{r}$ to distinguish untwisted matter so that they appear in three categories under orbifolding, i.e. differing in sub-lattice shifts,[4]

$$U_1: \quad \tilde{s} + \tilde{r} = (-1, 0, 0), \qquad U_2: \quad \tilde{s} + \tilde{r} = (0, 1, 0), \qquad U_3: \quad \tilde{s} + \tilde{r} = (0, 0, 1). \tag{22}$$

There are fixed points in field theory orbifolds. In string orbifolds also, we must consider physics related to fixed points. Massless strings can sit at fixed points, which is found by the mass-shell condition at fixed point. But noting that some linear combinations of strings sitting at several fixed points may be taken to satisfy the orbifold condition, we consider twisted sectors. For $\mathbf{Z}_N$ orbifold, we consider $k = 1, \ldots, N-1$ twisted sectors, T_k. The $\mathcal{CPT}$ conjugates of T_k appear in T_{N-k}. Thus, in non-prime orbifolds $\mathbf{Z}_4, \mathbf{Z}_6, \mathbf{Z}_8, \mathbf{Z}_{12}$, the sector $T_{N/2}$ contains $\mathcal{CPT}$ conjugates also.

For untwisted matter, multiplicity ($\mathcal{P}$) is given just by counting all possible states. Multiplicity in the twisted sector is more involved. The method of linear combination can come with complex numbers. This is taken into account in the (generalized) GSO projector, which projects out non-physical states. It can be read off from the one loop partition function of string [7,31];

$$\mathcal{P}_k = \frac{1}{N} \sum_{l=0}^{N-1} \tilde{\chi}\left(\theta^k, \theta^l\right) e^{2\pi i l \Theta_0} \equiv \frac{1}{N} \sum_{l=0}^{N-1} \tilde{\chi}\left(\theta^k, \theta^l\right) \Delta^l, \tag{23}$$

where N is the order of $\mathbf{Z}_N$ orbifolds, and

$$\Theta_0 = \sum_j \left(N_j^L - N_j^R\right) \hat{\phi}_j - \frac{k}{2}\left(V^2 - \phi_s^2\right) + (P + kV) \cdot V - (\tilde{s} + k\phi_s) \cdot \phi_s + \text{integer}, \tag{24}$$

where j denotes the coordinates of the 6-dimensional compactified space running over $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$ in complexified coordinates, and $\hat{\phi}_i = \phi_{si}\, \text{sgn}(\hat{\phi}_i)$ where $\text{sgn}(\hat{\phi}_{\bar{i}}) = -\text{sgn}(\hat{\phi}_i)$ [29]. It turns out that $N_j^R = 0$ generically for the massless right mover states in our $\mathbf{Z}_{12-I}$ orbifold compactification. The $\tilde{\chi}(\theta^m, \theta^k)$ in Eq. (23) denotes the degenerate factor tabulated in Table 1 [7,32]. For the sectors wound by Wilson lines, the Wilson line modified shift vector V_f is used instead of V.

Table 1
Degeneracy factor $\tilde{\chi}(\theta^k, \theta^l)$ in the $\mathbf{Z}_{12-I}$ orbifold

$k \backslash l$	0	1	2	3	4	5	6	7	8	9	10	11
1	3	3	3	3	3	3	3	3	3	3	3	3
2	3	3	3	3	3	3	3	3	3	3	3	3
3	4	1	1	4	1	1	4	1	1	4	1	1
4	27	3	3	3	27	3	3	3	27	3	3	3
5	3	3	3	3	3	3	3	3	3	3	3	3
6	16	1	1	4	1	1	16	1	1	4	1	1

---

[4] For the same chirality, i.e. L, we use $(---)$ instead of $(+++)$. So, we take $U_1 = (-1, 0, 0)$.

In the presence of a Wilson lines ($\equiv a^I$), the GSO projector Eq. (23) needs to be modified as [29]

$$e^{2\pi i l \Theta_0} \rightarrow \frac{1}{N_W} \sum_{f=0}^{N_W-1} e^{2\pi i l \Theta_f}, \tag{25}$$

where $N_W = 3$ for Wilson line of order three in case of $\mathbf{Z}_{12-I}$, and Θ in Eq. (24) should also be modified as

$$\Theta_f = \sum_j \left(N_j^L - N_j^R\right)\hat{\phi}_j - \frac{k}{2}\left(V_f^2 - \phi_s^2\right) + (P + kV_f) \cdot V_f - (\tilde{s} + k\phi_s) \cdot \phi_s + \text{integer}, \tag{26}$$

where $V_f \equiv (V + m_f a_3)$, and Eq. (23) can be rewritten as $\mathcal{P}_k = \sum_{f=0}^{N_W-1} \mathcal{P}_k(f)$ where

$$\mathcal{P}_k(f) = \frac{1}{N N_W} \sum_{l=0}^{N-1} \tilde{\chi}\left(\theta^k, \theta^l\right) e^{2\pi i l \Theta_f}. \tag{27}$$

Note that in the $\mathbf{Z}_{12-I}$ model, $f = \{0, 1, 2\} \equiv \{f_0, f_+, f_-\}$ and $\mathcal{P}_k(f_0) = \mathcal{P}_k(f_+) = \mathcal{P}_k(f_-)$ for $k = 3, 6, 9$.

3. $\mathbf{Z}_{12-I}$ model

We choose the following shift vector and Wilson lines

$$V = \left(\frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{5}{12} \frac{6}{12} 0\right)\left(\frac{2}{12} \frac{2}{12} 0\,0\,0\,0\,0\,0\right), \tag{28}$$

$$a_3 = a_4 = \left(0^5\, 0\, \frac{-1}{3} \frac{1}{3}\right)\left(0\,0\, \frac{2}{3}\, 0^5\right), \tag{29}$$

$$a_1 = a_2 = a_5 = a_6 = 0,$$

which satisfies Eq. (7): it gives $V^2 - \phi_s^2 = \frac{1}{2}$ modular $2 \cdot$ (integer)$/12$. Since the Wilson line is a $\mathbf{Z}_3$ shift, it distinguishes three cases. These satisfy the following conditions [31,33]:

$$12\left(|V|^2 - |\phi_s|^2\right) = 0 \mod \text{ even integer},$$

$$12(V \cdot a_3) = 0 \mod \text{ integer},$$

$$12|a_3|^2 = 0 \mod \text{ even integer}. \tag{30}$$

Thus, we consider the following effective shifts distinguishing twisted sector T_k to T_k^0, T_k^+, and T_k^- ($k \neq 0, 3, 6, 9$) by

$$V_0 \equiv V, \tag{31}$$

$$V_+ \equiv V + a_3 = \left(\frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{5}{12} \frac{2}{12} \frac{4}{12}\right)\left(\frac{2}{12} \frac{2}{12} \frac{8}{12} 0^5\right), \tag{32}$$

$$V_- \equiv V - a_3 = \left(\frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{5}{12} \frac{10}{12} \frac{-4}{12}\right)\left(\frac{2}{12} \frac{2}{12} \frac{-8}{12} 0^5\right). \tag{33}$$

For future convenience, we list

$$V_0^2 - \phi_s^2 = \frac{1}{2},$$
$$V_+^2 - \phi_s^2 = \frac{5}{6},$$
$$V_-^2 - \phi_s^2 = \frac{3}{2}. \tag{34}$$

3.1. Gauge group from untwisted sector

For the gauge multiplet, we search for roots satisfying $P^2 = 2$, $P \cdot V = 0$ and $P \cdot a_3 = 0$, and obtain the following unbroken gauge group

$$\left[SU(5) \times U(1)_X \times U(1)^3 \right] \times \left[SU(2) \times SO(10) \times U(1)^2 \right]' \tag{35}$$

where we choose the $U(1)_X$ of flipped $SU(5)$ as

$$Q_X = (-2, -2, -2, -2, -2; 0, 0, 0)(0^8)'. \tag{36}$$

For example, one can see that the following is the roots of $SU(5)$:

$$(1 \, \underline{-1 \, 0 \, 0 \, 0}; \, 0 \, 0 \, 0) \quad \text{nonzero roots among } \mathbf{24} \text{ of } SU(5), \tag{37}$$

where the underlined entries allow permutations.

3.2. Matter from untwisted sector

3.2.1. Chirality

The chirality is determined by the 8 component $SO(8)$ spinor of the Ramond sector of right movers. It is labeled by $s = (s_0, \tilde{s}) = \{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\}$ with an even number of minus signs. We define the 4D chirality χ as the one originating from the first entry of s denoted by $\oplus$ or $\ominus$, i.e.

$$s = \left\{ \oplus \text{ or } \ominus, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2} \right\} \equiv \left\{ \frac{1}{2}\chi, \tilde{s} \right\}, \tag{38}$$

$$\chi = 2s_0 = 2\oplus \text{ or } 2\ominus. \tag{39}$$

Let us call $\chi = 1$ (-1) as 'right- (left-)handed', and $\tilde{s}$ has three components in terms of $\pm\frac{1}{2}$. In the untwisted sector U, we have U_1, U_2 and U_3 as defined in Eq. (22). These U_i correspond to the untwisted $\tilde{s} = (---)$, $(++-)$, $(+-+)$, respectively [or to $(+++)$, $(--+)$, $(-+-)$, respectively, since antiparticles can be used also].

The chirality in the twisted sector can be similarly defined by the 8 component $SO(8)$ spinor of the Ramond sector of right movers.

3.2.2. Spectrum

The massless matter fields are those with $P \cdot V \neq 0$ and satisfy the masslessness condition. P must satisfy $P^2 = 2$, $P \cdot a_3 = $ integer and $P \cdot V = \frac{k}{12}$ where $k = 1, 2, \ldots, 11$. We will consider $k = 1, 2, \ldots, 6$ only, since the rest will provide their $\mathcal{CTP}$ conjugates. Here, it is sufficient to look at 3 cases only, $k = 1, 4, 5$, to keep the GSO allowed states.

Table 2
Visible sector chiral fields from the U sector

$P \cdot V$	$\tilde{s}, U_i$	Visible states	$SU(5) \times U(1)_X$
$\frac{1}{12}$	$(+-+), U_3$	$(\underline{+----}; +++)$	$\mathbf{5}^L_3$
		$(\underline{+++--}; +--)$	$\overline{\mathbf{10}}^L_{-1}$
		$(+++++; +++)$	$\mathbf{1}^L_{-5}$
$\frac{4}{12}$	$(++-), U_2$	$(\underline{-1, 0, 0, 0, 0}; -1, 0, 0)$	$\bar{\mathbf{5}}^L_2$
$\frac{5}{12}$	$(+++), U_1$	$(\underline{++++-}; -++)$	$\bar{\mathbf{5}}^R_{-3}$
		$(\underline{++---}; ---)$	$\mathbf{10}^R_1$
		$(-----; -++)$	$\mathbf{1}^R_5$

Table 3
Hidden sector chiral fields from the U sector

$P \cdot V$	$\tilde{s}, U_i$	Hidden states	$[SU(2) \times SO(10)]'$	Label
$\frac{4}{12}$	$(++-), U_2$	$(1, 1; 0, 0, 0, 0, 0, 0)'$	$\mathbf{1_0}$	s^u

We have the convention that the highest weight of the complex conjugated representation is in fact the lowest one so that all the weights of the complex conjugated representation are obtained by adding simple roots.

Let α denote the phase $e^{2\pi i/12}$. For $k = 1$ or $P \cdot V = \frac{1}{12}$, the left movers obtain a phase α. We need an extra phase α^{-1} from the right movers, which is accomplished by $e^{-2\pi i s \cdot \phi_s}$ where $\phi_s = \frac{1}{12}(5\ 4\ 1)$ and $s = (\ominus+-+)$. It is left-handed and allows U_3. For $k = 4$, $s = (\ominus++-)$ provides the needed α^{-4}; thus it is left-handed and provides U_2. For $k = 5$, $s = (\oplus+++)$ provides α^{-5}; thus it right-handed and provides U_1. α from the right movers is provided by $s = (\oplus-+-)$ which thus will couple to $k = 11$. It is right-handed. α^4 from the right movers is provided by $s = (\oplus--+)$ which will couple to $k = 8$. It is right-handed. α^5 from the right movers is provided by $s = (\ominus+--)$ which will couple to $k = 7$. It is left-handed. These particles for $k > 6$ give the antiparticle spectra. The chiralities and U_is are shown in Tables 2 and 3.

Thus, the vectors for $p \cdot V = \frac{1}{12}, \frac{4}{12}$, and $\frac{7}{12}$ give

$$(\mathbf{1_{-5}} + \mathbf{5_3} + \overline{\mathbf{10}}_{-1})^L_{U_3}, \qquad (\bar{\mathbf{5}}_2)^L_{U_2}, \qquad (\mathbf{1_{-5}} + \mathbf{5_3} + \overline{\mathbf{10}}_{-1})^L_{U_1}, \qquad (\mathbf{1_0})^L_{U_2} \qquad (40)$$

and their $\mathcal{CTP}$ conjugates appear in $p \cdot V = \frac{11}{12}, \frac{8}{12}$, and $\frac{5}{12}$. Here we listed only $U(1)_X$ charges as boldfaced subscripts. Note that for the untwisted $k = 6$ sector there is no way to provide an additional α^6 at the massless level from the right movers.

4. Twisted sectors with Wilson line

The $\mathbf{Z}_{12-I}$ with the twist vector $\phi = \frac{1}{12}(5, 4, 1)$ has three fixed points in the second torus for the prime order θ^1 and θ^5 twists. This is because it is the same as $\mathbf{Z}_3$. For the first and the third torus the origin is the only fixed point, viz. Fig. 1. For the other twists such as $k = 4$ and 6, counting the number of massless states involves a more complicated nonvanishing projection operator P_{θ^k}. But addition of Wilson lines can distinguish some fixed points. In fact, for $\mathbf{Z}_{12-I}$ possible Wilson lines must satisfy $3a_3 = 0$, $3a_4 = 0$ and $a_i = 0$ ($i \neq 3, 4$) so that any combination

of $k(V + m_f a_i)$ is another shift vector. Since $\mathbf{Z}_{12-I}$ allows $3a_3 = 0$ modular integer, the Wilson line is a $\mathbf{Z}_3$ shift.

In the second torus, Wilson lines must be symmetric, $a_3 = a_4$. Then, the kth twisted sector is distinguished by kV, $k(V + a_3)$, and $k(V - a_3)$, which is denoted as $\tilde{V}$,

$$\tilde{V} = k(V + m_f a_3), \quad m_f = 0, \pm 1, \tag{41}$$

or

$$\tilde{V} = k\{V_0, V_+, V_-\}. \tag{42}$$

Since $3a_3 = 6a_3 = 0$, $2V_\pm$ and $5V_\pm$ in the T_2 and T_5 sectors are equivalent to $2V \mp a_3$ and $5V \mp a_3$, respectively. On the other hand, $4V_\pm$ in the T_4 sector is equivalent to $4V \pm a_3$.

The masslessness condition is

$$(P + \tilde{V})^2 = 2\tilde{c}_k - 2\sum_j N_j^L \tilde{\phi}_j. \tag{43}$$

For the θ^k twist ($k = 1, 2, \ldots, 6$), we have

$$2\tilde{c}_k = \begin{cases} \frac{210}{144}, & k = 1; & \frac{192}{144}, & k = 4; \\ \frac{216}{144}, & k = 2; & \frac{210}{144}, & k = 5; \\ \frac{234}{144}, & k = 3; & \frac{216}{144}, & k = 6, \end{cases} \tag{44}$$

for the left movers, and

$$2c_k = \begin{cases} \frac{11}{24}, & k = 1; & \frac{1}{3}, & k = 4; \\ \frac{1}{2}, & k = 2; & \frac{11}{24}, & k = 5; \\ \frac{5}{8}, & k = 3; & \frac{1}{2}, & k = 6, \end{cases} \tag{45}$$

for the right movers.

For the sectors wound by Wilson lines, $V_\pm$ are used instead of V_0. The untwisted sector $k = 0$ and twisted sectors for $k = 3, 6, 9$ are not affected by Wilson lines since the Wilson line condition, $3a_3 = 0$, makes it trivial. So, for $k = 3, 6, 9$, there is the additional condition, $(P + kV) \cdot a_i = 0$, which is applicable to T_6 only in our case. For $k \neq 3, 6, 9$, the multiplicity for each twisted sector $k(V + m_f a_3)$ is $\mathcal{P} = \frac{1}{3}\mathcal{P}_k$.

The formula for multiplicity, Eq. (23), is the GSO allowed number of states.[5] For non-prime orbifolds such as $\mathbf{Z}_{12}$, the multiplicity (23) is nonvanishing even if Δ were not 1. Only for those with pure $\mathbf{Z}_{12}$ twists, i.e. $k = 1, 2$ and 5, the multiplicity is counted by those with the vanishing phase.

The twisted sectors for $k = 3, 6, 9$ are not affected by the additional Wilson lines since $3a_3 = 0$. Note that the untwisted sector also is not distinguished by Wilson lines, but Wilson lines give the modular invariance condition $P \cdot a_i = 0$ in the untwisted sector. By the same token, in the sectors where $3a_3 = 0$ ($k = 0, 3, 6, 9$, with 0 corresponding to the untwisted sector), the modular invariance condition restricts Wilson lines [34],

$$(P + kV) \cdot a_3 = 0 \mod Z, \quad k = 0, 3, 6, 9. \tag{46}$$

---

[5] For the prime orbifold $\mathbf{Z}_3$, the multiplicity is just $\frac{1}{3}(1 + \Delta + \Delta^2)$ which can be either 1 for $\Delta = 1$ or 0 for $\Delta = e^{\pm 2\pi i/3}$. So in $\mathbf{Z}_3$ it is sufficient to count those with the vanishing phase, i.e. $(P + V) \cdot V - (s + \phi) \cdot \phi = 0$. It is so also in the kth twisted sector of $\mathbf{Z}_{12-I}$ if $\tilde{\chi}(\theta^k, \theta^l)$ are the same for all l.

Table 4
Left-handed massless states satisfying $(P + 6V) \cdot a_3 = 0$ mod Z in T_6

$P + 6V$	$(N^L)_j$	Θ_0	$\mathcal{P}_6$	χ	Labels
$(\mathbf{5_3}; \frac{1}{2}, 0, 0)(0^8)'$	0	$\frac{1}{2}$	2	L	$T6_5$
$(\overline{\mathbf{10}}_{-1}; \frac{1}{2}, 0, 0)(0^8)'$	0	0	4	L	$T6_{\overline{10}}$
$(\mathbf{1}_{-5}; \frac{1}{2}, 0, 0)(0^8)'$	0	$\frac{1}{2}$	2	L	$T6_{1I}$
$(\overline{\mathbf{5}}_{-3}; \frac{-1}{2}, 0, 0)(0^8)'$	0	$\frac{-1}{6}$	2	L	$T6_{\overline{5}}$
$(\mathbf{10_1}; \frac{-1}{2}, 0, 0)(0^8)'$	0	$\frac{1}{3}$	3	L	$T6_{10}$
$(\mathbf{1_5}; \frac{-1}{2}, 0, 0)(0^8)'$	0	$\frac{-1}{6}$	2	L	$T6_{1J}$
$(\mathbf{1_0}; 0, \frac{1}{2}, \frac{1}{2})(0^8)'$	1_3	0	4	L	h_1
	$1_{\bar{3}}$	$\frac{-1}{6}$	2	L	h_2
	$1_{\bar{1}}$	$\frac{1}{2}$	2	L	h_3
	1_1	$\frac{1}{3}$	3	L	h_4
$(\mathbf{1_0}; 0, \frac{-1}{2}, \frac{-1}{2})(0^8)'$	1_3	$\frac{1}{2}$	2	L	$\bar{h}_1$
	$1_{\bar{3}}$	$\frac{1}{3}$	3	L	$\bar{h}_2$
	$1_{\bar{1}}$	0	4	L	$\bar{h}_3$
	1_1	$\frac{-1}{6}$	2	L	$\bar{h}_4$

But other twisted sectors are not affected, in particular the $k = 1, 2, 4, 5$ sectors. For these sectors, the multiplicity for each of $k(V + m_f a_3)$ is $\frac{1}{3}P_{\theta^k}$ where P_{θ^k} is given in Eq. (23).

Let us present twisted sectors for $k = 6, 1, 2, 3, 4, 5$ in order. The $k = 6$ twisted sector T_6 contains the $\mathcal{CTP}$ conjugates in T_6 again. The spectra in the $k = 1, \ldots, 5$ twisted sectors accompany their $\mathcal{CTP}$ conjugates in $k = 11, \ldots, 7$.

4.1. Twisted sector T_6

The massless condition for the left mover is $\frac{1}{2}|P + 6V|^2 + \sum_j (N^L)_j (\tilde{\phi})_j = \frac{3}{4}$, where P is the $E_8 \times E_8'$ weight vectors. The left mover states satisfying the massless condition always appear vector-like in the T_6 sector. In general, however, they carry different phases from those of the counterpart states with opposite quantum numbers. Thus, chiral matter spectrum is possible even in T_6 after imposing the GSO projection by Eq. (23). In view of (46), we additionally require

$$(P + 6V) \cdot a_3 = 0 \mod Z. \tag{47}$$

The massless states with left-handed chirality satisfying this constraint, and their multiplicity numbers determined by $\mathcal{P}_6$ are listed in Table 4.

For simplicity, here we employed the following abbreviations for the $SO(10)$ spinors and neutral singlets under $SU(5) \times U(1)_X$,

$$\mathbf{5_3} \equiv (\underline{+----}), \qquad \overline{\mathbf{10}}_{-1} \equiv (\underline{+++--}), \qquad \mathbf{1}_{-5} \equiv (\underline{+++++}), \tag{48}$$

$$\overline{\mathbf{5}}_{-3} \equiv (\underline{-++++}), \qquad \mathbf{10_1} \equiv (\underline{---++}), \qquad \mathbf{1_5} \equiv (\underline{-----}), \tag{49}$$

$$\mathbf{1_0} \equiv (0, 0, 0, 0, 0). \tag{50}$$

From the T_6 sector, thus we have the following massless states,

$$\overline{\mathbf{10}}^L_{-1} + 3\{\mathbf{10}^L_1, \overline{\mathbf{10}}^L_{-1}\} + 2\{\overline{\mathbf{5}}^L_{-3}, \mathbf{5}^L_3\} + 2\{\mathbf{1}^L_5, \mathbf{1}^L_{-5}\} + 22 \text{ neutral singlets}$$
$$+ \mathcal{CTP} \text{ conjugates.} \tag{51}$$

While the multiplicity number for $\overline{\mathbf{10}}^L_{-1}$ is 4, the multiplicity of $\mathbf{10}^L_1$ is 3. One of $\overline{\mathbf{10}}^L_{-1}$s provides one generation of the MSSM matter, $\{Q, d^c, \nu^c\}$. The remaining vector-like pairs of $\{\mathbf{10}^L_1, \overline{\mathbf{10}}^L_{-1}\}$ could be utilized to break $SU(5) \times U(1)_X$ into the MSSM gauge group.

We present the calculation in detail for the first row. The θ^6 twist vectors are

$$\tilde{\phi} = 6\phi_s = \left(\frac{1}{2}, 0, \frac{1}{2}\right),$$
$$\tilde{V} = 6V \equiv \left(\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,0\,0\right)(0\,0\,0\,0\,0\,0\,0\,0). \tag{52}$$

In the θ^6 twisted sector, consider $P = (0^8)(0^8)$. With some shifts, P can be $([-1]^5; -2, -3, 0) \times (-1, -1; -6, 0^5)'$. Then, from (43) and (44) we require $(P + 6V)^2 = 2\tilde{c}_6 = \frac{216}{144} = \frac{3}{2}$ which is certainly satisfied with $\tilde{N}_L = 0$. Therefore, the state $P = (0^8)(0^8)$ is massless. The GSO projection is given when combined with the right movers. The masslessness condition for right movers is $(s + 6\phi)^2 = \frac{1}{2}$,[6] from which we will determine the chirality.

Let us calculate the masslessness condition of the left movers first, the multiplicity and the chirality.

The masslessness condition for left movers becomes

$$(P + \tilde{V})^2 + 2\sum_j N^L_j \tilde{\phi}_j = \frac{3}{2}. \tag{53}$$

Thus, the vectors satisfying Eq. (53) with Eq. (52) constitute the representation $\overline{\mathbf{5}}_L$. These weights are

$$\begin{aligned}
\mathbf{1_5}: & \quad (0\,0\,0\,0\,0\,0\,0\,0), \\
\overline{\mathbf{5}}_{-3}: & \quad (\underline{-1\,0\,0\,0}\,-1\,0\,0), \\
\mathbf{10_1}: & \quad (\underline{-1\,-1\,-1\,0\,0}\,-1\,0\,0), \tag{54}
\end{aligned}$$

where we calculated $U(1)_X$ charges as $(P + \tilde{V}) \cdot Q_X$. Their $\mathcal{CTP}$ conjugates are

$$\begin{aligned}
& (-1\,-1\,-1\,-1\,-1\,-1\,0\,0), \\
& (\underline{-1\,-1\,-1\,-1\,0\,0\,0\,0}), \\
& (\underline{-1\,-1\,0\,0\,0}\,0\,0\,0). \tag{55}
\end{aligned}$$

Note that for $P = (0^8)(0^8)$,

$$(P + 6V) \cdot V = \frac{5}{6}, \qquad \tilde{\phi} \cdot \phi_s = \frac{1}{4}. \tag{56}$$

To obtain the chirality χ, we look at $s = (\frac{1}{2}\chi, \tilde{s})$ allowing nonvanishing multiplicities. Then the phase of Δ_{θ^6} is found as $(P + \tilde{V}) \cdot V - (\tilde{s} + \tilde{\phi}) \cdot \phi_s - \frac{1}{2}k(V^2 - \phi_s^2)$. With $k = 6$, we have

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[6] See Appendix D of [7].

Table 5
The multiplicity for $s^2 = 1$ and $(P + 6V) \cdot V = \frac{1}{6}$. + and − denote $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and $\oplus$ ($\ominus$) is R-(L-)handed. Note that $(P + \tilde{V}) \cdot V - \frac{1}{2}k(V^2 - \phi_s^2) = \frac{1}{3}$

$\frac{1}{2}\chi$	$\tilde{s} = (r + \tilde{\omega})$	$\tilde{s} \cdot \phi_s$	Δ phase	Multiplicity
$\oplus$	$+++$	$\frac{5}{12}$	$-\frac{4}{12} \cdot 2\pi$	3
$\oplus$	$+--$	0	$\frac{1}{12} \cdot 2\pi$	0
$\oplus$	$-+-$	$-\frac{1}{12}$	$\frac{2}{12} \cdot 2\pi$	2
$\oplus$	$--+$	$-\frac{4}{12}$	$\frac{5}{12} \cdot 2\pi$	0
$\ominus$	$---$	$-\frac{5}{12}$	$\frac{6}{12} \cdot 2\pi$	2
$\ominus$	$-++$	0	$\frac{1}{12} \cdot 2\pi$	0
$\ominus$	$+-+$	$\frac{1}{12}$	$0 \cdot 2\pi$	4
$\ominus$	$++-$	$\frac{4}{12}$	$-\frac{3}{12} \cdot 2\pi$	0

$\frac{1}{2}k(V^2 - \phi_s^2) = \frac{3}{2} \sim \frac{1}{2}$, and the phase of Δ is 2π times $(\frac{1}{12} - \tilde{s} \cdot \phi_s)$, and hence we obtain the multiplicity listed in Table 5. Then, we read the chirality χ, or $\oplus$ or $\ominus$, from the first entry of $s = (\frac{1}{2}\chi | s_1\ s_2\ s_3)$ if that chirality is allowed by the right mover condition, which is shown in the first column of Table 5. If the first column does not satisfy the right mover condition, we should search for higher s_0, which will be discussed shortly. The components of the vector $\tilde{s}$ are the last three $\pm\frac{1}{2}$'s of s. The number of massless states are given by P_{θ^6}. For these, we use the Euler numbers $\tilde{\chi}(\theta^k, \theta^l)$ given in Table 1 [7]. Therefore, we obtain

$$\mathcal{P}_6 = \mathcal{P}_{\theta^6} = \frac{1}{12}\left\{(1 + \Delta^6)(16 + 4\Delta^3) + \Delta(1 + \Delta)(1 + \Delta^3 + \Delta^6 + \Delta^9)\right\} \tag{57}$$

which becomes 4, 2, 2, and 3 for $\Delta = 1, -1, \Delta^3 = -1, +1$, respectively, and 0 for the other cases. We know that $s \cdot \tilde{\phi} = s \cdot \tilde{\phi} + $ (integer). Now consider the right mover condition. For $s^2 = 1$, the masslessness condition $(s + \tilde{\phi})^2 = \frac{1}{2}$ leads to $\tilde{s} = (-\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2})$ where we used the shifted $\tilde{\phi}$ of Eq. (52). The relevant ones appear in the third and fifth rows of Table 5. Among these one set is the $\mathcal{CTP}$ conjugates of the other. The $U(1)_X$ charge is $(P + 6V) \cdot Q_X = 5$. Thus, we obtain two singlets as shown in the third row of Table 4.

Consider $P = (-1\ 0\ 0\ 0\ 0\ -1\ 0\ 0)(0^8)$ and $P = (-1\ -1\ -1\ -1\ 0\ 0\ 0\ 0)(0^8)$. For $P = (-1\ 0\ 0\ 0\ 0\ -1\ 0\ 0)(0^8)$,

$$(P + 6V) \cdot V = \frac{1}{6}, \qquad \tilde{\phi} \cdot \phi_s = \frac{1}{4}. \tag{58}$$

For $k = 6$, the phase becomes $(P + \tilde{V}) \cdot V - (\tilde{s} + \tilde{\phi}) \cdot \phi_s - \frac{1}{2}k(V^2 - \phi_s^2) = (-\frac{7}{12} - \tilde{s} \cdot \phi_s)$. We add $-\frac{8}{12} \cdot 2\pi$ to the fourth column entries of Table 5. The masslessness condition chooses $\tilde{s} = (-\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2})$, the third and fifth rows again, leading to the multiplicity 2. $(\underline{-1\ 0\ 0\ 0\ 0\ -1\ 0\ 0})$ is $\bar{\mathbf{5}}$ whose $U(1)_X$ charge is 3. For $P = (-1\ -1\ -1\ -1\ 0\ 0\ 0\ 0)(0^8)$,

$$(P + 6V) \cdot V = -\frac{1}{6}, \qquad \tilde{\phi} \cdot \phi_s = \frac{1}{4}. \tag{59}$$

Now the phase becomes $(P + \tilde{V}) \cdot V - (\tilde{s} + \tilde{\phi}) \cdot \phi_s - \frac{1}{2}k(V^2 - \phi_s^2) = (-\frac{11}{12} - \tilde{s} \cdot \phi_s) = (\frac{1}{12} - \tilde{s} \cdot \phi_s)$. Thus we obtain the fourth column entries of Table 5. The masslessness condition chooses $\tilde{s} = (-\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2})$, the third and fifth rows again, leading to the multiplicity 2.

$(\underline{-1\,-1\,-1\,-1}\,0\,0\,0\,0)$ is **5** whose $U(1)_X$ charge is -3. The above four cases are shuffled to list the $\mathcal{CTP}$ conjugates together,

$$\mathbf{5}^L_{-3} \begin{cases} \mathbf{5}: & (P+6V) = \left(-\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,0\,0\right),\ \oplus,\quad Q_X = 3, \\ \bar{\mathbf{5}}: & (P+6V) = \left(-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,0\,0\right),\ \ominus,\quad Q_X = -3, \end{cases} \tag{60}$$

$$\mathbf{5}^L_{3} \begin{cases} \bar{\mathbf{5}}: & (P+6V) = \left(-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,0\,0\right),\ \oplus,\quad Q_X = -3, \\ \mathbf{5}: & (P+6V) = \left(-\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,0\,0\right),\ \ominus,\quad Q_X = 3. \end{cases} \tag{61}$$

Thus we obtain the first and fourth row entries of Table 4.

Next consider $\overline{\mathbf{10}}$s and $\mathbf{10}$s. So consider $P = (-1\,-1\,0\,0\,0\,0\,0\,0)(0^8)$ and $P = (-1\,-1\,-1\,0\,0\,-1\,0\,0)(0^8)$. For $P = (-1\,-1\,0\,0\,0\,0\,0\,0)(0^8)$, we have

$$(P+6V)\cdot V = \frac{1}{3}, \qquad \tilde{\phi}\cdot\phi_s = \frac{1}{4}. \tag{62}$$

The phase becomes $(P+\tilde{V})\cdot V - (\tilde{s}+\tilde{\phi})\cdot\phi_s - \tfrac{1}{2}k(V^2-\phi_s^2) = (-\tfrac{5}{12}-\tilde{s}\cdot\phi_s) = (\tfrac{7}{12}-\tilde{s}\cdot\phi_s)$. We add $\tfrac{6}{12}\cdot 2\pi$ to the fourth column entries of Table 5. The masslessness condition for right movers chooses $\tilde{s} = (-\tfrac{1}{2},\pm\tfrac{1}{2},-\tfrac{1}{2})$, the third and fifth rows again, leading to the multiplicity 3 and 4, respectively. For $P = (-1\,-1\,-1\,0\,0\,-1\,0\,0)(0^8)$, we have

$$(P+6V)\cdot V = -\frac{1}{3}, \qquad \tilde{\phi}\cdot\phi_s = \frac{1}{4}. \tag{63}$$

The phase becomes $(P+\tilde{V})\cdot V - (\tilde{s}+\tilde{\phi})\cdot\phi_s - \tfrac{1}{2}k(V^2-\phi_s^2) = (-\tfrac{1}{12}-\tilde{s}\cdot\phi_s) = (\tfrac{11}{12}-\tilde{s}\cdot\phi_s)$. We add $-\tfrac{2}{12}\cdot 2\pi$ to the fourth column entries of Table 5. The masslessness condition chooses $\tilde{s} = (-\tfrac{1}{2},\pm\tfrac{1}{2},-\tfrac{1}{2})$, the third and fifth rows again, leading to the multiplicity 4 and 3, respectively. These four cases are shuffled to list the $\mathcal{CTP}$ conjugates together,

$$\mathbf{10}^L_{1} \begin{cases} \overline{\mathbf{10}}: & (P+6V) = \left(-\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,0\,0\right),\ \oplus,\ \mathcal{P}_6 = 3,\quad Q_X = -1, \\ \mathbf{10}: & (P+6V) = \left(-\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,0\,0\right),\ \ominus,\ \mathcal{P}_6 = 3,\quad Q_X = 1, \end{cases} \tag{64}$$

$$\overline{\mathbf{10}}^L_{-1} \begin{cases} \mathbf{10}: & (P+6V) = \left(-\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,0\,0\right),\ \oplus,\ \mathcal{P}_6 = 4,\quad Q_X = 1, \\ \overline{\mathbf{10}}: & (P+6V) = \left(-\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,0\,0\right),\ \ominus,\ \mathcal{P}_6 = 4,\quad Q_X = -1. \end{cases} \tag{65}$$

Thus we obtain the second and fifth row entries of Table 4.

Other singlets with nonvanishing oscillators are also allowed, which are shown in Table 4.

4.2. Twisted sector T_1

The massless condition for the right mover in the T_1 sector is $(s+\phi_s)^2 = \tfrac{11}{24}$. It allows only one right-handed state $s = (\ominus---)$, which gives $(\tilde{s}+\phi_s)\cdot\phi_s = -\tfrac{1}{8}$.

Since $-\tfrac{k}{2}(V^2-\phi^2) = -\tfrac{1}{4}$ for $k=1$, the phase in Eq. (24) is given by $\Theta_0 = (P+V)\cdot V + \sum_j (N^L)_j(\tilde{\phi})_j - \tfrac{1}{8}$.

The T_1 sector is distinguished by Wilson lines: $V_0 = V$, $V_+ = V + a_1$, and $V_- = V - a_1$. These sectors are denoted as T_1^0, T_1^+ and T_1^-.

Note the degeneracy factor $\tilde{\chi}(\theta^k, \theta^l)$ for $k = 1, 2, 5$ in Table 1. They take the same value 3 along each horizontal line, as in the prime orbifolds such as $\mathbf{Z}_3$ and $\mathbf{Z}_7$. Thus, only the states with vanishing phase turn out to survive the projection by Eq. (23) in the T_1, T_2, T_5 (and also in T_{11}, T_{10}, T_7) sectors.

The masslessness condition for the left movers (43) gives

$$(P + kV)^2 = -2 \sum_j N_j^L \tilde{\phi}_j + 2\tilde{c}_k. \tag{66}$$

For $k = 1$ we have $2\tilde{c}_1 = \frac{35}{24}$. The states satisfying the massless condition and $(P + V_0) \cdot V_0 + \sum_j (N^L)_j (\hat{\phi})_j = \frac{1}{8}$ for T_1^0 are listed in Table 6. The multiplicity 3 reduces to 1 due to the distinction by Wilson lines: V_0, V_+, and V_-.

In the T_1^+ sector with $V_+ = V + a_3$, only the states with $(P + V_+) \cdot V_+ + (N^L)_j (\hat{\phi})_j = \frac{7}{24}$ survives the GSO projection by Eq. (23), which are listed in the middle part of Table 6.

In the T_1^- sector with $V_- = V - a_3$, only the states with $(P + V_-) \cdot V_- + (N^L)_j (\hat{\phi})_j = \frac{5}{8}$ survives the GSO projection by Eq. (23), which are listed in the lower part of Table 6.

The $\mathcal{CTP}$ conjugates of T_1 appear in T_{11}.

Table 6
Chiral matter fields satisfying $\Theta_{0,\pm} = 0$ in the T_1^0 and $T_1^\pm$ sectors. Here, $\bar{\mathbf{5}}_{-1/2} \equiv (\frac{-3}{4}, [\frac{1}{4}]^4)$, $\mathbf{5}_{1/2} \equiv (\frac{3}{4}, [\frac{-1}{4}]^4)$, and $\mathbf{1}_{\pm 5/2} \equiv (\pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4})$. There are two $SU(2)'$ doublets with $(\frac{-5}{6}, \frac{1}{6}; \cdots)'$

$P + V$	$(N^L)_j$	$\mathcal{P}_1(f_0)$	χ
$(\bar{\mathbf{5}}_{-1/2}; \frac{-7}{12}, \frac{6}{12}, 0)(\frac{1}{6}, \frac{1}{6}; 0^6)'$	0	1	L
$(\bar{\mathbf{5}}_{-1/2}; \frac{5}{12}, \frac{-6}{12}, 0)(\frac{1}{6}, \frac{1}{6}; 0^6)'$	1_3	1	L
$(\mathbf{5}_{1/2}; \frac{-1}{12}, 0, \frac{6}{12})(\frac{1}{6}, \frac{1}{6}; 0^6)'$	2_3	1	L
$(\mathbf{1}_{-5/2}; \frac{-7}{12}, \frac{-6}{12}, 0)(\frac{1}{6}, \frac{1}{6}; 0^6)'$	3_3	1	L
$(\mathbf{1}_{-5/2}; \frac{5}{12}, \frac{6}{12}, 0)(\frac{1}{6}, \frac{1}{6}; 0^6)'$	$1_2, 4_3$	$1 + 1$	L
$(\mathbf{1}_{5/2}; \frac{11}{12}, 0, \frac{6}{12})(\frac{1}{6}, \frac{1}{6}; 0^6)'$	0	1	L
$(\mathbf{1}_{5/2}; \frac{-1}{12}, 0, \frac{-6}{12})(\frac{1}{6}, \frac{1}{6}; 0^6)'$	$1_1, 5_3, \{1_2 + 1_3\}$	$1 + 1 + 1$	L
$P + V_+$	$(N^L)_j$	$\mathcal{P}_1(f_+)$	χ
$(\mathbf{1}_{-5/2}; \frac{5}{12}, \frac{2}{12}, \frac{4}{12})(\frac{-5}{6}, \frac{1}{6}; \frac{-1}{3}; 0^5)'$	0	1	L
$(\mathbf{1}_{-5/2}; \frac{5}{12}, \frac{2}{12}, \frac{4}{12})(\frac{1}{6}, \frac{1}{6}; \frac{2}{3}; 0^5)'$	2_3	1	L
$(\mathbf{1}_{5/2}; \frac{-1}{12}, \frac{-4}{12}, \frac{-2}{12})(\frac{-5}{6}, \frac{1}{6}; \frac{-1}{3}; 0^5)'$	1_3	1	L
$(\mathbf{1}_{5/2}; \frac{-1}{12}, \frac{-4}{12}, \frac{-2}{12})(\frac{1}{6}, \frac{1}{6}; \frac{2}{3}; 0^5)'$	3_3	1	L
$P + V_-$	$(N^L)_j$	$\mathcal{P}_1(f_-)$	χ
$(\mathbf{5}_{1/2}; \frac{-1}{12}, \frac{4}{12}, \frac{2}{12})(\frac{1}{6}, \frac{1}{6}; \frac{-2}{3}; 0^5)'$	0	1	L
$(\mathbf{1}_{-5/2}; \frac{5}{12}, \frac{-2}{12}, \frac{8}{12})(\frac{1}{6}, \frac{1}{6}; \frac{-2}{3}; 0^5)'$	0	1	L
$(\mathbf{1}_{-5/2}; \frac{-7}{12}, \frac{-2}{12}, \frac{-4}{12})(\frac{1}{6}, \frac{1}{6}; \frac{-2}{3}; 0^5)'$	1_3	1	L
$(\mathbf{1}_{5/2}; \frac{-1}{12}, \frac{-8}{12}, \frac{2}{12})(\frac{1}{6}, \frac{1}{6}; \frac{-2}{3}; 0^5)'$	1_3	1	L

4.3. Twisted sector T_2

The massless condition for the right mover in the T_2 sector is $(s + 2\phi_s)^2 = \frac{1}{2}$. It allows only one right-handed state $s = (\ominus---)$, which gives $(\tilde{s} + 2\phi_s) \cdot \phi_s = \frac{1}{6} = -\frac{5}{6}$.

Since $-\frac{k}{2}(V^2 - \phi^2) = -\frac{1}{2}$ for $k = 2$, the phase of Δ in Eq. (24) is given by $\Theta_0 = (P + 2V) \cdot V + \sum_j (N^L)_j (\hat{\phi})_j + \frac{1}{3}$. The T_2 sector is also distinguished by Wilson lines: $V_0 = V$, $V_+ = V + a_3$, and $V_- = V - a_3$. These sectors are denoted as T_2^0, T_2^+ and T_2^-.

Since $\tilde{\chi}(\theta^k, \theta^l)$ with $k = 2$ in Eq. (23) takes the same value 3 along the horizontal line, only the states with $\Theta_{0,\pm} = 0$ survive the projection operator in the T_2 sector. In the T_2^0 sector, hence, the massless states satisfying the condition $(P + 2V) \cdot V + \sum_j (N^L)_j (\hat{\phi})_j = -\frac{1}{3}$ are selected. Similarly, in the T_2^+ and T_2^- sectors, $(P + 2V_+) \cdot V_+ + \sum_j (N^L)_j (\hat{\phi})_j = 0$ and $(P + 2V_-) \cdot V_- + \sum_j (N^L)_j (\hat{\phi})_j = -\frac{1}{3}$ should be chosen. The massless condition for the left mover is $|P + 2V_{(\pm)}|^2 + 2\sum_j (N^L)_j (\tilde{\phi})_j = \frac{3}{2}$. The allowed shifted $E_8 \times E_8'$ weight vectors $(P + 2V)$s in the T_2 sector are shown in Table 7.

Consider the first row of Table 7. Since

$$2V = \left(\frac{1}{2} \; \frac{1}{2} \; \frac{1}{2} \; \frac{1}{2} \; \frac{1}{2}; \; \frac{5}{6} \; 1 \; 0 \right) \left(\frac{1}{3} \; \frac{1}{3} \; 0^6 \right)',$$

Table 7
Chiral matter fields satisfying $\Theta_0 = 0$, $\Theta_+ = 0$, and $\Theta_- = 0$ in the T_2^0, T_2^+, and T_2^- sectors, respectively. Here $\mathbf{1_0} \equiv (0, 0, 0, 0, 0)$ and the hidden sector $\mathbf{16}'$ and $\mathbf{10}'$ are not bold-faced not to be confused with observable sector representations. There are four $SU(2)'$ doublets of $D_{1,2}^{\pm}$

$P + 2V$	$(N^L)_j$	$\mathcal{P}_2(f_0)$	χ	Labels
$(\mathbf{5_3}; \frac{-1}{6}, 0^2)(\frac{1}{3}, \frac{1}{3}; 0^6)'$	0	1	L	$T2_5$
$(\mathbf{1_{-5}}; \frac{-1}{6}, 0^2)(\frac{1}{3}, \frac{1}{3}; 0^6)'$	0	1	L	$T2_1$
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{2}, \frac{-1}{2})(\frac{1}{3}, \frac{1}{3}; 0^6)'$	$2_{\bar{1}}, 2_3$	$1 + 1$	L	C_1^0, C_2^0
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{2}, \frac{1}{2})(\frac{1}{3}, \frac{1}{3}; 0^6)'$	$1_2, \{1_{\bar{1}} + 1_3\}$	$1 + 1$	L	C_3^0, C_4^0
$(\mathbf{1_0}; \frac{-2}{3}, \frac{1}{2}, \frac{1}{2})(\frac{1}{3}, \frac{1}{3}; 0^6)'$	$1_{\bar{1}}$	1	L	C_5^0
$(\mathbf{1_0}; \frac{-2}{3}, \frac{-1}{2}, \frac{-1}{2})(\frac{1}{3}, \frac{1}{3}; 0^6)'$	1_3	1	L	C_6^0
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{2}, \frac{1}{2})(\frac{-2}{3}, \frac{-2}{3}; 0^6)'$	0	1	L	C_7^0
$P + 2V_+$	$(N^L)_j$	$\mathcal{P}_2(f_+)$	χ	Labels
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{6}, \frac{1}{6})(\frac{-1}{6}, \frac{-1}{6}; \frac{-1}{6}; \overline{16})'$	0	1	L	$T2_{O\overline{16}}^+$
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{6}, \frac{1}{6})(\frac{-2}{3}, \frac{1}{3}; \frac{1}{3}; 0^5)'$	$1_2, \{1_{\bar{1}} + 1_3\}$	$1 + 1$	L	D_1^+, D_2^+
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{6}, \frac{1}{6})(\frac{1}{3}, \frac{1}{3}; \frac{-2}{3}; 0^5)'$	$2_{\bar{1}}, 2_3$	$1 + 1$	L	C_1^+, C_2^+
$P + 2V_-$	$(N^L)_j$	$\mathcal{P}_2(f_-)$	χ	Labels
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{6}, \frac{-1}{6})(\frac{1}{3}, \frac{1}{3}; \frac{-1}{3}; 10)'$	0	1	L	$T2_{O10}^-$
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{6}, \frac{-1}{6})(\frac{-2}{3}, \frac{1}{3}; \frac{-1}{3}; 0^5)'$	$2_{\bar{1}}, 2_3$	$1 + 1$	L	D_1^-, D_2^-
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{6}, \frac{-1}{6})(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}; 0^5)'$	$1_2, \{1_{\bar{1}} + 1_3\}$	$1 + 1$	L	C_1^-, C_2^-
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{6}, \frac{-1}{6})(\frac{-2}{3}, \frac{-2}{3}; \frac{2}{3}; 0^5)'$	0	1	L	C_3^-

$P = (\underline{-1 \, -1 \, -1 \, -1 \, 0}; \; -1 \, -1 \, 0)(0^8)'$ satisfies the masslessness condition. The $SU(5)$ representation is **5**. Then, we have

$$P + 2V = \left(-\frac{1}{2} \, -\frac{1}{2} \, -\frac{1}{2} \, -\frac{1}{2} \, +\frac{1}{2}; \; -\frac{1}{6} \, 0 \, 0 \right) \left(\frac{1}{3} \, \frac{1}{3} \, 0^6 \right)' \tag{67}$$

which gives $Q_X = 3$. From the previous discussion on the right mover condition, we obtain the chirality $2\ominus$.

For the $T_2^{\pm}$ sectors, only neutral fields under $SU(5) \times U(1)_X$ arise, which are tabulated in Table 7. Thus, from T_2 we obtain the following $SU(5) \times U(1)_X$ representations

$$\{\mathbf{5_3}, \mathbf{1_{-5}}\} + 12 \text{ neutral singlets.} \tag{68}$$

The $\mathcal{CTP}$ conjugates are provided from T_{10}.

There are four $SU(2)'$ doublets and one $\overline{\mathbf{16}}'$ and one $\mathbf{10}'$ of $SO(10)'$.

4.4. Twisted sector T_3

The shifted momenta in the T_3 sector must satisfy $(P + 3V) \cdot a_3 = 0 \mod Z$, viz. Eq. (46). It turns out that there is no massless states satisfying this condition.

4.5. Twisted sector T_4

The massless condition for the right movers in the T_4 sector is $(s + 4\phi_s)^2 = \frac{1}{3}$. Taking a shifted $4\phi_s$ as $(\frac{2}{3} \, \frac{1}{3} \, \frac{1}{3})$, only the right-handed state $s = (\ominus - - -)$ satisfies this condition. So it is left-handed. Now, $(\tilde{s} + 4\phi_s) \cdot \phi_s = 0$.

Since $-\frac{k}{2}(V^2 - \phi^2) = $ (integer) for $k = 4$, it does not contribute to the phase. So, Δ of Eq. (24) is given by $\Theta_f = (P + 4V_{(\pm)}) \cdot V_{(\pm)} + \sum_j (N^L)_j (\hat{\phi})_j$. The T_4 sector is again distinguished by Wilson lines: $V_0 = V$, $V_+ = V + a_3$ and $V_- = V - a_3$. These sectors are denoted as T_4^0, T_4^+ and T_4^-.

$\tilde{\chi}(\theta^4, \theta^k)$ of the T_4 sector are $(27, 3, 3, 3)^3$, hence the multiplicity is

$$\mathcal{P}_4 = \frac{3}{12} \left(1 + \Delta^4 + \Delta^8 \right) \left(8 + \left[1 + \Delta + \Delta^2 + \Delta^3 \right] \right). \tag{69}$$

So, $\Delta = e^{2\pi i/12}$, $e^{2\pi i/6}$ give $\mathcal{P}_4 = 0$. $\mathcal{P}_4 = 9, 6, 6$ for $\Delta = 1, -1, e^{\pm 2\pi i/4}$. Considering Wilson lines, the nonvanishing multiplicities are 3, 2, 2 in each T_4. The massless fields of T_4^0 are listed in Table 8.

Consider first two rows of Table 8. They are left-handed. The massless condition for the left movers is $(P + 4V)^2 = \frac{4}{3}$. Since

$$4V = \left(1 \, 1 \, 1 \, 1 \, 1; \; \frac{5}{3} \, 2 \, 0 \right) \left(\frac{2}{3} \, \frac{2}{3} \, 0^6 \right)',$$

the state

$$\left(0, [-1]^4; -2, -2, 0 \right) \left(-1, -1; 0^6 \right)'$$

satisfies the masslessness condition. It is **5** and $Q_X = -2$ since $P + 4V = (\underline{1 \, 0 \, 0 \, 0 \, 0} \cdots)()'$. Since $(P + 4V) \cdot V = 0$, we obtain $\Delta = e^{2\pi i \cdot 0}$ from (23) and the multiplicity is 3. The state $\overline{\mathbf{5}}$

Table 8
Chiral matter fields in the T_4^0, T_4^+ and T_4^- sectors. There are twelve $SU(2)'$ doublets of $d_{1,2,3}^\pm$

$P + 4V$	$(N^L)_j$	Θ_0	$\mathcal{P}_4(f_0)$	χ	Labels
$(\mathbf{5}_{-2}; \frac{-1}{3}, 0^2)(\frac{-1}{3}, \frac{-1}{3}; 0^6)'$	0	0	3	L	$T4_5$
$(\mathbf{\bar{5}}_2; \frac{-1}{3}, 0^2)(\frac{-1}{3}, \frac{-1}{3}; 0^6)'$	0	$\frac{1}{2}$	2	L	$T4_{\bar 5}$
$(\mathbf{1}_0; \frac{2}{3}, 0^2)(\frac{-1}{3}, \frac{-1}{3}; 0^6)'$	1_3	$\frac{1}{4}$	2	L	s_1^0
	1_2	$\frac{1}{2}$	2	L	s_2^0
	$1_{\bar 1}$	$\frac{-1}{4}$	2	L	s_3^0
$(\mathbf{1}_0; \frac{2}{3}, 0^2)(\frac{2}{3}, \frac{2}{3}; 0^6)'$	0	$\frac{1}{2}$	2	L	s_4^0
$(\mathbf{1}_0; \frac{-1}{3}, \pm1, 0)(\frac{-1}{3}, \frac{-1}{3}; 0^6)'$	0	$\frac{1}{4}$	$2+2$	L	s_5^0, s_6^0
$(\mathbf{1}_0; \frac{-1}{3}, 0, \pm1)(\frac{-1}{3}, \frac{-1}{3}; 0^6)'$	0	$\frac{-1}{4}$	$2+2$	L	s_7^0, s_8^0
$P + 4V_+$	$(N^L)_j$	Θ_+	$\mathcal{P}_4(f_+)$	χ	Labels
$(\mathbf{1}_0; \frac{2}{3}, \frac{-1}{3}, \frac{1}{3})(\underline{\frac{2}{3}, \frac{-1}{3}}; \frac{-1}{3}; 0^5)'$	0	$\frac{1}{2}$	2	L	d_1^+
$(\mathbf{1}_0; \frac{2}{3}, \frac{-1}{3}, \frac{1}{3})(\frac{-1}{3}, \frac{-1}{3}; \frac{2}{3}; 0^5)'$	0	0	3	L	s_1^+
$(\mathbf{1}_0; \frac{-1}{3}, \frac{2}{3}, \frac{1}{3})(\underline{\frac{2}{3}, \frac{-1}{3}}; \frac{-1}{3}; 0^5)'$	0	$\frac{1}{4}$	2	L	d_2^+
$(\mathbf{1}_0; \frac{-1}{3}, \frac{2}{3}, \frac{1}{3})(\frac{-1}{3}, \frac{-1}{3}; \frac{2}{3}; 0^5)'$	0	$\frac{-1}{4}$	2	L	s_2^+
$(\mathbf{1}_0; \frac{-1}{3}, \frac{-1}{3}, \frac{-2}{3})(\underline{\frac{2}{3}, \frac{-1}{3}}; \frac{-1}{3}; 0^5)'$	0	$\frac{-1}{4}$	2	L	d_3^+
$(\mathbf{1}_0; \frac{-1}{3}, \frac{-1}{3}, \frac{-2}{3})(\frac{-1}{3}, \frac{-1}{3}; \frac{2}{3}; 0^5)'$	0	$\frac{1}{4}$	2	L	s_3^+
$P + 4V_-$	$(N^L)_j$	Θ_-	$\mathcal{P}_4(f_-)$	χ	Labels
$(\mathbf{1}_0; \frac{2}{3}, \frac{1}{3}, \frac{-1}{3})(\underline{\frac{2}{3}, \frac{-1}{3}}; \frac{1}{3}; 0^5)'$	0	$\frac{1}{2}$	2	L	d_1^-
$(\mathbf{1}_0; \frac{2}{3}, \frac{1}{3}, \frac{-1}{3})(\frac{-1}{3}, \frac{-1}{3}; \frac{-2}{3}; 0^5)'$	0	0	3	L	s_1^-
$(\mathbf{1}_0; \frac{-1}{3}, \frac{-2}{3}, \frac{-1}{3})(\underline{\frac{2}{3}, \frac{-1}{3}}; \frac{1}{3}; 0^5)'$	0	$\frac{1}{4}$	2	L	d_2^-
$(\mathbf{1}_0; \frac{-1}{3}, \frac{-2}{3}, \frac{-1}{3})(\frac{-1}{3}, \frac{-1}{3}; \frac{-2}{3}; 0^5)'$	0	$\frac{-1}{4}$	2	L	s_2^-
$(\mathbf{1}_0; \frac{-1}{3}, \frac{1}{3}, \frac{2}{3})(\underline{\frac{2}{3}, \frac{-1}{3}}; \frac{1}{3}; 0^5)'$	0	$\frac{-1}{4}$	2	L	d_3^-
$(\mathbf{1}_0; \frac{-1}{3}, \frac{1}{3}, \frac{2}{3})(\frac{-1}{3}, \frac{-1}{3}; \frac{-2}{3}; 0^5)'$	0	$\frac{1}{4}$	2	L	s_3^-

with $Q_X = 2$,

$$\left(\underline{-2, [-1]^4}; -2, -2, 0\right)\left(-1, -1; 0^6\right)'$$

also satisfies the masslessness condition. Since $(P + 4V) \cdot V = -\frac{1}{2}$, we obtain $\Delta = e^{2\pi i \cdot (-\frac{1}{2})}$ from (23) and the multiplicity is 2.

In the $T_4^\pm$ sectors, the phases in Eq. (26) are respectively given by

$$\Theta_+ = (P + 4V_+) \cdot V_+ + \sum_j (N^L)_j (\hat\phi)_j + \frac{1}{3}, \tag{70}$$

$$\Theta_- = (P + 4V_-) \cdot V_- + \sum_j (N^L)_j (\hat\phi)_j, \tag{71}$$

where "$\frac{1}{3}$" in Eq. (70) comes from "$-\frac{4}{2}(V_+^2 - \phi_s^2)$".

It turns out that from the $4(V \pm a_3)$ sectors, only neutral fields under $SU(5) \times U(1)_X$ arise, which are listed in Table 8.

Thus, the massless states in $T_4(+T_8)$ sectors are

$$\mathbf{5}_{-2} + 2\{\mathbf{5}_{-2}, \bar{\mathbf{5}}_2\} + 30 \text{ neutral singlets} + \mathcal{CTP} \text{ conjugates.} \tag{72}$$

There are twelve $SU(2)'$ doublets.

4.6. Twisted sector T_5

In the T_5 (and $T_5^{\pm}$) sector of the $\mathbf{Z}_{12-\mathrm{I}}$ orbifold, only the right-handed chirality states appear as massless states from the right mover condition.

The massless condition for the right mover in the T_5 sector is $(s + 5\phi_s)^2 = \frac{11}{24}$. Taking a shifted $5\phi_s$ as $(\frac{1}{12} - \frac{4}{12} \frac{5}{12})$, only the right mover $s = (\oplus - + -)$ satisfies this condition. So it is right-handed. Now, $(\tilde{s} + 5\phi_s) \cdot \phi_s = -\frac{1}{8}$.

Note that $-\frac{k}{2}(V^2 - \phi^2) = -\frac{1}{4}$ for $k = 5$. So, the phase in Eq. (24) is given by $\Theta_0 = (P + 5V) \cdot V + \sum_j (N^L)_j (\hat{\phi})_j - \frac{1}{8}$. The T_5 sector is again distinguished by Wilson lines: $V_0 = V$, $V_+ = V + a_3$, and $V_- = V - a_3$. These sectors are denoted as T_5^0, T_5^+ and T_5^-. In the T_5 sectors, only the states with $\Theta_{0,\pm} = 0$ survive the GSO projection Eqs. (23), (25) since all $\tilde{\chi}(5, l)$s are the same.

Only neutral and vector-like pairs of $SU(5)$ singlets arise in T_5. In the T_5^+ sector, only the states with $(P + 5V_+) \cdot V_+ + \sum_j (N^L)_j (\hat{\phi})_j = -\frac{1}{24}$ survive the GSO projection Eq. (23). In the T_5^- sector, only the states with $(P + 5V_-) \cdot V_- + \sum_j (N^L)_j (\hat{\phi})_j = -\frac{3}{8}$ survive. These are shown in Table 9.

The $\mathcal{CTP}$ conjugates of T_5 appear in T_7.

5. Summary of matter spectra

Collecting all the flipped $SU(5)$ model fields, we obtain the following:

$$U: \quad (\mathbf{1}_{-5} + \mathbf{5}_3 + \overline{\mathbf{10}}_{-1})^L_{U_3}, \quad (\bar{\mathbf{5}}_2)^L_{U_2}, \quad (\mathbf{1}_{-5} + \mathbf{5}_3 + \overline{\mathbf{10}}_{-1})^L_{U_1}, \quad (\mathbf{1}_0)^L_{U_2}, \tag{73}$$

from the untwisted sector, and

$$T_6: \quad \overline{\mathbf{10}}^L_{-1} + \{2(\mathbf{1}_5 + \mathbf{1}_{-5} + \bar{\mathbf{5}}_{-3} + \mathbf{5}_3) + 3(\mathbf{10}_1 + \overline{\mathbf{10}}_{-1})\}^L + 22\{\mathbf{1}_0\}, \tag{74}$$

$$T_2: \quad \mathbf{1}^L_{-5} + \mathbf{5}^L_3 + 12\{\mathbf{1}_0\}, \tag{75}$$

$$T_4: \quad \mathbf{5}^L_{-2} + 2(\mathbf{5}_{-2} + \bar{\mathbf{5}}_2)^L + 30\{\mathbf{1}_0\}, \tag{76}$$

from the twisted sectors. The chiral matter resulting from this spectra constitutes three families of quarks and leptons.

In addition, we obtain G-exotics and E-exotics from T_1 and T_5 sectors,

$$T_1: \quad 2(\bar{\mathbf{5}}_{-1/2})^L + 2(\mathbf{5}_{+1/2})^L + 7(\mathbf{1}_{-5/2})^L + 7(\mathbf{1}_{+5/2})^L, \tag{77}$$

$$T_5: \quad 2(\mathbf{5}_{+1/2})^R + 2(\bar{\mathbf{5}}_{-1/2})^R + 7(\mathbf{1}_{+5/2})^R + 7(\mathbf{1}_{-5/2})^R. \tag{78}$$

From (2), we note that G-exotics carry $Q_{\mathrm{em}} = \pm\frac{1}{6}$ quarks and $Q_{\mathrm{em}} = \pm\frac{1}{2}$ E-exotics and E-exotics with $X = \pm\frac{5}{2}$ have $Q_{\mathrm{em}} = \pm\frac{1}{2}$. All these exotics can be removed if all $U(1)$s except the

Table 9

Chiral matter fields satisfying $\Theta_{0,\pm} = 0$ in the T_5^0 and $T_5^\pm$ sectors. There are two $SU(2)'$ doublets with $(\frac{5}{6}, \frac{-1}{6}; \cdots)'$

$P + 5V$	$(N^L)_j$	$\mathcal{P}_5(f_0)$	χ
$(\mathbf{5}_{1/2}; \frac{7}{12}, 0, \frac{6}{12})(\frac{-1}{6}, \frac{-1}{6}; 0^6)'$	0	1	R
$(\mathbf{5}_{1/2}; \frac{-5}{12}, 0, \frac{-6}{12})(\frac{-1}{6}, \frac{-1}{6}; 0^6)'$	1_1	1	R
$(\bar{\mathbf{5}}_{-1/2}; \frac{1}{12}, \frac{6}{12}, 0)(\frac{-1}{6}, \frac{-1}{6}; 0^6)'$	2_1	1	R
$(\mathbf{1}_{5/2}; \frac{7}{12}, 0, \frac{-6}{12})(\frac{-1}{6}, \frac{-1}{6}; 0^6)'$	3_1	1	R
$(\mathbf{1}_{5/2}; \frac{-5}{12}, 0, \frac{6}{12})(\frac{-1}{6}, \frac{-1}{6}; 0^6)'$	$1_{\bar{2}}, 4_1$	$1 + 1$	R
$(\mathbf{1}_{-5/2}; \frac{-11}{12}, \frac{6}{12}, 0)(\frac{-1}{6}, \frac{-1}{6}; 0^6)'$	0	1	R
$(\mathbf{1}_{-5/2}; \frac{1}{12}, \frac{-6}{12}, 0)(\frac{-1}{6}, \frac{-1}{6}; 0^6)'$	$1_3, 5_1, \{1_1 + 1_{\bar{2}}\}$	$1 + 1 + 1$	R
$P + 5V_+$	$(N^L)_j$	$\mathcal{P}_5(f_+)$	χ
$(\mathbf{1}_{5/2}; \frac{-5}{12}, \frac{4}{12}, \frac{2}{12})(\frac{5}{6}, \frac{-1}{6}; \frac{1}{3}; 0^5)'$	0	1	R
$(\mathbf{1}_{5/2}; \frac{-5}{12}, \frac{4}{12}, \frac{2}{12})(\frac{-1}{6}, \frac{-1}{6}; \frac{-2}{3}; 0^5)'$	2_1	1	R
$(\mathbf{1}_{-5/2}; \frac{1}{12}, \frac{-2}{12}, \frac{-4}{12})(\frac{5}{6}, \frac{-1}{6}; \frac{1}{3}; 0^5)'$	1_1	1	R
$(\mathbf{1}_{-5/2}; \frac{1}{12}, \frac{-2}{12}, \frac{-4}{12})(\frac{-1}{6}, \frac{-1}{6}; \frac{-2}{3}; 0^5)'$	3_1	1	R
$P + 5V_-$	$(N^L)_j$	$\mathcal{P}_5(f_-)$	χ
$(\bar{\mathbf{5}}_{-1/2}; \frac{1}{12}, \frac{2}{12}, \frac{4}{12})(\frac{-1}{6}, \frac{-1}{6}; \frac{2}{3}; 0^5)'$	0	1	R
$(\mathbf{1}_{5/2}; \frac{-5}{12}, \frac{8}{12}, \frac{-2}{12})(\frac{-1}{6}, \frac{-1}{6}; \frac{2}{3}; 0^5)'$	0	1	R
$(\mathbf{1}_{5/2}; \frac{7}{12}, \frac{-4}{12}, \frac{-2}{12})(\frac{-1}{6}, \frac{-1}{6}; \frac{2}{3}; 0^5)'$	1_1	1	R
$(\mathbf{1}_{-5/2}; \frac{1}{12}, \frac{2}{12}, \frac{-8}{12})(\frac{-1}{6}, \frac{-1}{6}; \frac{2}{3}; 0^5)'$	1_1	1	R

$U(1)_Y$ are broken at the GUT scale and a sufficient number of neutral singlets develop VEVs, and hence there are not left with dangerous half-integer charged fields below the GUT scale. It will be explained in Section 6.1. The lightest of these half-integer charged fields (LHIC) is absolutely stable since all the light observable SM fields (including color singlet composites) are integer charged. If the mass of LHIC is much larger than the reheating temperature after inflation, we expect that most of LHIC are diluted away by inflation.

For the hidden sector, there appear 20 $SU(2)'$ doublets and one $\overline{\mathbf{16}}'$ and one $\mathbf{10}'$ of the hidden $SO(10)'$.

6. Yukawa couplings

Nonvanishing couplings of vertex operators are constructed by satisfying the $\mathbf{Z}_{12-I}$ symmetry [29,35,37]. It is summarized in [7]. In our notation for the shift vector, basically it amounts for the operator $\mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \cdots$ to satisfy

Invariance under the group space shift V,   and   (79)

Invariance under the internal space shift ϕ_s.   (80)

For the shift V, it is easy to check the modular invariance: The relevant vertex operators have only to satisfy the gauge invariance. The invariance under the shift ϕ_s belongs to a generalized Lorentz shift and the condition is sometimes called the H-momentum conservation. The (bosonic) H-

momentum is defined as

$$R_i = (\tilde{s} + k\phi_s + \tilde{r})_i - (N_i^L - N_{\tilde{i}}^L), \quad i = \{1, 2, 3\}, \tag{81}$$

where $\tilde{r}$ is $(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2})$ for left-handed states ($\equiv \tilde{r}_-$) and $(\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$ for right-handed states ($\equiv \tilde{r}_+$). As discussed earlier, $\tilde{s}$ should satisfy the mass-shell condition, $|(\frac{1}{2}\chi, \tilde{s}) + k\phi_s|^2 = 2c_k$. R_i can be interpreted as a discrete R-charge. Thus, neglecting oscillator numbers, the H-momenta for $\mathbf{Z}_{12-I}$ twist are

$$
\begin{aligned}
&U_1: \quad (-1\ 0\ 0), \qquad U_2: \quad (0\ 1\ 0), \qquad U_3: \quad (0\ 0\ 1), \\
&T_1: \quad \left(\tfrac{-7}{12}\ \tfrac{4}{12}\ \tfrac{1}{12}\right), \qquad T_2: \quad \left(\tfrac{-1}{6}\ \tfrac{4}{6}\ \tfrac{1}{6}\right), \qquad T_3: \quad \left(\tfrac{-3}{4}\ 0\ \tfrac{1}{4}\right), \\
&T_4: \quad \left(\tfrac{-1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\right), \qquad T_5: \quad \left(\tfrac{1}{12}\ \tfrac{-4}{12}\ \tfrac{-7}{12}\right), \qquad T_6: \quad \left(\tfrac{-1}{2}\ 0\ \tfrac{1}{2}\right),
\end{aligned}
\tag{82}
$$

which are used to check the generalized Lorentz invariance.

As an example, consider the T_6 H-momentum, $(\frac{-1}{2}\ 0\ \frac{1}{2})$. It is derived in the following way. The right mover mass-shell condition is

$$\frac{1}{2} M_R^2 = (s + 6\phi_s)^2 = 2c = \frac{1}{2} \tag{83}$$

where $6\phi_s = (\frac{30}{12}\ \frac{24}{12}\ \frac{6}{12})$. There are two solutions of Eq. (83),

$$s_+ = \left(\oplus; \frac{-5}{2}, \frac{-3}{2}, \frac{-1}{2}\right) \quad \text{and} \quad s_- = \left(\ominus; \frac{-5}{2}\ \frac{-5}{2}\ \frac{-1}{2}\right).$$

For the left-handed states appearing in Table 4, let us focus on s_- whose corresponding sum-to-odd-integer solution is $(-3\ -2\ 0)$. So bosonization of $\tilde{s}_-$ is $\tilde{s}_- + \tilde{r}_-$. Thus, H-momentum, in analogy with $P + kV$, is $(\tilde{s}_- + \tilde{r}_-) + k\phi_s = (\frac{-1}{2}\ 0\ \frac{1}{2})$ appearing in Eq. (82).

The H-momenta conservation conditions with $\phi_s = (\frac{5}{12}, \frac{4}{12}, \frac{1}{12})$ can be reformulated only in terms of the bosonic H-momenta as follows:

$$\sum_z R_1(z) = -1 \mod 12, \qquad \sum_z R_2(z) = 1 \mod 3, \qquad \sum_z R_3(z) = 1 \mod 12, \tag{84}$$

where $z\ (\equiv A, B, C, \ldots)$ denotes the index of states participating in a vertex operator. In addition, space group selection rules requires a vertex operator with z-states in $T_k^{m_f}$ sector ($k = 0$ for the untwisted sector) should satisfy

$$\sum_z k(z) = 0 \mod 12, \tag{85}$$

$$\sum_z [km_f](z) = 0 \mod 3. \tag{86}$$

One can easily check the following cubic couplings are allowed in $\mathbf{Z}_{12-I}$ orbifold models fulfilling Eqs. (84) and (85), if $N_i^L = N_{\tilde{i}}^L = 0$ [35,37]:

$$U_1 U_2 U_3, \qquad T_6 T_6 U_2, \qquad T_4 T_4 T_4, \qquad T_2 T_4 T_6, \qquad T_1 T_4 T_7. \tag{87}$$

Note that in considering the superpotential couplings, one should consider only the same chirality, and in our model there is no massless states from the T_3 and T_9 sectors.

Table 10
H-momenta for some combinations of neutral singlets under $SU(5) \times U(1)_X$ appearing in our model. All the combinations are neutral under all gauge symmetries in this model, and fulfill the space group selection rules

Comb. of singlets	H-momenta	Comb. of singlets	H-momenta
$h_1 \bar{h}_1$	$(-1\,0\,-1)$	$h_1 \bar{h}_2$	$(-1\,0\,1)$
$h_2 \bar{h}_2$	$(-1\,0\,3)$	$h_1 \bar{h}_4$	$(-2\,0\,0)$
$h_3 \bar{h}_3$	$(1\,0\,1)$	$h_2 \bar{h}_3$	$(0\,0\,2)$
$h_4 \bar{h}_4$	$(-3\,0\,1)$	$h_2 \bar{h}_4$	$(-2\,0\,2)$
Comb. of doublets	H-momenta	Comb. of doublets	H-momenta
$D_1^+ d_2^+ \bar{h}_1$	$(-1\,3\,0)$	$D_1^+ d_2^+ \bar{h}_3$	$(0\,3\,1)$
$D_1^+ d_2^+ \bar{h}_2$	$(-1\,3\,2)$	$D_1^+ d_2^+ \bar{h}_4$	$(-2\,3\,1)$
Comb. of singlets	H-momenta	Comb. of singlets	H-momenta
$C_1^+ s_2^+ \bar{h}_1$	$(1\,1\,0)$	$C_1^- s_2^- h_1$	$(-1\,3\,0)$
$C_1^+ s_2^+ \bar{h}_2$	$(1\,1\,2)$	$C_1^- s_2^- h_2$	$(-1\,3\,2)$
$C_1^+ s_2^+ \bar{h}_3$	$(2\,1\,1)$	$C_1^- s_2^- h_3$	$(0\,3\,1)$
$C_1^+ s_2^+ \bar{h}_4$	$(0\,1\,1)$	$C_1^- s_2^- h_4$	$(-2\,3\,1)$
$C_2^+ s_2^+ \bar{h}_1$	$(-1\,1\,-2)$	$C_2^- s_2^- h_1$	$(0\,1\,-1)$
$C_2^+ s_2^+ \bar{h}_2$	$(-1\,1\,0)$	$C_2^- s_2^- h_2$	$(0\,1\,1)$
$C_2^+ s_2^+ \bar{h}_3$	$(0\,1\,-1)$	$C_2^- s_2^- h_3$	$(1\,1\,0)$
$C_2^+ s_2^+ \bar{h}_4$	$(-2\,1\,-1)$	$C_2^- s_2^- h_4$	$(-1\,1\,0)$

For future convenience, we display the H-momenta for combinations of some singlets appearing in our model in Table 10.

In Ref. [30], it has been shown that one can always find a vacuum where only neutral singlets under a symmetry of interest develop VEVs of string scale preserving the F and D flatness conditions. This is possible because (a) once a superpotential term W is allowed by the selection rules, then W^{N+1} is also allowed in the $\mathbf{Z}_N$ orbifold model, and (b) there exists in general a transformation rescaling the VEVs leaving intact the above F flatness conditions but making accordingly D flatness conditions satisfied by adjusting VEVs. In our model, there are enough superpotential terms constructed with only neutral singlets, e.g. $W = W_S + W_S^{13} + W_S^{25} + \cdots$, where

$$
\begin{aligned}
W_S = {} & \left(C_1^+ s_2^+ \bar{h}_1\right)\left(C_1^- s_2^- h_4\right) + \left(C_1^+ s_2^+ \bar{h}_4\right)\left(C_1^- s_2^- h_1\right) \\
& + \left(C_1^- s_2^- h_2\right)\left(C_2^- s_2^- h_1\right) + \left(C_1^- s_2^- h_1\right)\left(C_2^- s_2^- h_2\right) + \cdots
\end{aligned}
$$

where the string scale is set $M_{\text{string}} = 1$. We assume a vacuum where only neutral singlets develop large string scale VEVs.

6.1. Flipped $SU(5)$ spectrum

There exist $(\mathbf{10_1}, \overline{\mathbf{10}}_{-1})$ whose VEV (in the $SU(2)$ singlet direction ν^c) breaks the flipped $SU(5)$ down to the standard model. Also, there exist the needed electroweak Higgs fields $(\bar{\mathbf{5}}_2, \mathbf{5}_{-2})$.

In T_1 and T_5, there appear G-exotics and E-exotics. These are removed by $T_1 T_4 T_7$ couplings of Eq. (87) via $\langle \mathbf{1_0}(T_4) \rangle$ and other singlet's VEVs. For instance, $(\mathbf{1_{5/2}}; \frac{-1}{12}, 0, \frac{-6}{12})(\frac{1}{6}\,\frac{1}{6};\,0^6)'$ with $N_j^L = 1_1$ in T_1^0, and $(\mathbf{1_{-5/2}}; \frac{-7}{12}, \frac{4}{12}, \frac{2}{12})(\frac{-1}{6}\,\frac{-1}{6};\,\frac{2}{3};\,0^5)'$ with $N_j^L = 1_{\bar{1}}$ in T_7^- get a mass from

the coupling with $\langle s_1^+ \rangle$ in T_4^+. Similarly $(\mathbf{1}_{-5/2}; \frac{-7}{12}, \frac{-2}{12}, \frac{-4}{12})(\frac{1}{6} \frac{1}{6}; \frac{-2}{3}; 0^5)'$ with $N_j^L = 1_3$ in T_1^-, and $(\mathbf{1}_{5/2}; \frac{-1}{12}, \frac{6}{12}, 0)(\frac{1}{6} \frac{1}{6}; 0^6)'$ with $N_j^L = 1_{\bar{3}}$ in T_7^0 achieve a mass also from s_1^+. It turns out that the contributions of oscillator numbers carried by the other states to the H-momenta can be always cancelled by (multi-) products of $C_1^+ s_3^+ h_3$, $C_2^+ s_2^+ \bar{h}_3$, $C_1^- s_2^- h_3$, $C_1^- s_2^- \bar{h}_1$, etc. When $N_j^L = \{1_2 + 1_3\}$ is involved in a $T_1 T_7 T_4$ vertex operator, for instance, one could additionally multiply $C_1^+ s_2^+ \bar{h}_4$ to the vertex operator in order to cancel the oscillator number contribution. When $N_j^L = 4_3$ is involved, e.g. $(C_1^- s_2^- h_3)^4$ needs to be multiplied.

In T_6 and T_4, there appear vector-like representations $(\mathbf{10}_1 + \overline{\mathbf{10}}_{-1})$s, $(\mathbf{1}_5 + \mathbf{1}_{-5})$s, and $(\mathbf{5}_{-2} + \bar{\mathbf{5}}_2)$s. These are removed by the survival hypothesis that all vector-like representations are removed at the GUT scale [36]. Indeed, it is so in our model by allowing large VEVs to neutral singlets. Thus, we obtain just only one $\mathbf{5}_{-2}^L$ from T_4 and $\bar{\mathbf{5}}_2^L$ from the untwisted sector. One can anticipate that this may happen if the twisted sectors provide large Yukawa couplings. Indeed, such couplings are present.

The fields $2(\mathbf{5}_{-2} + \bar{\mathbf{5}}_2)^L$ of T_4 can be removed by $T_4 T_4 T_4$ couplings where one T_4 is $\langle s_4^0 \rangle$ in Table 8. The vector-like representations in T_6 are also removed by the couplings, e.g. with $(C_1^+ s_2^+ \bar{h}_1)(C_1^- s_2^- h_1)$ and $(C_2^+ s_2^+ \bar{h}_3)(C_1^- s_2^- h_3)$, etc. Thus, $\{2(\mathbf{1}_5 + \mathbf{1}_{-5} + \bar{\mathbf{5}}_{-3} + \mathbf{5}_3) + 3(\mathbf{10}_1 + \overline{\mathbf{10}}_{-1})\}^L$ and $2(\mathbf{5}_{-2} + \bar{\mathbf{5}}_2)^L$ are expected to be removed at the GUT scale. In all these, several singlets with $Q_X = 0$ are expected to develop GUT scale VEVs. Then, there result the following light fields,

$$U: \quad (\mathbf{1}_5 + \bar{\mathbf{5}}_{-3} + \mathbf{10}_1)_{U_1}^R + (\mathbf{1}_5 + \bar{\mathbf{5}}_{-3} + \mathbf{10}_1)_{U_3}^R + (\mathbf{5}_{-2})_{U_2}^R, \tag{88}$$

$$T_6: \quad \mathbf{10}_1^R, \tag{89}$$

$$T_2^0: \quad \mathbf{1}_5^R + \bar{\mathbf{5}}_{-3}^R, \tag{90}$$

$$T_4: \quad \bar{\mathbf{5}}_2^R. \tag{91}$$

These constitute the three families and one pair of Higgs quintets. Relabeling R to L, we can obtain the standard form of left-handed $W^\pm$ interactions. In the remainder, however, we will keep R to compare with the entries of tables.

It is interesting to note that the pair of Higgs quintets, $(\mathbf{5}_{-2} + \bar{\mathbf{5}}_2)$, survives the above analysis. Certainly, it is not allowed to write $M_{\text{GUT}} \mathbf{5}_{-2} \bar{\mathbf{5}}_2$ since there is no coupling of the form $U_2 T_4$. In fact, it is not easy to construct a $\mathbf{5}_{-2} \bar{\mathbf{5}}_2$ consistent with the H-momentum conservation. Thus Higgs doublet mass can be far below the GUT scale, which is a good thing. But the colored scalars in $(\mathbf{5}_{-2} + \bar{\mathbf{5}}_2)$ must be removed at high energy scale toward MSSM. It is achieved by the doublet–triplet splitting mechanism we discuss below.

6.2. GUT breaking

The GUT breaking in the flipped $SU(5)$ model proceeds by $\langle \mathbf{10}_1 \rangle = \langle \overline{\mathbf{10}}_{-1} \rangle = M_{\text{GUT}}$, which is a D-flat direction. This vector-like representation, $\mathbf{10}_1 + \overline{\mathbf{10}}_{-1}$, is present in the T_6 sector. Note that the H-momentum of $(\mathbf{10}_1 \overline{\mathbf{10}}_{-1})^L$ is $(-1, 0, 1)$.

The T_6 sector also contains twenty two $Q_{\text{em}} = 0$ singlets. Some combinations of them, e.g. $h_1 \bar{h}_2$ in Table 10 also give the H-momentum of $(-1, 0, 1)$. $h_3 \bar{h}_4$, $h_2 \bar{h}_1$, $h_4 \bar{h}_3$ also provide the same H-momentum. The GUT scale VEV of them could induce $\langle \mathbf{10}_1 \rangle = \langle \overline{\mathbf{10}}_{-1} \rangle = M_{\text{GUT}}$ along an F-flat direction, for instance, through a non-renormalizable superpotential,

$$W = (D_1^+ d_2^+ \bar{h}_1)(C_1^+ s_2^+ \bar{h}_1)[\mathbf{10}_1^H \overline{\mathbf{10}}_{-1}^H - h_1 \bar{h}_2 - h_3 \bar{h}_4 + \cdots] \tag{92}$$

with $\langle D_1^+ \rangle$ or $\langle d_2^+ \rangle$ vanishing. Note that there are 20 $SU(2)'$ doublets. Thus, the gauge coupling of $SU(2)'$ would not blow up at lower energies. The neutrino direction of $\langle \mathbf{10_1} \rangle$, $\langle \overline{\mathbf{10}}_{-1} \rangle$ allows the symmetry breaking

$$SU(5) \times U(1)_X \to SU(3)_c \times SU(2) \times U(1)_Y. \tag{93}$$

6.3. Doublet–triplet splitting

Let us call the three vector-like ten and tenbars of Eq. (74) as Higgs fields, $\mathbf{10}_1^H$ and $\overline{\mathbf{10}}_{-1}^H$. Among the three two are purely vector-like and removed at the GUT scale. One remaining vector-like pair joins the Higgs mechanism. Let us consider this remaining pair.

The Higgs $\mathbf{10}_1^H$ and $\overline{\mathbf{10}}_{-1}^H$ contain $\{q, d^c, \nu^c\} + \{\bar{q}, \bar{d}^c, \bar{\nu}^c\}$ in terms of the standard model quantum numbers. $\{\nu^c, \bar{\nu}^c\}$ obtain GUT scale masses from Eq. (92) when $SU(5) \times U(1)_X$ is broken. $\{q, \bar{q}\}$ contained in $\mathbf{10}_1^H$ and $\overline{\mathbf{10}}_{-1}^H$ are absorbed by the heavy gauge sector. On the other hand, $\{d^c, \bar{d}^c\}$ still remain light. In order to make the standard model vacuum stabilized, somehow they should get superheavy masses.

Let us consider $W = \mathbf{10}_1^H \mathbf{10}_1^H \mathbf{5}_h + \overline{\mathbf{10}}_{-1}^H \overline{\mathbf{10}}_{-1}^H \bar{\mathbf{5}}_h$, where $\mathbf{5}_h$ and $\bar{\mathbf{5}}_h$ indicate the five-plets inducing electroweak symmetry breaking [38,39]. When $\mathbf{10}_1^H$ and $\overline{\mathbf{10}}_{-1}^H$ develop VEVs in the $\langle \nu^c \rangle = \langle \bar{\nu}^c \rangle = M_{\rm GUT}$ direction, $\{d^c, \bar{d}^c\}$ in $\mathbf{10}_1^H$, $\overline{\mathbf{10}}_{-1}^H$ and triplets $(D, \bar{D})$ included in $\mathbf{5}_h$, $\bar{\mathbf{5}}_h$ pair up to be superheavy [40]. This achieves the doublet–triplet splitting in flipped $SU(5)$.

This mechanism can be realized also in our model. The $[\overline{\mathbf{10}}_{-1}^H \overline{\mathbf{10}}_{-1}^H \bar{\mathbf{5}}_h]^L$ term arises from $T_6 T_6 U_2$, which satisfies the H-momentum conservation. The $[\mathbf{10}_1^H \mathbf{10}_1^H \mathbf{5}_h]^L$ is still also possible from highly non-renormalizable interactions, e.g. $[\mathbf{10}_1^H \mathbf{10}_1^H \mathbf{5}_h]^L (s_1^+ s_1^- s^u)(C_2^+ s_2^+ \bar{h}_3)(C_2^- s_2^- h_3)$, which satisfies all selection rules discussed above. Thus we should assume cutoff scale VEVs for the neutral singlets.

Thus, we obtain the so-called MSSM spectra with one pair of Higgs doublets at cubic level. However, the Higgs doublets would obtain mass since there exist a lot of singlets. Indeed, there exist many couplings. The simplest two terms for μ are

$$\left[(\bar{\mathbf{5}}_2; -1, 0, 0)(0^8)'\right]_{U_2} \left[\left(\mathbf{5}_{-2}; \frac{-1}{3}, 0, 0\right)\left(\frac{-1}{3}, \frac{-1}{3}; 0^6\right)'\right]_{T_4^0} \left[s^u (s_2^0)^2 + f_\mu s_2^0 s_4^0\right]_{[T_4^0]^2} \tag{94}$$

where f_μ is a relative strength. Before, s_4^0 was needed for charged lepton masses. On the other hand, s_2^0 was needed for mixing of charged leptons. Let us set s_2^0 a free parameter. Assuming that $\bar{\mathbf{5}}_2$ and $\mathbf{5}_{-2}$ obtain VEVs, the F-flat direction chooses $s_2^0 = -f_\mu s_4^0 / 2s^u$ so that the μ term is of order $-f_\mu^2 s_4^{02}/4s^u$. This can be linked to the charged lepton masses. For example, if we choose $s_4^0 \sim 10^{-5}$ from electron mass and $s^u \sim O(1)$, we require $f_\mu = O(10^{-2})$ to obtain a TeV scale μ term. This fine tuning is a huge improvement over a fine tuning of 10^{-15}. Since we neglected many higher order terms, this is just an illustration of smoothing the fine tuning problem. The s_2^0 mimics the axion multiplet of the μ solution [41].

6.4. Fermion masses

• There is one cubic coupling relevant for u-type quarks and Dirac neutrinos, $T_6 T_2 T_4$, which is interpreted as the top quark and (Dirac) tau neutrino Yukawa coupling,

$$t \text{ quark} + \text{(Dirac) } \tau \text{ neutrino coupling:} \quad [\mathbf{10}_1 \bar{\mathbf{5}}_{-3} \bar{\mathbf{5}}_2]^R. \tag{95}$$

Non-renormalizable terms allow the other u-type quark (Dirac neutrino) Yukawa couplings. Most of all, through the couplings with C_5^0, C_6^0 in T_2^0 and h_1, $\bar{h}_3$ in T_6, $[\overline{\mathbf{10}}_{-\mathbf{1}}(T_6)\mathbf{5}_{\mathbf{3}}(U_1)\mathbf{5}_{-\mathbf{2}}(T_4^0)]^L$, $[\overline{\mathbf{10}}_{-\mathbf{1}}(T_6)\mathbf{5}_{\mathbf{3}}(U_3)\mathbf{5}_{-\mathbf{2}}(T_4^0)]^L$, $[\overline{\mathbf{10}}_{-\mathbf{1}}(U_3)\mathbf{5}_{\mathbf{3}}(T_2^0)\mathbf{5}_{-\mathbf{2}}(T_4^0)]^L$, and $[\overline{\mathbf{10}}_{-\mathbf{1}}(U_1)\mathbf{5}_{\mathbf{3}}(T_2^0)\mathbf{5}_{-\mathbf{2}}(T_4^0)]^L$ are allowed. With $h_1 C_5^0$, $\bar{h}_3 C_6^0$, $\bar{h}_3 C_5^0$, and $h_1 C_6^0$, the masses and mixing for the first two families of u-type quarks and Dirac neutrinos are possible.

- The bottom quark mass arises in terms of U_2 Higgs doublet. Indeed, such a coupling is present from $T_6 T_6 U_2$. If the coupling strength of $T_6 T_2 T_4$ and $T_6 T_6 U_2$ are comparable, a large $\tan\beta$ is needed to obtain $m_t/m_b \sim 35$. But the couplings depend on the location of the respective fields and one may treat the ratio as a free parameter of order 1 [42].

In fact $[\mathbf{10_1}(U_1)\mathbf{10_1}(U_3)\mathbf{5}_{-\mathbf{2}}(U_2)]^R$ exists. For smallness of it we could assume a proper volume of the 6-dimensional compact space. As an alternative way, one could consider $(1, 1)$ and $(2, 2)$ components of the d-quark mass matrix. They can be induced with $\langle h_1 h_1 \rangle$ and $\langle \bar{h}_3 \bar{h}_3 \rangle$. By $\langle h_1 \rangle$ and $\langle \bar{h}_3 \rangle$, the mixing between the first two and the third families of d-type quarks are also permitted.

- For the mass of charged lepton in the twisted sector T_2^0, we need a coupling containing $[\mathbf{1}_{-\mathbf{5}}(T_2^0)\mathbf{5}_{\mathbf{3}}(T_2^0)\bar{\mathbf{5}}_{\mathbf{2}}(U_2)]^L$. This coupling is possible by being supplemented with $(s_2^0)^2$. On the other hand, cubic couplings $[\mathbf{1}_{\mathbf{5}}(U_1)\bar{\mathbf{5}}_{-\mathbf{3}}(U_3)\mathbf{5}_{-\mathbf{2}}(U_2)]^R$ $[\mathbf{1}_{\mathbf{5}}(U_3)\bar{\mathbf{5}}_{-\mathbf{3}}(U_1)\mathbf{5}_{-\mathbf{2}}(U_2)]^R$ are possible. It may be that τ is placed in the untwisted sector. In this sense, the leptonic sector does not go parallel to the quark sector. The $b - \tau$ unification is not achieved in this model. $(1, 1)$ and $(2, 2)$ components of the charged lepton mass matrix are generated via, e.g. $\langle \bar{h}_1 \bar{h}_1 \rangle$ and $\langle h_3 h_3 \rangle$. The mixing terms between the first two and the third families of charged leptons should be mediated by $\langle h_3 s_2^0 \rangle$ and $\langle \bar{h}_1 s_2^0 \rangle$.

- In flipped $SU(5)$, Majorana neutrino masses arise from $[\langle \mathbf{10_1} \rangle \cdot \langle \mathbf{10_1} \rangle \cdot \overline{\mathbf{10}}_{-\mathbf{1}} \cdot \overline{\mathbf{10}}_{-\mathbf{1}}]^L$. The H-momentum of this operator is $(-2, 0, 2)$. Thus this operator can be induced when supplemented, e.g. by the coupling with $(C_1^+ s_2^+ \bar{h}_3)(C_2^+ s_2^+ \bar{h}_1)[(C_1^- s_2^- h_1)(C_2^- s_2^- h_3)]^2$. The other components of the Majorana neutrino mass matrix should require additional singlet VEVs such as h_1, $\bar{h}_3$, $(h_1)^2$, and $(\bar{h}_3)^2$. If heavy Majorana neutrino masses are around 10^{14} GeV, thus we obtain the ν_τ mass of order 0.1 eV,

$$m_{\nu_\tau} \simeq \frac{m_{\text{top}}^2}{(10^{14}\ \text{GeV})} \sim 0.1\ \text{eV} \tag{96}$$

for $m_{\text{top}} = m_{\text{Dirac neutrino}}$, which is valid in flipped $SU(5)$.

6.5. R-parity

If we consider cubic couplings Eq. (87), we can define an R-parity in the standard way, $R = -1$ for matter fermions and $R = +1$ for Higgs bosons.

Firstly, consider the coupling $\mathbf{10_1}(U_1)\mathbf{10_1}(U_3)\mathbf{5}_{-\mathbf{2}}(U_2)$. A nontrivial parity can be defined as

$$R = -1 \quad \text{for } \mathbf{10_1}(U_1),\ \mathbf{10_1}(U_3),\ \bar{\mathbf{5}}_{-\mathbf{3}}(U_1),\ \bar{\mathbf{5}}_{-\mathbf{3}}(U_3),\ \mathbf{1_5}(U_1),\ \mathbf{1_5}(U_3), \tag{97}$$

$$R = +1 \quad \text{for } \mathbf{5}_{-\mathbf{2}}(U_2). \tag{98}$$

As discussed above, mixing between the first two families of matter in the untwisted sector and the third family of matter in T_5 and T_2^0 are always possible if VEVs of some neutral singlets are supposed. Such neutral singlets should also be inert under all symmetries relevant at low energies. Otherwise, symmetries must be broken at low energies by their VEVs. Even with the mixing terms between untwisted and twisted matter fields, the R-parity relevant in low energies

can still be defined by assigning $R = 1$ for the neutral singlets developing VEVs and

$$R = -1 \quad \text{for } \overline{\mathbf{10}}^L_{-1}(T_6),\ \mathbf{5}^L_3(T_2),\ \mathbf{1}^L_{-5}(T_2). \tag{99}$$

Then, the allowed Yukawa coupling $T_6 T_2 T_4$ determines

$$R = +1 \quad \text{for } \mathbf{5}_{-2}(T_4). \tag{100}$$

Thus, R-parity can survive down to low energies and hence R-parity conservation for proton longevity is fulfilled in the present model.

7. Electroweak hypercharge in flipped $SU(5)$

7.1. GUT value of weak mixing angle

Flipped $SU(5)$ was originally considered as a subgroup of $SO(10)$ [9], which can be called $SO(10)$-flipped $SU(5)$. In this case, the GUT value for $\sin^2 \theta_W$ should be $\frac{3}{8}$. In this case, symmetry breaking must proceed via adjoint Higgs field. The shift vector must take a form

$$V = \begin{cases} (0\ 0\ 0\ 0\ 0\ x\ y\ z)(\cdots)', & \text{or} \\ (\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ x\ y\ z)(\cdots)' \end{cases} \tag{101}$$

so that an $SO(10)$ group is obtained.

String flipped $SU(5)$, for example Eq. (28) with five $\frac{1}{4}$s, is basically different from $SO(10)$ flipped $SU(5)$ even though it includes $SO(10)$ flipped $SU(5)$ if the shift vector takes the form Eq. (101). If the electroweak $SU(2) \times U(1)_Y$ is embedded in a simple group in a field theory GUT, the GUT value of $\sin^2 \theta_W$ is $\frac{3}{8}$ as calculated from

$$\sin^2 \theta_W = \frac{\text{Tr}\, T_3^2}{\text{Tr}\, Q_{\text{em}}^2}. \tag{102}$$

Many possibilities of $\sin^2 \theta_W$ appear because of many possibilities of embedding the electroweak hypercharge Y in an Abelian group. In the $E_8 \times E'_8$ heterotic string, the embedding of Y in $U(1)$s is basically looked from the untwisted sector spectrum, which sets the embedding in E_8. Thus, in string compactification, $\sin^2 \theta_W$ depends on the spectrum in the untwisted sector [7]. Therefore, if the untwisted sector spectrum includes $\mathbf{16}_{\text{flip}}$, then $\sin^2 \theta_W$ is the same as that of $SO(10)$. On the other hand, if $\mathbf{16}_{\text{flip}}$ cannot be obtained from the fields in the U sector, $\sin^2 \theta_W = \frac{3}{8}$ is not warranted. In the present model, a complete multiplet $\mathbf{16}_{\text{flip}}$ appears in the U sector, we obtain $\sin^2 \theta_W = \frac{3}{8}$. This is explicitly shown below.

The electroweak hypercharge Y is a combination of $SU(5)$ and $U(1)_X$ generators, $Y = \frac{1}{5}(X + Y_5)$. As will be shown in the next section, the $U(1)_Y$ gauge coupling is

$$g_Y^{-2} = \frac{g_5^{-2}}{15} + \frac{g_X^{-2}}{25u^2}, \tag{103}$$

where u denotes an employed unit of $U(1)_X$ charges. So far we tacitly supposed $u = 1$, but its absolute value should be determined by the string theory with $g_5 = g_X$ at the compactification scale. Along the standard model direction, Eq. (103) is still valid above the flipped $SU(5)$ breaking scale. Thus at the compactification scale, where $SU(2)$ gauge coupling g_2 is identified

with g_5, the Weinberg angle is given by

$$\sin^2 \theta_W^0 = \frac{1}{1 + \frac{g_5^2}{g_Y^2}} = \frac{1}{1 + (\frac{1}{15} + \frac{1}{25u^2})}. \tag{104}$$

Including the unit factor 'u', the $U(1)_X$ charge operator, $Q_X^A \equiv -2(1, 1, 1, 1, 1, 0, 0, 0) \times (0^8)' \times u$ can be expressed in terms of a proper orthonormal basis vector q_5 [43],

$$Q_X^A = \frac{1}{\sqrt{2}} c_5 q_5, \tag{105}$$

where

$$c_5 \equiv -2\sqrt{10}u \quad \text{and} \quad q_5 \equiv \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1, 0, 0, 0)(0^8). \tag{106}$$

With this expression, g_5^2/g_X^2 can be calculated in the string theory framework [7],

$$\frac{g_5^2}{g_X^2} = c_5^2 = 40u^2. \tag{107}$$

Hence, the string theory determines the absolute values of $U(1)_X$ charges with $u^2 = \frac{1}{40}$ from $g_5 = g_X$ at the compactification scale. With this unit, the bare value of the Weinberg angle is also determined from Eq. (104):

$$\sin^2 \theta_W^0 = \frac{3}{8}. \tag{108}$$

7.2. Embedding of Y in flipped SU(5)

A covariant derivative in flipped $SU(5)$ includes the term

$$\sqrt{\frac{3}{5}} Y_5 g_5 A_5 + X u g_X A_x, \tag{109}$$

where $Y_5 = \text{diag}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{-1}{2}, \frac{-1}{2})$ and X $[= x \cdot \text{diag}(1, 1, 1, 1, 1)]$ denotes the $U(1)_X$ charge operator employed in this paper, u a unit of $U(1)_X$ charge. So far u is tacitly assumed to be unity, but generically it is not necessarily 1. $\sqrt{\frac{3}{5}} Y_5$ is one of $SU(5)$ generator normalized with $\text{Tr} \, T_{SU(5)} T_{SU(5)} = \frac{1}{2}$. A_5 and A_x stand for the $SU(5)$ and $U(1)_X$ gauge fields, respectively. After symmetry breaking, one linear combination of A_5 and A_x becomes the $U(1)_Y$ gauge field B_μ of the standard model. We introduce a mixing angle ϕ,

$$\begin{pmatrix} A_5 \\ A_x \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} B_\mu \\ C_\mu \end{pmatrix}, \tag{110}$$

Eq. (109) is recast in the following form,

$$\left[\frac{1}{5} Y_5 + \frac{1}{5} X \sqrt{\frac{5}{3}} \frac{g_X}{g_5} \tan\phi u \right] \sqrt{15} \cos\phi \, g_5 B_\mu + [C_\mu \text{ terms}], \tag{111}$$

where C_μ achieves a superheavy mass from the 10-dimensional Higgs VEVs $\langle v_H^c \rangle$, $\langle \bar{v}_H^c \rangle$. From this expression, one can read off the $U(1)_Y$ charges and coupling,

$$Y = \frac{1}{5}Y_5 + \frac{1}{5}X\sqrt{\frac{5}{3}}\frac{g_X}{g_5}\tan\phi u, \tag{112}$$

$$\frac{g_Y}{g_5} = \sqrt{15}\cos\phi. \tag{113}$$

Since $Y(v^c) = 0$ is the SM direction, Eq. (112) should be

$$Y = \frac{1}{5}Y_5 + \frac{1}{5}X, \quad \text{or} \tag{114}$$

$$\frac{g_5}{g_X} = \sqrt{\frac{5}{3}}\tan\phi u. \tag{115}$$

With Eqs. (113) and (115), the relation between g_5, g_X, and g_Y is derived:

$$g_Y^{-2} = \frac{g_5^{-2}}{15} + \frac{g_X^{-2}}{25u^2}. \tag{116}$$

8. Conclusion

We constructed a supersymmetric flipped $SU(5)$ model from a $\mathbf{Z}_{12-I}$ orbifold compactification. The notable features of the model are:

- From E_8, the only non-Abelian group is the needed $SU(5)$. This is possible only for one shift vector in $\mathbf{Z}_{12-I}$. In this sense, it is the unique $\mathbf{Z}_{12-I}$ flipped $SU(5)$ model.
- Three families are obtained. The third family is located in twisted sectors while the first two families are in the untwisted sector. Separating the third family from the first two families enables a mass hierarchy of fermions.
- There exists a doublet–triplet splitting from a kind of the missing partner mechanism.
- There results only one pair of Higgs doublets.
- The R-parity is present.
- Allowed Yukawa couplings can generate a GUT scale VEVs of $\mathbf{10}_1^H$ and $\overline{\mathbf{10}}_{-1}^H$ for the GUT breaking down to the standard model.
- There exist $Q_{\rm em} = \pm\frac{1}{2}$ particles. But these form vector-like representations and most are removed at the GUT scale. Thus, $\sin^2\theta_W$ is similar to that of $SO(10)$ GUT.

In this paper, all the relevant Yukawa couplings are derived from string construction. So far, we have not encountered any serious phenomenological problem. Successful Yukawa couplings may be generated by appropriate GUT scale VEVs of singlet fields which are treated here as free parameters. Finally, it is expected that a standard model can be derived from $\mathbf{Z}_{12}$ without going through the intermediate stage of the flipped $SU(5)$ with features discovered in the present model [44]. In a future communication, we will tabulate all computer-searched $\mathbf{Z}_{12}$ orbifold models.

Appendix A. Flipped $SU(5)$ with $SU(4)'$

One can eliminate some exotic particles carrying $Q_X = \pm\frac{1}{2}, \pm\frac{5}{2}$ observed in the model discussed in the main text (Model I) by employing more complicated shift vector and Wilson line,

$$V = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}; \frac{5}{12}, \frac{6}{12}, 0\right)\left(\frac{2}{12}, \frac{2}{12}; 1, 0; 0^4\right)', \tag{A.1}$$

$$a_3 = \left(0, 0, 0, 0, 0; 0, \frac{-1}{3}, \frac{1}{3}\right)\left(0, 0; 0, 0; \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3}\right)', \tag{A.2}$$

which satisfy the modular invariance conditions. Model II, constructed with these shift vector and Wilson line, eliminates in particular all $\mathbf{5}_{1/2}$ and $\bar{\mathbf{5}}_{-1/2}$ from the massless spectrum, leaving intact the MSSM fields obtained in Model I. Here, the gauge group is further broken down to

$$\left[\{SU(5) \times U(1)_X\} \times U(1)^3\right] \times \left[SU(4) \times SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times U(1)^2\right]'. \tag{A.3}$$

Model II gives the same spectrum as Model I for the T_6, T_2^0, and T_4^0 sectors and visible sector of U in Model I. As in Model I, there is no massless states satisfying $(P + 3V) \cdot a_3 = 0 \mod Z$ in the T_3 sector. The spectrum of Model II is summarized as follows:

- Fields of flipped $SU(5)$: $3 \times \mathbf{16}_{\text{flip}} + 1 \times \{\mathbf{5}_{-2}, \bar{\mathbf{5}}_2,\} + 1 \times \{\mathbf{10}_1, \overline{\mathbf{10}}_{-1}\}$, where $\mathbf{16}_{\text{flip}} \equiv \{\mathbf{10}_1, \bar{\mathbf{3}}_{-3}, \mathbf{1}_5\}$.
- (Regularly charged) vector-like fields: $2 \times \{\mathbf{16}_{\text{flip}}, \overline{\mathbf{16}}_{\text{flip}}\} + 2 \times \mathbf{10}_{\text{flip}}$, where $\mathbf{10}_{\text{flip}} \equiv \{\mathbf{5}_{-2}, \bar{\mathbf{5}}_2\}$.

Table 11
Chiral fields from the U sector. $+ (-)$ denotes $\frac{1}{2}$ $(\frac{-1}{2})$

$P \cdot V$	$\tilde{s}, U_i$	Visible states	$SU(5) \times U(1)_X$
$\frac{1}{12}$	$(+-+), U_3$	$(\underline{+----}; +++)$	$\mathbf{5}_3^L$
		$(\underline{+++--}; +--)$	$\overline{\mathbf{10}}_{-1}^L$
		$(\underline{+++++}; +++)$	$\mathbf{1}_{-5}^L$
$\frac{4}{12}$	$(++-), U_2$	$(\underline{-1, 0, 0, 0, 0}; -1, 0, 0)$	$\mathbf{5}_2^L$
$\frac{5}{12}$	$(+++), U_1$	$(\underline{++++-}; -++)$	$\bar{\mathbf{5}}_{-3}^R$
		$(\underline{++---}; ---)$	$\mathbf{10}_1^R$
		$(-----; -++)$	$\mathbf{1}_5^R$
$P \cdot V$	$\tilde{s}, U_i$	Hidden states	$[SU(4) \times SU(2)^3]'$
$\frac{4}{12}$	$(++-), U_2$	$(--; ++; \pm\pm\mp\mp)', (--; --; \pm\pm\mp\mp)'$	
		$(--; ++; \pm\mp\pm\mp)', (--; --; \pm\mp\pm\mp)'$	$(\mathbf{6, 2, 1, 1})_L'$
		$(--; ++; \pm\mp\mp\pm)', (--; --; \pm\mp\mp\pm)'$	
$\frac{4}{12}$	$(++-), U_2$	$(1, 1; 0, 0; 0, 0, 0, 0)'$	singlet

Table 12
Massless states satisfying $(P + 6V) \cdot a_3 = 0 \bmod Z$ in T_6. The definitions of $\mathbf{5_3}$, $\mathbf{\bar{5}_{-3}}$, $\mathbf{10_1}$, $\mathbf{\overline{10}_{-1}}$, and $\mathbf{1_{\pm 5,0}}$ are found in the main text

$P + 6V$	$(N^L)_j$	Θ_0	$\mathcal{P}_6$	χ
$(\mathbf{5_3}; \frac{1}{2}, 0, 0)(0^8)'$	0	$\frac{1}{2}$	2	L
$(\mathbf{\overline{10}_{-1}}; \frac{1}{2}, 0, 0)(0^8)'$	0	0	4	L
$(\mathbf{1_{-5}}; \frac{1}{2}, 0, 0)(0^8)'$	0	$\frac{1}{2}$	2	L
$(\mathbf{\bar{5}_{-3}}; \frac{-1}{2}, 0, 0)(0^8)'$	0	$\frac{-1}{6}$	2	L
$(\mathbf{10_1}; \frac{-1}{2}, 0, 0)(0^8)'$	0	$\frac{1}{3}$	3	L
$(\mathbf{1_5}; \frac{-1}{2}, 0, 0)(0^8)'$	0	$\frac{-1}{6}$	2	L
$(\mathbf{1_0}; 0, \frac{1}{2}, \frac{1}{2})(0^8)'$	1_3	0	4	L
	$1_{\bar{3}}$	$\frac{-1}{6}$	2	L
	$1_{\bar{1}}$	$\frac{1}{2}$	2	L
	1_1	$\frac{1}{3}$	3	L
$(\mathbf{1_0}; 0, \frac{-1}{2}, \frac{-1}{2})(0^8)'$	1_3	$\frac{1}{2}$	2	L
	$1_{\bar{3}}$	$\frac{1}{3}$	3	L
	$1_{\bar{1}}$	0	4	L
	1_1	$\frac{-1}{6}$	2	L

Table 13
Chiral matter fields satisfying $\Theta_{0,+,-} = 0$ in the $T_2^{0,+,-}$ sectors

$P + 2V$	$(N^L)_j$	$\mathcal{P}_2(f_0)$	χ
$(\mathbf{5_3}; \frac{-1}{6}, 0^2)(\frac{1}{3}, \frac{1}{3}; 0^6)'$	0	1	L
$(\mathbf{1_{-5}}; \frac{-1}{6}, 0^2)(\frac{1}{3}, \frac{1}{3}; 0^6)'$	0	1	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{2}, \frac{-1}{2})(\frac{1}{3}, \frac{1}{3}; 0^6)'$	$2_{\bar{1}}, 2_3$	$1 + 1$	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{2}, \frac{1}{2})(\frac{1}{3}, \frac{1}{3}; 0^6)'$	$1_{\bar{2}}, \{1_{\bar{1}} + 1_3\}$	$1 + 1$	L
$(\mathbf{1_0}; \frac{-2}{3}, \frac{1}{2}, \frac{1}{2})(\frac{1}{3}, \frac{1}{3}; 0^6)'$	$1_{\bar{1}}$	1	L
$(\mathbf{1_0}; \frac{-2}{3}, \frac{-1}{2}, \frac{-1}{2})(\frac{1}{3}, \frac{1}{3}; 0^6)'$	1_3	1	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{2}, \frac{1}{2})(\frac{-2}{3}, \frac{-2}{3}; 0^6)'$	0	1	L
$P + 2V_+$	$(N^L)_j$	$\mathcal{P}_2(f_+)$	χ
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{6}, \frac{1}{6})(\mathbf{\bar{4}, 1, 1, 2})'_{2+}$	0	1	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{6}, \frac{1}{6})(\mathbf{1, 1, 2, 2})'_{2+}$	0	1	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{6}, \frac{1}{6})(\mathbf{1, 2, 1, 1})'_{2+}$	$2_{\bar{1}}, 2_3$	$1 + 1$	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{-1}{6}, \frac{1}{6})(\mathbf{1, 1, 1, 1})'_{2+}$	$2_{\bar{1}}, 2_3$	$1 + 1$	L
$P + 2V_-$	$(N^L)_j$	$\mathcal{P}_2(f_-)$	χ
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{6}, \frac{-1}{6})(\mathbf{6, 1, 1, 1})'_{2-}$	0	1	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{6}, \frac{-1}{6})(\mathbf{\bar{4}, 1, 2, 1})'_{2-}$	0	1	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{6}, \frac{-1}{6})(\mathbf{1, 1, 1, 1})'_{2-}$	0	1	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{6}, \frac{-1}{6})(\mathbf{1, 2, 1, 1})'_{2-}$	$1_{\bar{2}}, \{1_{\bar{1}} + 1_3\}$	$1 + 1$	L
$(\mathbf{1_0}; \frac{1}{3}, \frac{1}{6}, \frac{-1}{6})(\mathbf{1, 1, 1, 1})''_{2-}$	$1_{\bar{2}}, \{1_{\bar{1}} + 1_3\}$	$1 + 1$	L

- Exotic particles: $16 \times \{\mathbf{1}_{5/2}, \mathbf{1}_{-5/2}\}$.
- A lot of neutral singlets under $SU(5) \times U(1)_X$.

The full massless spectrum of Model II is presented in Tables 11–15.

In Table 13, the abbreviated symbols denote

$$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{2})'_{2+} \equiv \left(\tfrac{-2}{3}, \tfrac{1}{3}; 0, 0; \tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)',$$

$$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})'_{2+} \equiv \left(\tfrac{5}{6}, \tfrac{-1}{6}; \tfrac{1}{2}, \tfrac{-1}{2}; \tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{-1}{6}\right)',$$

$$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{2+} \equiv \left(\tfrac{-1}{6}, \tfrac{-1}{6}; \pm\tfrac{1}{2}, \pm\tfrac{1}{2}; \tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{-1}{6}\right)',$$

$$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{2+} \equiv \left(\tfrac{1}{3}, \tfrac{1}{3}; 0, 0; \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)',$$

Table 14

Chiral matter fields in the T_4^0, T_4^+ and T_4^- sectors. In the T_4^+ and T_4^- sectors, $(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{4+} = (\tfrac{1}{6}, \tfrac{1}{6}; \pm\pm; \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6})'$, $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{4+} = (\tfrac{-1}{3}, \tfrac{-1}{3}; 0, 0; \tfrac{-1}{3}, \tfrac{-1}{3}, \tfrac{-1}{3}, \tfrac{-1}{3})'$, $(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{4-} = (\tfrac{1}{6}, \tfrac{1}{6}; \pm\pm; \tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{-1}{6}, \tfrac{-1}{6})'$ and $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{4-} = (\tfrac{-1}{3}, \tfrac{-1}{3}; 0, 0; \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})'$

$P + 4V$	$(N^L)_j$	Θ_0	$\mathcal{P}_4(f_0)$	χ
$(\mathbf{5}_{-2}; \tfrac{-1}{3}, 0^2)(\tfrac{-1}{3}, \tfrac{-1}{3}; 0^6)'$	0	0	3	L
$(\bar{\mathbf{5}}_2; \tfrac{-1}{3}, 0^2)(\tfrac{-1}{3}, \tfrac{-1}{3}; 0^6)'$	0	$\tfrac{1}{2}$	2	L
$(\mathbf{1}_0; \tfrac{2}{3}, 0^2)(\tfrac{-1}{3}, \tfrac{-1}{3}; 0^6)'$	$1_{\bar{1}}$	$\tfrac{-1}{4}$	2	L
	1_2	$\tfrac{1}{2}$	2	L
	1_3	$\tfrac{1}{4}$	2	L
$(\mathbf{1}_0; \tfrac{2}{3}, 0^2)(\tfrac{2}{3}, \tfrac{2}{3}; 0^6)'$	0	$\tfrac{1}{2}$	2	L
$(\mathbf{1}_0; \tfrac{-1}{3}, \pm 1, 0)(\tfrac{-1}{3}, \tfrac{-1}{3}; 0^6)'$	0	$\tfrac{1}{4}$	$2 + 2$	L
$(\mathbf{1}_0; \tfrac{-1}{3}, 0, \pm 1)(\tfrac{-1}{3}, \tfrac{-1}{3}; 0^6)'$	0	$\tfrac{-1}{4}$	$2 + 2$	L
$P + 4V_+$	$(N^L)_j$	Θ_+	$\mathcal{P}_4(f_+)$	χ
$(\mathbf{1}_0; \tfrac{2}{3}, \tfrac{-1}{3}, \tfrac{1}{3})(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{4+}$	0	0	3	L
$(\mathbf{1}_0; \tfrac{2}{3}, \tfrac{-1}{3}, \tfrac{1}{3})(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{4+}$	0	0	3	L
$(\mathbf{1}_0; \tfrac{-1}{3}, \tfrac{-1}{3}, \tfrac{-2}{3})(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{4+}$	0	$\tfrac{1}{4}$	2	L
$(\mathbf{1}_0; \tfrac{-1}{3}, \tfrac{-1}{3}, \tfrac{-2}{3})(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{4+}$	0	$\tfrac{1}{4}$	2	L
$(\mathbf{1}_0; \tfrac{-1}{3}, \tfrac{2}{3}, \tfrac{1}{3})(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{4+}$	0	$\tfrac{-1}{4}$	2	L
$(\mathbf{1}_0; \tfrac{-1}{3}, \tfrac{2}{3}, \tfrac{1}{3})(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{4+}$	0	$\tfrac{-1}{4}$	2	L
$P + 4V_-$	$(N^L)_j$	Θ_-	$\mathcal{P}_4(f_-)$	χ
$(\mathbf{1}_0; \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-1}{3})(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{4-}$	0	0	3	L
$(\mathbf{1}_0; \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-1}{3})(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{4-}$	0	0	3	L
$(\mathbf{1}_0; \tfrac{-1}{3}, \tfrac{1}{3}, \tfrac{2}{3})(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{4-}$	0	$\tfrac{1}{4}$	2	L
$(\mathbf{1}_0; \tfrac{-1}{3}, \tfrac{1}{3}, \tfrac{2}{3})(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{4-}$	0	$\tfrac{1}{4}$	2	L
$(\mathbf{1}_0; \tfrac{-1}{3}, \tfrac{-2}{3}, \tfrac{-1}{3})(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{4-}$	0	$\tfrac{-1}{4}$	2	L
$(\mathbf{1}_0; \tfrac{-1}{3}, \tfrac{-2}{3}, \tfrac{-1}{3})(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{4-}$	0	$\tfrac{-1}{4}$	2	L

Table 15
Chiral matter fields satisfying $\Theta_{0,+} = 0$ in the T_1^0, T_1^+, T_5^0, and T_5^+ sectors. There are no massless states in T_1^- and T_5^-. Here, $\mathbf{1}_{\mp 5/2} \equiv (\pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4})$. In T_1^0 and T_1^+, $(\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{1+} \equiv (\frac{1}{6}, \frac{1}{6}; 0, 0; \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3})'$, $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})'_{1+} \equiv (\frac{-1}{3}, \frac{-1}{3}; \pm\mp; \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})'$. In T_5^0 and T_5^+, $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})_5 = (\frac{5}{6}, \frac{-1}{6}; 0, 0; 0, 0, 0, 0)'$, $(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{5+} = (\frac{-1}{6}, \frac{-1}{6}; 0, 0; \underline{\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}})'$, and $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})'_{5+} = (\frac{1}{3}, \frac{1}{3}; \pm\mp; \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6})'$

$P + V$		$(N^L)_j$	$\mathcal{P}_1(f_0)$	χ
$(\mathbf{1}_{-5/2}; \frac{5}{12}, \frac{6}{12}, 0)(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})'_1$		0	1	L
$(\mathbf{1}_{5/2}; \frac{-1}{12}, 0, \frac{-6}{12})(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})'_1$		1_3	1	L
$P + V_+$		$(N^L)_j$	$\mathcal{P}_1(f_+)$	χ
$(\mathbf{1}_{-5/2}; \frac{5}{12}, \frac{2}{12}, \frac{4}{12})(\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{1+}$		0	1	L
$(\mathbf{1}_{-5/2}; \frac{5}{12}, \frac{2}{12}, \frac{4}{12})(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})'_{1+}$		0	1	L
$(\mathbf{1}_{5/2}; \frac{-1}{12}, \frac{-4}{12}, \frac{-2}{12})(\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{1+}$		1_3	1	L
$(\mathbf{1}_{5/2}; \frac{-1}{12}, \frac{-4}{12}, \frac{-2}{12})(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})'_{1+}$		1_3	1	L
$P + 5V$		$(N^L)_j$	$\mathcal{P}_5(f_0)$	χ
$(\mathbf{1}_{5/2}; \frac{1}{12}, \frac{-6}{12}, 0)(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})'_5$		1_1	1	R
$(\mathbf{1}_{-5/2}; \frac{-5}{12}, 0, \frac{6}{12})(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})'_5$		0	1	R
$P + 5V_+$		$(N^L)_j$	$\mathcal{P}_5(f_+)$	χ
$(\mathbf{1}_{5/2}; \frac{1}{12}, \frac{-2}{12}, \frac{-4}{12})(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{5+}$		1_1	1	R
$(\mathbf{1}_{5/2}; \frac{1}{12}, \frac{-2}{12}, \frac{-4}{12})(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})'_{5+}$		1_1	1	R
$(\mathbf{1}_{-5/2}; \frac{-5}{12}, \frac{4}{12}, \frac{2}{12})(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{5+}$		0	1	R
$(\mathbf{1}_{-5/2}; \frac{-5}{12}, \frac{4}{12}, \frac{2}{12})(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})'_{5+}$		0	1	R

$$(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{2-} \equiv \begin{cases} \left(\frac{1}{3}, \frac{1}{3}; 0, 0; \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3}\right)', \\ \left(\frac{1}{3}, \frac{1}{3}; 0, 0; \frac{-1}{3}, \frac{-1}{3}, \frac{2}{3}, \frac{2}{3}\right)', \\ \left(\frac{1}{3}, \frac{1}{3}; 0, 0; \underline{\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}}\right)', \end{cases}$$

$$(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{1})'_{2-} \equiv \left(\frac{-1}{6}, \frac{-1}{6}; \pm\mp; \underline{\frac{-5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}}\right)',$$

$$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})'_{2-} \equiv \left(\frac{-2}{3}, \frac{-2}{3}; 0, 0; \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3}\right)',$$

$$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})'_{2-} \equiv \left(\frac{-1}{6}, \frac{-1}{6}; \pm\pm; \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)',$$

$$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})''_{2-} \equiv \left(\frac{1}{3}, \frac{1}{3}; 0, 0; \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3}\right)'.$$

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